Bertrand-Chebyshev Theorem and small gaps between primes

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Abstract

The final solution to the problem about small gaps between primes lies in Bertrand-Chebyshev Theorem. we construct a pair of intervals [3n, 6n], [6n, 12n], and a set: { p_{6n-}^{max} , 6n, p_{6n+}^{min} }, where p_{6n-}^{max} denotes the largest prime in [3n, 6n] and p_{6n+}^{min} denotes the smallest prime in [6n, 12n]. Analyzing and dealing with them by the combination of Bertrand-Chebyshev Theorem, T. Tao's result and the elemental property of primes reveal that, there can't be the case: $\{p_{6n-}^{max} \neq (6n-1),\ 6n,\ p_{6n+}^{min} \neq (6n+1)\}$, there are only three possible cases: ① $\{p_{6n-}^{max} = (6n-1),\ 6n,\ p_{6n+}^{min} \neq (6n+1)\}$, ② $\{p_{6n-}^{max} = (6n-1),\ 6n,\ p_{6n+}^{min} = (6n+1)\}$, ③ $\{p_{6n-}^{max} \neq (6n-1),\ 6n,\ p_{6n+}^{min} = (6n+1)\}$. For each 6n, there must be one of the three cases.
As $6n \to \infty$, each case $\to$ infinitely often. Hence, ① $2 < \liminf_{6n\to\infty}(\ p_{6n+}^{min} - p_{6n-}^{max}\ ) \leq 246$ (in case ① and ③); ② $\liminf_{6n\to\infty}(p_{6n+}^{min} - p_{6n-}^{max}) = 2$ (in case ②).

Keywords. Bertrand-Chebyshev Theorem, Twin Prime Conjecture

AMS subject classifications. 11A41, 11N05

1 Introduction

One of the most famous problems in mathematics is the Twin Prime Conjecture. It arose from an open question about the "distribution of prime number". The conjecture states that there exist infinitely many primes P such that P+2 is a prime. The twin prime 181 and 179, for instance, have a gap of $181 - 179 = 2$. No one know how old the Twin Prime Conjecture is, but it was certainly considered by de Polignac over 174 years ago[1]. Since 1900, tens of thousands of mathematicians all over the world have devoted to solve this problem, which still has been attracting interests of a lot of researchers over the past decade [2] [3] [4] [5] [6] [7] [8] [9].

Up to now, Y. Zhang [10], T. Tao and dozens of mathematicians [11] have succeeded in making dramatic new progress. Y. Zhang proved that there are infinitely many consecutive primes with a distance of 7×10^7 at most, and it was afterward lessened down to 246.

Methods used to achieve these rather deep results above include sieve method and circle method. Nevertheless, there are key limitations inherent in these methods. This conjecture is still very much open and very significant new ideas are required for the final proof. In fact, the final solution lies in Bertrand-Chebyshev Theorem.

2 The small gap between primes

BERTRAND-CHEBYSHEV THEOREM. *For all integers $X \geq 2$, there exists at least one prime P: $X < P < 2X$.*

2.1 For integer n ≥ 1, we construct a pair of intervals:

$$[3n,\ 6n],\ [6n,\ 12n]$$

By Bertrand-Chebyshev Theorem, there exist primes in both intervals. Let p_{6n-}^{max} denote the largest prime in [3n, 6n] and p_{6n+}^{min} denote the smallest prime in [6n, 12n]. Then,

$$p_{6n-}^{max} = (6n-1) - 2k_1 \quad (k_1,\ integer \geq 0)$$
$$p_{6n+}^{min} = (6n+1) + 2k_2 \quad (k_2,\ integer \geq 0)$$
$$|p_{6n+}^{min} - p_{6n-}^{max}| = 2k \quad (k,\ integer \geq 1)$$

2.2 We introduce and consider a sequence:

$$\{\ p_{6n-}^{max},\ 6n,\ p_{6n+}^{min}\ \}$$

By the elemental property of primes, every prime greater than 3 is of either the form "6n+1"or the form "6n−1", so are p_{6n-}^{max} and p_{6n+}^{min}. Thus,
there can't be the following case:

$$\{\ p_{6n-}^{max} \neq 6n-1,\ 6n,\ p_{6n+}^{min} \neq 6n+1\ \}$$

there are only three cases:

case ①
$$\{\ p_{6n-}^{max} = 6n-1,\ 6n,\ p_{6n+}^{min} \neq 6n+1\ \}$$

case ②
$$\{\ p_{6n-}^{max} = 6n-1,\ 6n,\ p_{6n+}^{min} = 6n+1\ \}$$

case ③
$$\{\ p_{6n-}^{max} \neq 6n-1,\ 6n,\ p_{6n+}^{min} = 6n+1\ \}$$

For each 6n, there must be one of the three cases.
As $6n \to \infty$, each case $\to$ infinitely often. Hence

$$(in\ case\ ②\ ) \quad \liminf_{6n\to\infty}(\ p_{6n+}^{min} - p_{6n-}^{max}\ ) = 2$$
$$(in\ case\ ①\ and\ ③\ ) \quad \liminf_{6n\to\infty}(\ p_{6n+}^{min} - p_{6n-}^{max}\ ) > 2$$

Thus, Twin Prime Conjecture holds.

It has been proved that there are infinitely many consecutive primes with a distance of 246 at most. Therefore:

$$(in\ case\ ①\ and\ ③\ ) \quad 2 < \liminf_{6n\to\infty}(\ p_{6n+}^{min} - p_{6n-}^{max}\ ) \leq 246$$

.

3 Conclusion

Wir müssen wissen
Wir werden wissen *(D. Hilbert)*

In 1900, D. Hilbert listed Twin Prime Conjecture in the 8th mathematical problems at the International Mathematical Conference held in Paris. Today, it can be proved by way of Bertrand-Chebyshev Theorem.

Had J. Bertrand and P. Chebyshev proposed Twin Prime Conjecture, and written:

Cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.

We should believe them.

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