Considerations on the 3n+1 problem

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Abstract
This paper presents some considerations on the 3n+1 problem. In particular on the next odd elements in the sequence lower than the starting number.

3n + 1 problem (or conjecture)

In the $3\cdot n+1$ problem[1] it is possible to define the function:

$$f(x)=\begin{cases} 3\cdot x+1 & if\ x\equiv 1 (mod\, 2) \\ \frac{x}{2} & if\ x\equiv 0 (mod\, 2) \end{cases}$$

The sequence obtained using the function is as follows:

$$n_i=\begin{cases} n & for\ i=0 \\ f(n_{i-1}) & for\ i>0 \end{cases}$$

the sequence can be rewritten considering only the odd terms, then starting from an odd number n if x is an odd integer we have $f(x)=\frac{3\cdot x+1}{2^a}$ (Syracuse function[1]) with $a\geq 1$ and $\frac{3\cdot x+1}{2^a}$ odd, therefore:

$$n_1=\frac{3\cdot n+1}{2^{a_1}}$$

$$n_2=\frac{3\cdot n_1+1}{2^{a_2}}=\frac{3\cdot\frac{3\cdot n+1}{2^{a_1}}+1}{2^{a_2}}$$

$\ldots$

if $k>1$

$$n_k=\frac{3\cdot n_{k-1}+1}{2^{a_k}}=\frac{3^k\cdot n+3^{k-1}+\sum_{j=0}^{k-2} 3^j\cdot 2^{\sum_{i=1}^{k-1-j} a_i}}{2^{\sum_{i=1}^{k} a_i}}$$

using this formula for n_k to find the next odd number lower than n it is possible to obtain a pattern given a certain classes of numbers modulo 2^c example for $n=43$ we have

$43\rightarrow 130\rightarrow 65\rightarrow 196\rightarrow 98\rightarrow 49\rightarrow 148\rightarrow 74\rightarrow 37\rightarrow\ldots$

then the next odd number in the sequence lower than 43 is $37=\frac{27\cdot 43+23}{32}$

and for $n=107$ the next odd number in the sequence less than 107 is $91=\frac{27\cdot 107+23}{32}$,

we have that the next odd number in the sequence less than $n\equiv 43(mod\, 64)$ is $\frac{27\cdot n+23}{32}$.

As it is possible to analyze also in OEIS A177789[2] the study based on the residue classes modulo 2^d with d obtained from the sequence OEIS A020914[3] and if we also consider the corresponding 3^k result from the sequence OEIS A020914 we can obtain the following results:

for the numbers $n\equiv 1(mod\ 4)$ we have $d=2$ and $k=1$ then

$n\rightarrow\frac{3\cdot n+1}{2^{2\cdot x+2}}$ if $n\equiv\frac{2^{2\cdot x+2}-1}{3}(mod\ 2^{2\cdot x+3})$

$n\rightarrow\frac{3\cdot n+1}{2^{2\cdot x+3}}$ if $n\equiv\frac{5\cdot 2^{2\cdot x+3}-1}{3}(mod\ 2^{2\cdot x+4})$

for the numbers $n\equiv 3(mod\ 16)$ we have $d=4$ and $k=2$ then

$n\rightarrow\frac{9\cdot n+5}{2^{6\cdot x+4}}$ if $n\equiv\frac{11\cdot 2^{6\cdot x+4}-5}{9}(mod\ 2^{6\cdot x+5})$

$n\rightarrow\frac{9\cdot n+5}{2^{6\cdot x+5}}$ if $n\equiv\frac{2^{6\cdot x+5}-5}{9}(mod\ 2^{6\cdot x+6})$

$n\rightarrow\frac{9\cdot n+5}{2^{6\cdot x+6}}$ if $n\equiv\frac{5\cdot 2^{6\cdot x+6}-5}{9}(mod\ 2^{6\cdot x+7})$

$n\rightarrow\frac{9\cdot n+5}{2^{6\cdot x+7}}$ if $n\equiv\frac{7\cdot 2^{6\cdot x+7}-5}{9}(mod\ 2^{6\cdot x+8})$

$n\rightarrow\frac{9\cdot n+5}{2^{6\cdot x+8}}$ if $n\equiv\frac{17\cdot 2^{6\cdot x+8}-5}{9}(mod\ 2^{6\cdot x+9})$

$n\rightarrow\frac{9\cdot n+5}{2^{6\cdot x+9}}$ if $n\equiv\frac{13\cdot 2^{6\cdot x+9}-5}{9}(mod\ 2^{6\cdot x+10})$

for the numbers $n\equiv 23(mod\ 32)$ we have $d=5$ and $k=3$ then

$n\rightarrow\frac{27\cdot n+19}{2^{18\cdot x+5}}$ if $n\equiv\frac{47\cdot 2^{18\cdot x+5}-19}{27}(mod\ 2^{18\cdot x+6})$

$$n \rightarrow \frac{27\cdot n+19}{2^{18\cdot x+6}} \quad \text{if} \quad n \equiv \frac{37\cdot 2^{18\cdot x+6}-19}{27} \left(mod\ 2^{18\cdot x+7}\right)$$

$$n \rightarrow \frac{27\cdot n+19}{2^{18\cdot x+7}} \quad \text{if} \quad n \equiv \frac{5\cdot 2^{18\cdot x+7}-19}{27} \left(mod\ 2^{18\cdot x+8}\right)$$

$$n \rightarrow \frac{27\cdot n+19}{2^{18\cdot x+8}} \quad \text{if} \quad n \equiv \frac{43\cdot 2^{18\cdot x+8}-19}{27} \left(mod\ 2^{18\cdot x+9}\right)$$

$$n \rightarrow \frac{27\cdot n+19}{2^{18\cdot x+9}} \quad \text{if} \quad n \equiv \frac{35\cdot 2^{18\cdot x+9}-19}{27} \left(mod\ 2^{18\cdot x+10}\right)$$

$$n \rightarrow \frac{27\cdot n+19}{2^{18\cdot x+10}} \quad \text{if} \quad n \equiv \frac{31\cdot 2^{18\cdot x+10}-19}{27} \left(mod\ 2^{18\cdot x+11}\right)$$

$$n \rightarrow \frac{27\cdot n+19}{2^{18\cdot x+11}} \quad \text{if} \quad n \equiv \frac{29\cdot 2^{18\cdot x+11}-19}{27} \left(mod\ 2^{18\cdot x+12}\right)$$

$$n \rightarrow \frac{27\cdot n+19}{2^{18\cdot x+12}} \quad \text{if} \quad n \equiv \frac{2^{18\cdot x+12}-19}{27} \left(mod\ 2^{18\cdot x+13}\right)$$

$$n \rightarrow \frac{27\cdot n+19}{2^{18\cdot x+13}} \quad \text{if} \quad n \equiv \frac{41\cdot 2^{18\cdot x+13}-19}{27} \left(mod\ 2^{18\cdot x+14}\right)$$

$$n \rightarrow \frac{27\cdot n+19}{2^{18\cdot x+14}} \quad \text{if} \quad n \equiv \frac{7\cdot 2^{18\cdot x+14}-19}{27} \left(mod\ 2^{18\cdot x+15}\right)$$

$$n \rightarrow \frac{27\cdot n+19}{2^{18\cdot x+15}} \quad \text{if} \quad n \equiv \frac{17\cdot 2^{18\cdot x+15}-19}{27} \left(mod\ 2^{18\cdot x+16}\right)$$

$$n \rightarrow \frac{27\cdot n+19}{2^{18\cdot x+16}} \quad \text{if} \quad n \equiv \frac{49\cdot 2^{18\cdot x+16}-19}{27} \left(mod\ 2^{18\cdot x+17}\right)$$

$$n \rightarrow \frac{27\cdot n+19}{2^{18\cdot x+17}} \quad \text{if} \quad n \equiv \frac{11\cdot 2^{18\cdot x+17}-19}{27} \left(mod\ 2^{18\cdot x+18}\right)$$

$$n \rightarrow \frac{27\cdot n+19}{2^{18\cdot x+18}} \quad \text{if} \quad n \equiv \frac{19\cdot 2^{18\cdot x+18}-19}{27} \left(mod\ 2^{18\cdot x+19}\right)$$

$$n \rightarrow \frac{27\cdot n+19}{2^{18\cdot x+19}} \quad \text{if} \quad n \equiv \frac{23\cdot 2^{18\cdot x+19}-19}{27} \left(mod\ 2^{18\cdot x+20}\right)$$

$$n \rightarrow \frac{27\cdot n+19}{2^{18\cdot x+20}} \quad \text{if} \quad f\,n \equiv \frac{25\cdot 2^{18\cdot x+20}-19}{27} \left(mod\ 2^{18\cdot x+21}\right)$$

$$n \rightarrow \frac{27\cdot n+19}{2^{18\cdot x+21}} \quad \text{if} \quad n \equiv \frac{53\cdot 2^{18\cdot x+21}-19}{27} \left(mod\ 2^{18\cdot x+22}\right)$$

$$n \rightarrow \frac{27\cdot n+19}{2^{18\cdot x+22}} \quad \text{if} \quad n \equiv \frac{13\cdot 2^{18\cdot x+22}-19}{27} \left(mod\ 2^{18\cdot x+23}\right)$$

for the numbers $n \equiv 11 (mod\ 32)$ we have $d=5$ and $k=3$ then

$$n \rightarrow \frac{27 \cdot n+23}{2^{18 \cdot x+5}} \quad \text{if} \quad n \equiv \frac{37 \cdot 2^{18 \cdot x+5}-23}{27} (mod\ 2^{18 \cdot x+6})$$

$$n \rightarrow \frac{27 \cdot n+23}{2^{18 \cdot x+6}} \quad \text{if} \quad n \equiv \frac{5 \cdot 2^{18 \cdot x+6}-23}{27} (mod\ 2^{18 \cdot x+7})$$

...

for the numbers $n \equiv 15 (mod\ 128)$ we have $d=7$ and $k=4$ then

$$n \rightarrow \frac{81 \cdot n+65}{2^{54 \cdot x+7}} \quad \text{if} \quad n \equiv \frac{91 \cdot 2^{54 \cdot x+7}-65}{81} (mod\ 2^{54 \cdot x+8})$$

$$n \rightarrow \frac{81 \cdot n+65}{2^{54 \cdot x+8}} \quad \text{if} \quad n \equiv \frac{5 \cdot 2^{54 \cdot x+8}-65}{81} (mod\ 2^{54 \cdot x+9})$$

...

We can conjecture that for any value of $k>1$ integer we can find $n_k<n$

$$n_k = \frac{3^k \cdot n+3^{k-1}+\sum_{j=0}^{k-2} 3^j \cdot 2^{\sum_{i=1}^{k-1-j} a_i}}{2^{\sum_{i=1}^{k} a_i}} = \frac{3^k \cdot n+s}{2^e}$$

with s odd integer not divisible by 3 and $s \geq 3^k-2^k$.

It should be noted that we are interested in the first odd number n_k in the sequence with $n_k<n$ therefore we assume that there is a value $k_1<k$ for which $n_{k_1}<n$ then

$$n_{k_1} = \frac{3^{k_1} \cdot n+s_{k_1}}{2^{e_{k_1}}} \quad \text{and} \quad n_k = n_{k_2} = \frac{3^{k_2} \cdot n_{k_1}+s_{k_2}}{2^{e_{k_2}}} = \frac{3^{k_2} \cdot \frac{3^{k_1} \cdot n+s_{k_1}}{2^{e_{k_1}}}+s_{k_2}}{2^{e_{k_2}}} = \frac{3^{k_1+k_2} \cdot n+3^{k_2} \cdot s_{k_1}+2^{e_{k_1}} \cdot s_{k_2}}{2^{e_{k_1}+e_{k_2}}}$$

with $k=k_1+k_2$, $e=e_{k_1}+e_{k_2}$ and $s=3^{k_2} \cdot s_{k_1}+2^{e_{k_1}} \cdot s_{k_2}$

therefore if we want that k to be the smallest value for which $n_k<n$ then it must be $s \neq 3^{k_2} \cdot s_{k_1}+2^{e_{k_1}} \cdot s_{k_2}$ for s_{k_1} and e_{k_1} values already associated with residue classes obtained with $k_1<k$.

In conclusion we have:

$$n \rightarrow \frac{3^k \cdot n+s}{2^{m \cdot b+d+r}} \quad \text{if} \quad n \equiv \frac{t \cdot 2^{m \cdot b+d+r}-s}{3^k} (mod\ 2^{m \cdot b+d+r+1})$$

with $b \geq 0$ integer, t every odd integer not divisible by 3 and $0< t <2 \cdot 3^k$, $m=2 \cdot 3^{k-1}$, $0 \leqslant r<m$ integer and d number of digits in the base-2 representation of 3^k .

Fixed k and then fixed $d=\lfloor 1+k\cdot\log_2(3)\rfloor$ for find the residue class $n\equiv h(mod\ 2^d)$ for $s=3^k-2^k$, which is obtained when $a_i=1$ for $1\leqslant i<k$, chosen $b=0$ and $r=0$ we need to find the possible values of t odd integer not divisible by 3 and $0<t<2\cdot 3^k$ such that $(t\cdot 2^d-3^k+2^k)$ it is divisible by 3^k from which $h=\left(\frac{t\cdot 2^d+2^k}{3^k}-1\right)\ mod\ 2^d$ is obtained.

To find the other values of s relative to the others residue classes $n\equiv h_i(mod\ 2^d)$, obtained as described in OEIS A177789, chosen $t=1$ and $b=0$ for some value of $0\leqslant r<m$ we need to find the possible values of s odd integer not divisible by 3 and $3^k-2^k<s<2\cdot 3^k$ such that $(2^{d+r}-s)$ it is divisible by 3^k and $\frac{2^{d+r}-s}{3^k}\equiv h_i(mod\ 2^d)$.

Stopping time

In the $3\cdot n+1$ problem it is possible to define the concept of stopping time[1]:

$$f(n)=\begin{cases}3\cdot n+1 & if\ n\equiv 1(mod\,2)\\ \frac{n}{2} & if\ n\equiv 0(mod\,2)\end{cases}$$

the sequence obtained using the function $f(n)$ is as follows:

$$a_i=\begin{cases}n & for\ i=0\\ f(a_{i-1}) & for\ i>0\end{cases}$$

the smallest i such that $a_i<a_0$ is called the stopping time of n .

If n is even $n\equiv 0(mod\,2)$ then the stopping time $i=1$ and $a_1=\frac{n}{2}$.

If $n\equiv 1(mod\,2^2)$ then the stopping time $i=3$ and $a_3=\frac{3\cdot n+1}{2^2}$.

Fixed an integer o with $o>1$ if n odd and $n\equiv r(mod\,2^{k-o})$ then for some classes with $r>1$ it is possible to observe that

$$a_i=\frac{3^o\cdot n+b}{2^{k-o}}$$

with $b\geq 3^o-2^o$ and $b=3^{o-1}+\sum_{j=0}^{o-2}3^j\cdot 2^{\sum_{i=1}^{o-1-j}c_i}$

therefore from the definition of stopping time we have $a_i<n$ i.e if $n=r+q\cdot 2^{k-o}$

$$\frac{3^o\cdot n+b}{2^{k-o}}=\frac{3^o\cdot r+b}{2^{k-o}}+q\cdot 3^o<r+q\cdot 2^{k-o}$$ for any integer q with $q\geq 0$

obviously it has to be $3^o<2^{k-o}$ and if $q=0$ then $b<r\cdot(2^{k-o}-3^o)$

but from the definition of stopping time it also must be $a_{k-1}=2\cdot a_k>n$ then

$$2\cdot(\frac{3^o\cdot r+b}{2^{k-o}}+q\cdot 3^o)>r+q\cdot 2^{k-o}$$

and if $q=r$

$$3^o\cdot r+b+r\cdot 3^o\cdot 2^{k-o}>r\cdot 2^{k-o-1}+q\cdot 2^{k-o}\cdot 2^{k-o-1}$$

but $b<r\cdot(2^{k-o}-3^o)$ then

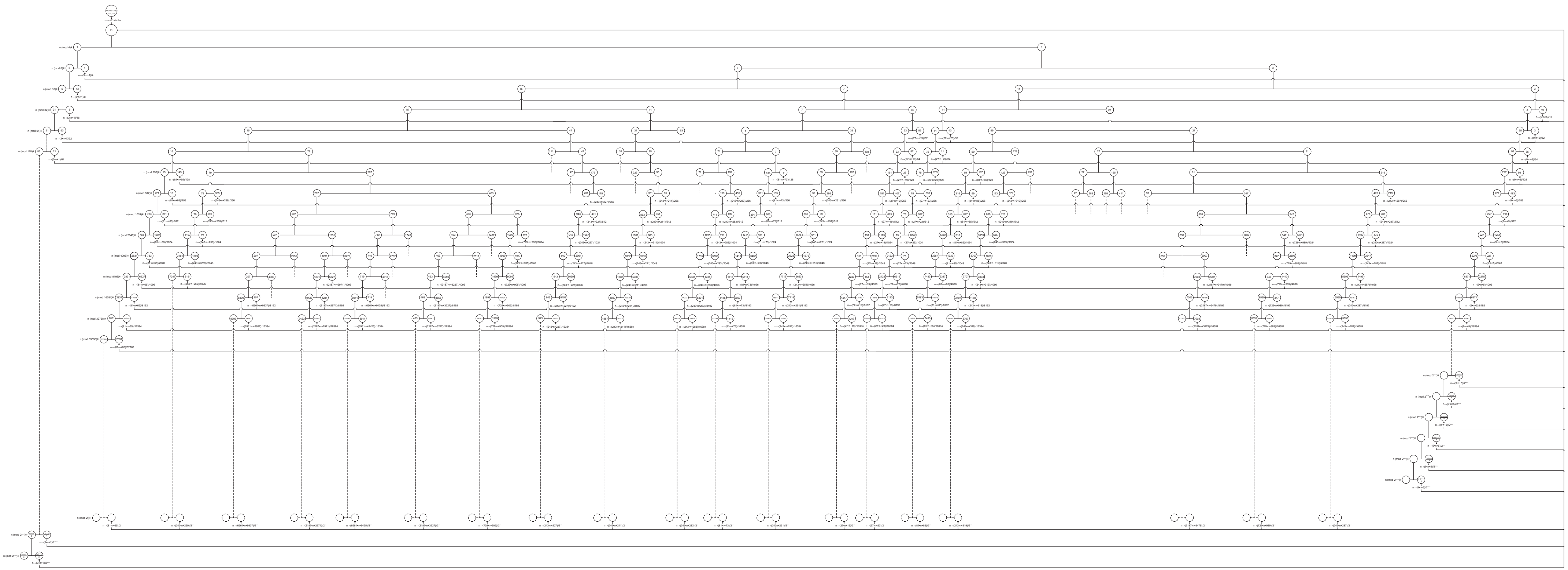
$$3^o\cdot r+r\cdot(2^{k-o}-3^o)+r\cdot 3^o\cdot 2^{k-o}>r\cdot 2^{k-o-1}+r\cdot 2^{k-o}\cdot 2^{k-o-1}$$

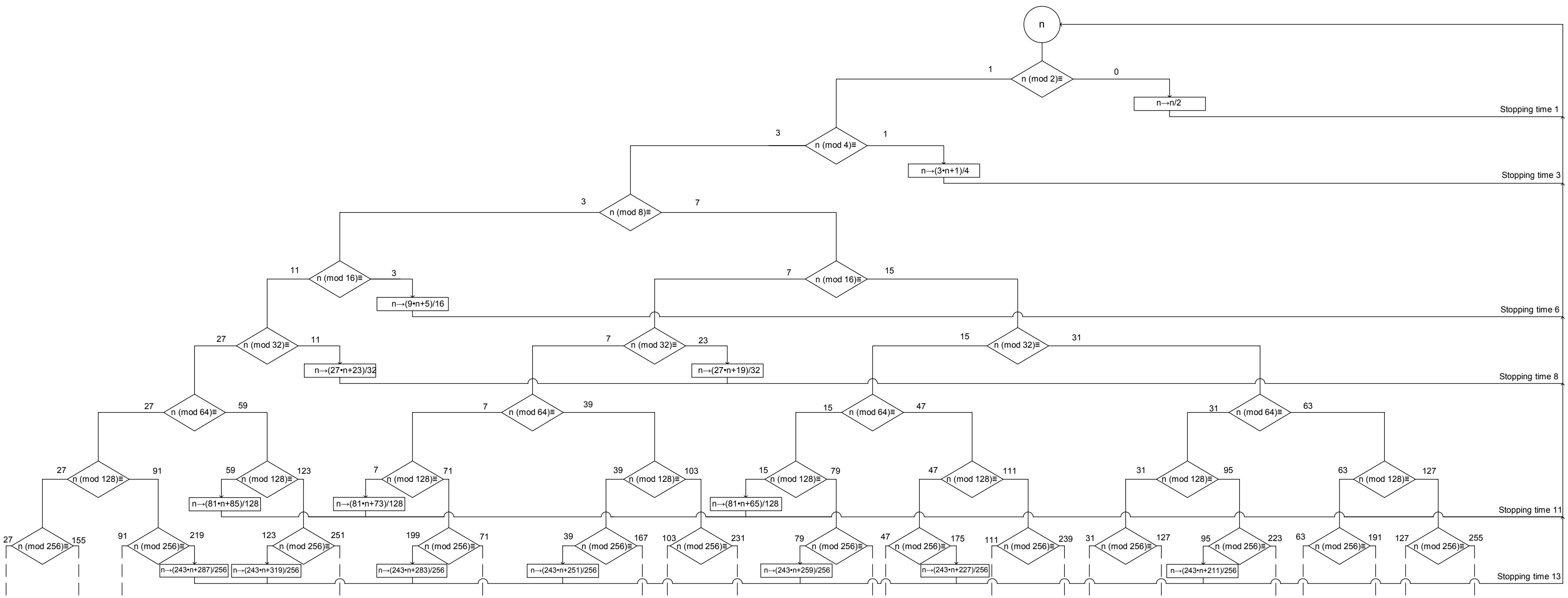
$$2^{k-o}+3^o\cdot 2^{k-o}>2^{k-o-1}+2^{k-o}\cdot 2^{k-o-1}$$

$$1+3^o>2^{-1}+2^{k-o-1}$$

from which the condition $2^{k-o-1}<3^o<2^{k-o}$ is found.

We can create two flowcharts (shown on the following pages) where the different residue classes are shown for each row of the graph.




n
n (mod 2)≡
1
0
n→n/2
Stopping time 1
n (mod 4)≡
3
1
n→(3•n+1)/4
Stopping time 3
n (mod 8)≡
3
7
n (mod 16)≡
11
3
n→(9•n+5)/16
n (mod 16)≡
7
15
Stopping time 6
n (mod 32)≡
27
11
n→(27•n+23)/32
n (mod 32)≡
7
23
n→(27•n+19)/32
n (mod 32)≡
15
31
Stopping time 8
n (mod 64)≡
27
59
n (mod 64)≡
7
39
n (mod 64)≡
15
47
n (mod 64)≡
31
63
n (mod 128)≡
27
91
n (mod 128)≡
59
123
n→(81•n+85)/128
n (mod 128)≡
7
71
n→(81•n+73)/128
n (mod 128)≡
39
103
n (mod 128)≡
15
79
n→(81•n+65)/128
n (mod 128)≡
47
111
n (mod 128)≡
31
95
n (mod 128)≡
63
127
Stopping time 11
n (mod 256)≡
27
155
n (mod 256)≡
91
219
n→(243•n+287)/256
n (mod 256)≡
123
251
n→(243•n+319)/256
n (mod 256)≡
199
71
n→(243•n+283)/256
n (mod 256)≡
39
167
n→(243•n+251)/256
n (mod 256)≡
103
231
n (mod 256)≡
79
n→(243•n+259)/256
n (mod 256)≡
47
175
n→(243•n+227)/256
n (mod 256)≡
111
239
n (mod 256)≡
31
127
n (mod 256)≡
95
223
n→(243•n+211)/256
n (mod 256)≡
63
191
n (mod 256)≡
127
255
Stopping time 13

References

[1] https://en.wikipedia.org/wiki/Collatz_conjecture

[2] https://oeis.org/A177789

[3] https://oeis.org/A020914