Bertrand's Postulate and the Sum of Primes

by
YUNG ZHAO
Tianjin University
Tianjin, 300072
China
E-mail: yzhao@tju.edu.cn *

Abstract

It is deduced from Bertrand's Postulate that every even integer greater than 4 is the sum of two primes.

MSC 2020 11A41, 11N05

1 Introduction

One of the most famous problems in mathematics is Goldbach Conjecture. Tens of thousands of mathematicians all over the world have devoted to solve this problem, which still has been attracting interests of a lot of researchers over the past decade [1] [2] [3] [4] [5] [6] [7] [8].

Up to now, the best result on Goldbach Conjecture has been due to Jinrun Chen [9]. He proved that every sufficiently large even integer *2N* can be presented as the sum of a prime p and a number q that is divisible by at most two different primes. Methods used to address Goldbach Conjecture include sieve method and circle method. Nevertheless, there are key limitations inherent in these methods. For example, the barrier from Chen's theorem to the Goldbach Conjecture has been well known as the parity problem in sieve theory: no one can be sure whether $q = 2N - p$ has exactly one or two prime divisors. The conjecture is still very much open and very significant new ideas are required for the proof.

We discover that the final solution lies in Bertrand's Postulate.

2 The Sum of Primes

Bertrand's Postulate *For all integers $N \geq 2$, there is always at least one prime P such that $N < P < 2N$*

We construct an interval:

$$(\ N,\ \ 2N\ )$$

where integer N ≥ 4. In this interval, there exists at least one prime, and let P_n denote it.

We construct another interval:

$$(\frac{2N-P_n}{2}, \quad 2N-P_n\ )$$

By Bertrand's Postulate, there exists at least one prime in this interval. Let p_{max} denote the largest prime, then,

$$\frac{(2N-P_n)}{2} < p_{max} < (2N-P_n)$$

$$\Downarrow$$

$$p_{max} \leq (2N-P_n)-2$$

which means two possibilities:

$$\text{①}\ p_{max} < (2N-P_n)-2$$

$$\text{②}\ p_{max} = (2N-P_n)-2$$

• If it is case ② , $\Longrightarrow 2(N-1) = P_n + p_{max}$

It indicates that even integer $2(N-1)$ is the sum of prime P_n and prime p_{max}

Thus, Goldbach Conjecture holds.

• If it is case ① , we go on to construct the following interval:

$$(\frac{2N-P_n+2}{2}, \quad 2N-P_n+2\ )$$

By Bertrand's Postulate,

$$\frac{(2N-P_n+2)}{2} < p_{max} < (2N-P_n+2)$$

$$\Downarrow$$

$$p_{max} \leq (2N-P_n)$$

which means two possibilities:

$$\text{①}\ p_{max} < (2N-P_n)-2$$

$$\text{③}\ p_{max} = 2N-P_n$$

• If it is case ③ , $\Longrightarrow 2N = P_n + p_{max}$

It indicates that even integer $2N$ is the sum of prime P_n and prime p_{max}

Thus, Goldbach Conjecture holds.

• If it is case ① , we go on to construct the following interval:

$$(\frac{2N-P_n+4}{2}, \quad 2N-P_n+4\ )$$

.

.

.

$$(\frac{2N-P_n+2k}{2}, \quad 2N-P_n+2k\ )$$

and it can be proved that: $2(N+k) = P_n + p_{max}$ $(k,\ integer \geq 1)$

Thus, Goldbach Conjecture holds.

The maximum k is $\frac{P_n-N}{2}$.

when $k = \frac{P_n-N}{2}$, $N = P_n \Longrightarrow 2N = P_n + P_n$

3 Conclusion

Wir müssen wissen
Wir werden wissen *(D. Hilbert)*

In 1900, D. Hilbert listed Goldbach Conjecture in the 8th mathematical problems at the International Mathematical Conference held in Paris. Today, it is proved with Bertrand's Postulate.

References

[1] Shanks, Daniel. Solved and Unsolved Problems in Number Theory. New York: Spartan Books, p. 30, 1962

[2] Hayat Rezgui. Conjecture of Twin Prime (Still Unsolved Problems in number theory). An Expository Essay. *Surveys in Mathematics and Its Applications*, 12: 229-252, 2017

[3] Renato Betti. The Twin Primes Conjectures and other Curiosities Regarding Prime Numbers. *Lettera Matematica*, 5(4): 297-303, 2017

[4] J.J. Hoskins. Proofs of the Twin Primes and Goldbach Conjecture. *arXiv. 1901.09668v7*, 2019

[5] Andri Lopez. Twin Primes Conjecture and Two Problems More. *International Journal of Mathematics and Computation*, 29(4): 63-66, 2018

[6] Maria Suzuki. Alternative Formulations of the Twin Prime Problem. *The American Mathematical Monthly*, 107(1): 55-56, 2000

[7] M. Ram Mourty and Akshaa Vatwani. Twin Primes and the Purity Problem. *Journal of Number Theory*, 180: 643-659, 2017

[8] Stephen Ramon Garcia, Elvis Kahoro and Florian Luca. Primitive Root bios for Twin Primes, *Experimental Mathematics*, 28 (2): 151-160, 2019

[9] J. CHEN, On the representation of a large even integer as the sum of a prime and the product of at most two primes. *Science in China, Series A.* 2(1973), 111-128