

Considerations on the $3n+1$ problem

V. Barbera

Abstract

This paper presents some considerations on the $3n+1$ problem. In particular on the next odd elements in the sequence lower than the starting number.

$3n + 1$ problem (or conjecture)

In the $3n+1$ problem^[1] it is possible to define the function:

$$f(x) = \begin{cases} 3 \cdot x + 1 & \text{if } x \equiv 1 \pmod{2} \\ \frac{x}{2} & \text{if } x \equiv 0 \pmod{2} \end{cases}$$

The sequence obtained using the function is as follows:

$$n_i = \begin{cases} n & \text{for } i=0 \\ f(n_{i-1}) & \text{for } i>0 \end{cases}$$

the sequence can be rewritten considering only the odd terms, then starting from an odd number n if x is an odd integer we have $f(x) = \frac{3 \cdot x + 1}{2^a}$ (Syracuse function^[1]) with

$a \geq 1$ and $\frac{3 \cdot x + 1}{2^a}$ odd, therefore:

$$n_1 = \frac{3 \cdot n + 1}{2^{a_1}}$$
$$n_2 = \frac{3 \cdot n_1 + 1}{2^{a_2}} = \frac{3 \cdot \frac{3 \cdot n + 1}{2^{a_1}} + 1}{2^{a_2}}$$

...

if $k > 1$

$$n_k = \frac{3 \cdot n_{k-1} + 1}{2^{a_k}} = \frac{3^k \cdot n + 3^{k-1} + \sum_{j=0}^{k-2} 3^j \cdot 2^{\sum_{i=1}^{k-1-j} a_i}}{2^{\sum_{i=1}^k a_i}}$$

using this formula for n_k to find the next odd number lower than n it is possible to obtain a pattern given a certain classes of numbers modulo 2^c example for $n=43$ we have

$$43 \rightarrow 130 \rightarrow 65 \rightarrow 196 \rightarrow 98 \rightarrow 49 \rightarrow 148 \rightarrow 74 \rightarrow 37 \rightarrow \dots$$

then the next odd number in the sequence lower than 43 is $37 = \frac{27 \cdot 43 + 23}{32}$

and for $n=107$ the next odd number in the sequence less than 107 is $91 = \frac{27 \cdot 107 + 23}{32}$,

we have that the next odd number in the sequence less than $n \equiv 43 \pmod{64}$ is $\frac{27 \cdot n + 23}{32}$.

As it is possible to analyze also in OEIS A177789^[2] the study based on the residue classes modulo 2^d with d obtained from the sequence OEIS A020914^[3] and if we also consider the corresponding 3^k result from the sequence OEIS A020914 we can obtain the following results:

for the numbers $n \equiv 1 \pmod{4}$ we have $d=2$ and $k=1$ then

$$n \rightarrow \frac{3 \cdot n + 1}{2^{2 \cdot x + 2}} \text{ if } n \equiv \frac{2^{2 \cdot x + 2} - 1}{3} \pmod{2^{2 \cdot x + 3}}$$

$$n \rightarrow \frac{3 \cdot n + 1}{2^{2 \cdot x + 3}} \text{ if } n \equiv \frac{5 \cdot 2^{2 \cdot x + 3} - 1}{3} \pmod{2^{2 \cdot x + 4}}$$

for the numbers $n \equiv 3 \pmod{16}$ we have $d=4$ and $k=2$ then

$$n \rightarrow \frac{9 \cdot n + 5}{2^{6 \cdot x + 4}} \text{ if } n \equiv \frac{11 \cdot 2^{6 \cdot x + 4} - 5}{9} \pmod{2^{6 \cdot x + 5}}$$

$$n \rightarrow \frac{9 \cdot n + 5}{2^{6 \cdot x + 5}} \text{ if } n \equiv \frac{2^{6 \cdot x + 5} - 5}{9} \pmod{2^{6 \cdot x + 6}}$$

$$n \rightarrow \frac{9 \cdot n + 5}{2^{6 \cdot x + 6}} \text{ if } n \equiv \frac{5 \cdot 2^{6 \cdot x + 6} - 5}{9} \pmod{2^{6 \cdot x + 7}}$$

$$n \rightarrow \frac{9 \cdot n + 5}{2^{6 \cdot x + 7}} \text{ if } n \equiv \frac{7 \cdot 2^{6 \cdot x + 7} - 5}{9} \pmod{2^{6 \cdot x + 8}}$$

$$n \rightarrow \frac{9 \cdot n + 5}{2^{6 \cdot x + 8}} \text{ if } n \equiv \frac{17 \cdot 2^{6 \cdot x + 8} - 5}{9} \pmod{2^{6 \cdot x + 9}}$$

$$n \rightarrow \frac{9 \cdot n + 5}{2^{6 \cdot x + 9}} \text{ if } n \equiv \frac{13 \cdot 2^{6 \cdot x + 9} - 5}{9} \pmod{2^{6 \cdot x + 10}}$$

for the numbers $n \equiv 23 \pmod{32}$ we have $d=5$ and $k=3$ then

$$n \rightarrow \frac{27 \cdot n + 19}{2^{18 \cdot x + 5}} \text{ if } n \equiv \frac{47 \cdot 2^{18 \cdot x + 5} - 19}{27} \pmod{2^{18 \cdot x + 6}}$$

$$n \rightarrow \frac{27 \cdot n + 19}{2^{18 \cdot x + 6}} \text{ if } n \equiv \frac{37 \cdot 2^{18 \cdot x + 6} - 19}{27} \pmod{2^{18 \cdot x + 7}}$$

$$n \rightarrow \frac{27 \cdot n + 19}{2^{18 \cdot x + 7}} \text{ if } n \equiv \frac{5 \cdot 2^{18 \cdot x + 7} - 19}{27} \pmod{2^{18 \cdot x + 8}}$$

$$n \rightarrow \frac{27 \cdot n + 19}{2^{18 \cdot x + 8}} \text{ if } n \equiv \frac{43 \cdot 2^{18 \cdot x + 8} - 19}{27} \pmod{2^{18 \cdot x + 9}}$$

$$n \rightarrow \frac{27 \cdot n + 19}{2^{18 \cdot x + 9}} \text{ if } n \equiv \frac{35 \cdot 2^{18 \cdot x + 9} - 19}{27} \pmod{2^{18 \cdot x + 10}}$$

$$n \rightarrow \frac{27 \cdot n + 19}{2^{18 \cdot x + 10}} \text{ if } n \equiv \frac{31 \cdot 2^{18 \cdot x + 10} - 19}{27} \pmod{2^{18 \cdot x + 11}}$$

$$n \rightarrow \frac{27 \cdot n + 19}{2^{18 \cdot x + 11}} \text{ if } n \equiv \frac{29 \cdot 2^{18 \cdot x + 11} - 19}{27} \pmod{2^{18 \cdot x + 12}}$$

$$n \rightarrow \frac{27 \cdot n + 19}{2^{18 \cdot x + 12}} \text{ if } n \equiv \frac{2^{18 \cdot x + 12} - 19}{27} \pmod{2^{18 \cdot x + 13}}$$

$$n \rightarrow \frac{27 \cdot n + 19}{2^{18 \cdot x + 13}} \text{ if } n \equiv \frac{41 \cdot 2^{18 \cdot x + 13} - 19}{27} \pmod{2^{18 \cdot x + 14}}$$

$$n \rightarrow \frac{27 \cdot n + 19}{2^{18 \cdot x + 14}} \text{ if } n \equiv \frac{7 \cdot 2^{18 \cdot x + 14} - 19}{27} \pmod{2^{18 \cdot x + 15}}$$

$$n \rightarrow \frac{27 \cdot n + 19}{2^{18 \cdot x + 15}} \text{ if } n \equiv \frac{17 \cdot 2^{18 \cdot x + 15} - 19}{27} \pmod{2^{18 \cdot x + 16}}$$

$$n \rightarrow \frac{27 \cdot n + 19}{2^{18 \cdot x + 16}} \text{ if } n \equiv \frac{49 \cdot 2^{18 \cdot x + 16} - 19}{27} \pmod{2^{18 \cdot x + 17}}$$

$$n \rightarrow \frac{27 \cdot n + 19}{2^{18 \cdot x + 17}} \text{ if } n \equiv \frac{11 \cdot 2^{18 \cdot x + 17} - 19}{27} \pmod{2^{18 \cdot x + 18}}$$

$$n \rightarrow \frac{27 \cdot n + 19}{2^{18 \cdot x + 18}} \text{ if } n \equiv \frac{19 \cdot 2^{18 \cdot x + 18} - 19}{27} \pmod{2^{18 \cdot x + 19}}$$

$$n \rightarrow \frac{27 \cdot n + 19}{2^{18 \cdot x + 19}} \text{ if } n \equiv \frac{23 \cdot 2^{18 \cdot x + 19} - 19}{27} \pmod{2^{18 \cdot x + 20}}$$

$$n \rightarrow \frac{27 \cdot n + 19}{2^{18 \cdot x + 20}} \text{ if } n \equiv \frac{25 \cdot 2^{18 \cdot x + 20} - 19}{27} \pmod{2^{18 \cdot x + 21}}$$

$$n \rightarrow \frac{27 \cdot n + 19}{2^{18 \cdot x + 21}} \text{ if } n \equiv \frac{53 \cdot 2^{18 \cdot x + 21} - 19}{27} \pmod{2^{18 \cdot x + 22}}$$

$$n \rightarrow \frac{27 \cdot n + 19}{2^{18 \cdot x + 22}} \text{ if } n \equiv \frac{13 \cdot 2^{18 \cdot x + 22} - 19}{27} \pmod{2^{18 \cdot x + 23}}$$

for the numbers $n \equiv 11 \pmod{32}$ we have $d=5$ and $k=3$ then

$$n \rightarrow \frac{27 \cdot n + 23}{2^{18 \cdot x + 5}} \text{ if } n \equiv \frac{37 \cdot 2^{18 \cdot x + 5} - 23}{27} \pmod{2^{18 \cdot x + 6}}$$

$$n \rightarrow \frac{27 \cdot n + 23}{2^{18 \cdot x + 6}} \text{ if } n \equiv \frac{5 \cdot 2^{18 \cdot x + 6} - 23}{27} \pmod{2^{18 \cdot x + 7}}$$

...

for the numbers $n \equiv 15 \pmod{128}$ we have $d=7$ and $k=4$ then

$$n \rightarrow \frac{81 \cdot n + 65}{2^{54 \cdot x + 7}} \text{ if } n \equiv \frac{91 \cdot 2^{54 \cdot x + 7} - 65}{81} \pmod{2^{54 \cdot x + 8}}$$

$$n \rightarrow \frac{81 \cdot n + 65}{2^{54 \cdot x + 8}} \text{ if } n \equiv \frac{5 \cdot 2^{54 \cdot x + 8} - 65}{81} \pmod{2^{54 \cdot x + 9}}$$

...

We can conjecture that for any value of $k > 1$ integer we can find $n_k < n$

$$n_k = \frac{3^k \cdot n + 3^{k-1} + \sum_{j=0}^{k-2} 3^j \cdot 2^{\sum_{i=1}^{k-1-j} a_i}}{2^{\sum_{i=1}^k a_i}} = \frac{3^k \cdot n + s}{2^e}$$

with s odd integer not divisible by 3 and $s \geq 3^k - 2^k$.

It should be noted that we are interested in the first odd number n_k in the sequence with $n_k < n$ therefore we assume that there is a value $k_1 < k$ for which $n_{k_1} < n$ then

$$n_{k_1} = \frac{3^{k_1} \cdot n + s_{k_1}}{2^{e_{k_1}}} \text{ and } n_k = n_{k_2} = \frac{3^{k_2} \cdot n_{k_1} + s_{k_2}}{2^{e_{k_2}}} = \frac{3^{k_2} \cdot \frac{3^{k_1} \cdot n + s_{k_1}}{2^{e_{k_1}}} + s_{k_2}}{2^{e_{k_2}}} = \frac{3^{k_1+k_2} \cdot n + 3^{k_2} \cdot s_{k_1} + 2^{e_{k_1}} \cdot s_{k_2}}{2^{e_{k_1}+e_{k_2}}}$$

with $k = k_1 + k_2$, $e = e_{k_1} + e_{k_2}$ and $s = 3^{k_2} \cdot s_{k_1} + 2^{e_{k_1}} \cdot s_{k_2}$

therefore if we want that k to be the smallest value for which $n_k < n$ then it must be

$s \neq 3^{k_2} \cdot s_{k_1} + 2^{e_{k_1}} \cdot s_{k_2}$ for s_{k_1} and e_{k_1} values already associated with residue classes obtained with $k_1 < k$.

In conclusion we have:

$$n \rightarrow \frac{3^k \cdot n + s}{2^{m \cdot b + d + r}} \text{ if } n \equiv \frac{t \cdot 2^{m \cdot b + d + r} - s}{3^k} \pmod{2^{m \cdot b + d + r + 1}}$$

with $b \geq 0$ integer, t every odd integer not divisible by 3 and $0 < t < 2 \cdot 3^k$, $m = 2 \cdot 3^{k-1}$,

$0 \leq r < m$ integer and d number of digits in the base-2 representation of 3^k .

Fixed k and then fixed $d = \lceil 1 + k \cdot \log_2(3) \rceil$ for find the residue class $n \equiv h \pmod{2^d}$ for $s = 3^k - 2^k$, which is obtained when $a_i = 1$ for $1 \leq i < k$, chosen $b = 0$ and $r = 0$ we need to find the possible values of t odd integer not divisible by 3 and $0 < t < 2 \cdot 3^k$ such that $(t \cdot 2^d - 3^k + 2^k)$ it is divisible by 3^k from which $h = \left(\frac{t \cdot 2^d + 2^k}{3^k} - 1 \right) \pmod{2^d}$ is obtained.

To find the other values of s relative to the others residue classes $n \equiv h_i \pmod{2^d}$, obtained as described in OEIS A177789, chosen $t = 1$ and $b = 0$ for some value of $0 \leq r < m$ we need to find the possible values of s odd integer not divisible by 3 and $3^k - 2^k < s < 2 \cdot 3^k$ such that $(2^{d+r} - s)$ it is divisible by 3^k and $\frac{2^{d+r} - s}{3^k} \equiv h_i \pmod{2^d}$.

An alternative method for obtaining the value of k associated with a generic remainder h_i is to use the following algorithm: we consider a generic integer $h_i \equiv 3 \pmod{4}$ the search is done for all values of k starting from $k = 2$ until s satisfies the condition $s = 3^{k-1} + \sum_{j=0}^{k-2} 3^j \cdot 2^{\sum_{i=1}^{k-1-j} a_i}$. Therefore fixed k and then fixed $d = \lceil 1 + k \cdot \log_2(3) \rceil$ we have $s = 2^d - 3^k \cdot h_i \pmod{2^d}$ or $s = 2^{d+1} - 3^k \cdot h_i \pmod{2^d}$ if $2^d - 3^k \cdot h_i \pmod{2^d} < 3^k - 2^k$.

Example:

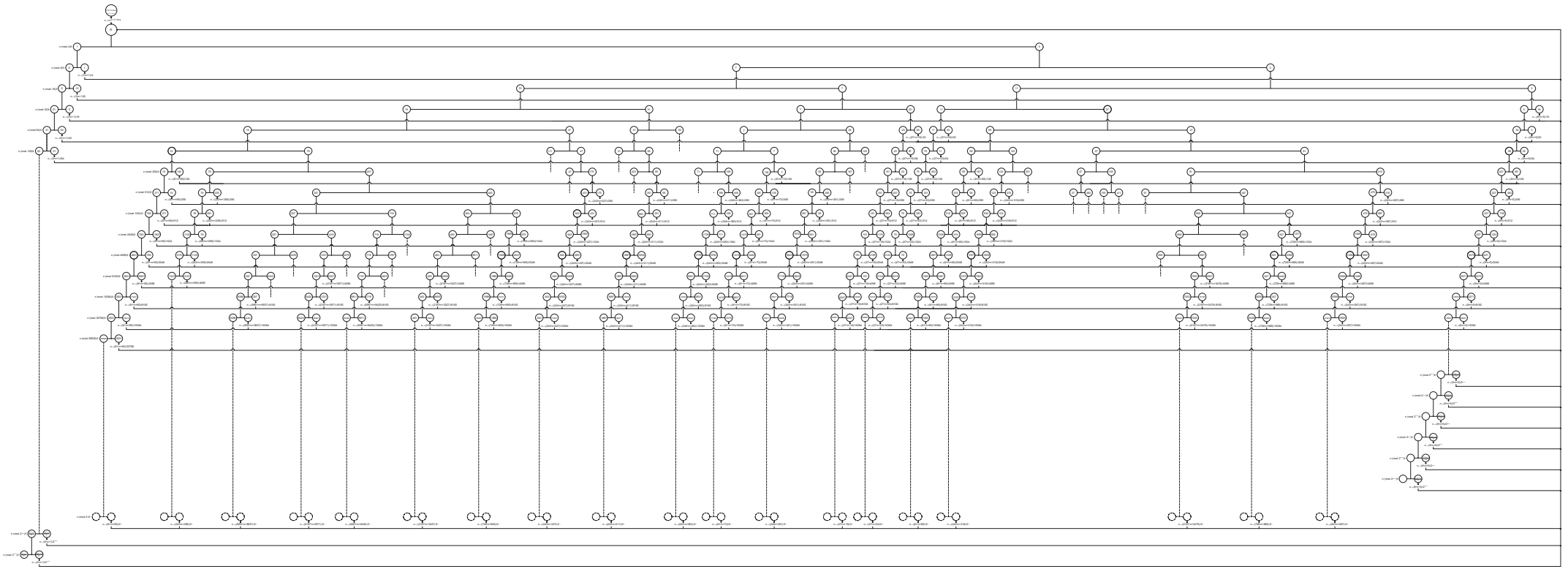
for $h_i = 3$ and for $k = 2$ we have $d = 4$ from which $s = 16 - (9 \cdot 3) \pmod{16} = 5$ and for $h_i = 79$ and for $k = 5$ we have $d = 8$ from which $s = 512 - (243 \cdot 79) \pmod{256} = 512 - 253 = 259$.

Finally, to check if s satisfies the condition $s = 3^{k-1} + \sum_{j=0}^{k-2} 3^j \cdot 2^{\sum_{i=1}^{k-1-j} a_i}$ we proceed as follows: the final goal is to verify if $A_{k-1} = \sum_{i=1}^{k-1} a_i < d$ to obtain A_{k-1} let's set $A_{k-1} = 0$ so one subtracts the value 3^{k-1} from s and we add to A_{k-1} the value $v_2(s)$ of the new s obtained, with $v_2()$ the 2-adic valuation, then we subtract 3^{k-2} and so on until subtracted 3^1 and we add to A_{k-1} the value $v_2(s)$ of the new s obtained, which must be equal to $2^{a_{k-1}}$.

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n_max = 1000; k_min = 2; k_max = 6; Nt = vector(n_max); i=0; forstep(n = 3, n_max, 4, i++; Nt[i] = n); n_max = i; for(i_n = 1, n_max, if(Nt[i_n] > 0, n = Nt[i_n]; k = 2; ck = 0; d = floor(1 + k * log(3) / log(2)); while(k <= k_max, s = 2^d - (3^k * n) % 2^d; if(s < 3^k - 2^k, s += 2^d); st = s; Ak = 0; for(j = 1, k - 1, s = s - 3^(k-j); if(s < 1, ck = 2); while(s % 2 == 0, Ak++; s /= 2);); if(s == 1 && Ak < d && ck == 0, if(k >= k_min, printf("%d mod %d, k = %d, s = %d \n", n, 2^d, k, st)); nt = i_n + 2^(d - 2); while(nt <= n_max, Nt[nt] = 0; nt += 2^(d - 2)); ck = 1); if(ck == 1, break); ck = 0; k++); d = floor(1 + k * log(3) / log(2)););)
```

PARI/GP code of the algorithm

We can create a flowchart (shown on the next page) where for each row of the graph the different residue classes modulo 2^c are shown.



References

[1] https://en.wikipedia.org/wiki/Collatz_conjecture

[2] <https://oeis.org/A177789>

[3] <https://oeis.org/A020914>