Model $\lambda\left(\varphi^{2n}\right)_4, n \geq 2$ Quantum Field Theory: A Nonstandard Approach Based on Nonstandard Pointwise-Defined Quantum Fields

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Abstract. A new non-Archimedean approach to interacted quantum fields is presented. In proposed approach, a field operator $\varphi(x,t)$ no longer a standard tempered operator-valued distribution, but a non-classical operator-valued function. We prove using this novel approach that the quantum field theory with Hamiltonian $P(\varphi)_4$ exists and that the corresponding C^*- algebra of bounded observables satisfies all the Haag-Kastler axioms except Lorentz covariance. We prove that the $\lambda(\varphi^{2n})_4, n \geq 2$ quantum field theory models are Lorentz covariant.

1. Introduction

Extending the real numbers $\mathbb{R}$ to include infinite and infinitesimal quantities originally enabled D. Laugwitz [1] to view the delta distribution $\delta(x)$ as a nonstandard point function. Independently A. Robinson [2] demonstrated that distributions could be viewed as generalized polynomials. Luxemburg [3] and Sloan [4] presented an alternate representative of distributions as internal functions within the context of canonical Robinson's theory of nonstandard analysis. For further information on classical model theoretical nonstandard analysis namely NSA, we refer to [5]-[8].

Abbreviation 1.1 In this paper we adopt the following canonical notations. For a standard set E we often write E_{st}. For a set E_{st} let ${}^{\sigma}E_{\mathrm{st}}$ be a set ${}^{\sigma}E_{\mathrm{st}} = \{{}^{*}x | x \in E_{\mathrm{st}}\}$. We identify z with ${}^{\sigma}z$ i.e., $z \equiv {}^{\sigma}z$ for all $z \in \mathbb{C}$. Hence, ${}^{\sigma}E_{\mathbf{st}} = E_{\mathrm{st}}$ if $E \subseteq \mathbb{C}$, e.g., ${}^{\sigma}\mathbb{C} = \mathbb{C}$, ${}^{\sigma}\mathbb{R} = \mathbb{R}$, ${}^{\sigma}P = P$, ${}^{\sigma}L_{+}^{\uparrow} = L_{+}^{\uparrow}$, etc.

Let ${}^{*}\mathbb{R}_{\approx}$, ${}^{*}\mathbb{R}_{\approx+}$, ${}^{*}\mathbb{R}_{\mathrm{fin}}$, ${}^{*}\mathbb{R}_{\infty}$, and ${}^{*}\mathbb{N}_{\infty}$ denote the sets of infinitesimal hyper-real numbers, positive infinitesimal hyper-real numbers, finite hyper-real numbers, infinite hyper-real numbers and infinite hyper natural numbers, respectively.

Note that: ${}^{*}\mathbb{R}_{\mathrm{fin}} = {}^{*}\mathbb{R} \backslash {}^{*}\mathbb{R}_{\infty}$, ${}^{*}\mathbb{C} = {}^{*}\mathbb{R} + \mathrm{i}{}^{*}\mathbb{R}$, ${}^{*}\mathbb{C}_{\mathrm{fin}} = {}^{*}\mathbb{R}_{\mathrm{fin}} + \mathrm{i}{}^{*}\mathbb{R}_{\mathrm{fin}}$.

Definition 1.1 Let $\{X, \|\cdot\|\}$ be a standard Banach space. For $x \in {}^{*}X$ and $\varepsilon > 0, \varepsilon \approx 0$ we define the open $\approx$-ball about x of radius ε to be the set $B_{\varepsilon}(x) = \{y \in {}^{*}X | {}^{*}\|x - y\| < \varepsilon\}$.

Definition 1.2 Let $\{\{X, \|\cdot\|\}$ be a standard Banach space, $Y \subset X$, thus ${}^{*}Y \subset {}^{*}X$ and let $x \in {}^{*}X$.Then x is an $*$- accumulation point of ${}^{*}Y$ if for any $\varepsilon \in {}^{*}\mathbb{R}_{\approx+}$ there is a hyper infinite sequence $\{x_n\}_{n=1}^{{}^{*}\infty}$in ${}^{*}Y$ such that $\{x_n\}_{n=1}^{{}^{*}\infty} \cap (B_{\varepsilon}(x) \backslash \{x\} \neq \emptyset)$.

Definition 1.3 Let $\{\{X, \|\cdot\|\}$ be a standard Banach space, let${}^{*}Y \subseteq {}^{*}X$,${}^{*}Y$ is $*$ -closed if any $*$-accumulation point of ${}^{*}Y$ is an element of ${}^{*}Y$.

Definition 1.4 Let $\{\{X, \|\cdot\|\}$ be a standard Banach space. We shall say that internal hyper infinite sequence $\{x_n\}_{n=1}^{{}^{*}\infty}$in ${}^{*}X$ is $*$ -converges to $x \in {}^{*}X$ as $n \to {}^{*}\infty$ if for any $\varepsilon \in {}^{*}\mathbb{R}_{\approx+}$ there is $N \in {}^{*}\mathbb{N}$ such that for any $n > N$: ${}^{*}\|x - y\| < \varepsilon$.

Definition 1.5 Let $\{\{X, \|\cdot\|_X\}$, $\{\{Y, \|\cdot\|_Y\}$ be a standard Banach spaces. A linear internal operator $A: D(A) \subseteq {}^{*}X \to {}^{*}Y$ is $*$ -closed if for every internal hyper infinite sequence $\{x_n\}_{n=1}^{{}^{*}\infty}$ in $D(A)$ $*$ -converging to $x \in {}^{*}X$ such that $Ax_n \to y \in {}^{*}Y$ as $n \to {}^{*}\infty$ one has $x \in D(A)$ and $Ax = y$. Equivalently , A is $*$-closed if its graph is $*$ -closed in the direct sum ${}^{*}X \oplus {}^{*}Y$.

Definition 1.6 Let H be a standard external Hilbert space. The graph of the internal linear transformation $T: {}^{*}H \to {}^{*}H$ is the set of pairs $\{\langle\varphi, T\varphi\rangle | \varphi \in D(T)\}$. The graph of T, denoted by $\Gamma(\mathrm{T})$, is thus a subset of ${}^{*}H \times {}^{*}H$ which is internal Hilbert space with inner product $(\langle\varphi_1, \psi_1\rangle, \langle\varphi_2, \psi_2\rangle) =$

$(\varphi_1,\varphi_2)+(\psi_1,\psi_2)$.The operator T is called a $*$-closed operator if $\Gamma(\mathrm{T})$ is a $*$ -closed subset of Cartesian product ${}^*H\times{}^*H$.

Definition 1.7 Let H be a standard Hilbert space. Let T_1 and T be internal operators on internal Hilbert space ${}^*H$. Note that if $\Gamma(T_1)\supset\Gamma(T)$, then T_1 is said to be an extension of T and we write $T_1\supset T$. Equivalently, $T_1\supset T$ if and only if $D(T_1)\supset D(T)$ and $T_1\varphi\ =T\varphi$ for all $\varphi\in D(T)$.

Definition 1.8 Any internal operator T on ${}^*H$ is $*$-closable if it has a $*$-closed extension. Every $*$-closable internal operator T has a smallest $*$-closed extension, called its $*$-closure, which we denote by $*$-$\bar{T}$.

Definition 1.9 Let H be a standard Hilbert space. Let T be a $*$-densely defined internal linear operator on internal Hilbert space ${}^*H$. Let $D(T^*)$ be the set of $\varphi\in{}^*H$ for which there is a vector $\xi\in{}^*H$ with $(T\psi,\varphi)=(\varphi,\xi)$ for all $\psi\in D(T)$, then for each $\varphi\in D(T^*)$, we define $T^*\varphi=\xi$. T^* is called the $*$ -adjoint of T. Note that $S\subset T$ implies $T^*\subset S^*$.

Definition 1.10 Let H is a standard Hilbert space. A $*$-densely defined internal linear operator T on internal Hilbert space ${}^*H$ is called symmetric (or Hermitian) if $T\subset T^*$. Equivalently, T is symmetric if and only if $(T\varphi,\psi)=(\varphi,T\psi)$ for all $\varphi,\psi\in D(T)$.

Definition 1.11 Let H be a standard Hilbert space. A symmetric internal linear operator T on internal Hilbert space ${}^*H$ is called essentially self- $*$-adjoint if its $*$-closure $*$-$\bar{T}$ is self- $*$-adjoint. If T is $*$-closed, a subset $D\subset D(T)$ is called a $*$-core for T if $*$- $\left(\overline{T\restriction D}\right)\ =T$. If T is essentially self- $*$-adjoint, then it has one and only one self -$*$-adjoint extension.

Let F be the standard Fock space [9],[10] for a massive, neutral scalar field in four-dimensional space-time [10]. The elements of ${}^*F$ are internal sequences of functions on internal momentum space${}^*\mathbb{R}^3$. Let the standard annihilation and creation operators be normalized by the relation

$$[a(k),a^\dagger(k')]=\delta^3(k-k'). \tag{1.1}$$

so that the free-field Hamiltonian with finite momentum cut-off $\sigma\in{}^\sigma\mathbb{R}$ is

$$H_{0,\sigma}=\int_{|k|\le\sigma}a^\dagger(k')a(k)\mu(k)d^3k\,,\mu(k)=\sqrt{k_1^2+k_2^2+k_3^2}\,. \tag{1.2}$$

From (1.1) by transfer one obtains

$$[{}^*a(k),{}^*a^\dagger(k')]={}^*\delta^3(k-k'), \tag{1.3}$$

so that internal free-field Hamiltonian with hyperfinite cut-off $\varkappa\in{}^*\mathbb{R}_{+;\infty}$ is

$${}^*H_{0,\varkappa}={}^*\!\!\int_{|k|\le\varkappa}{}^*a^\dagger(k')\left({}^*a(k)\right)\left({}^*\mu(k)\right)d^3k. \tag{1.4}$$

The $t=0$ internal field ${}^*\varphi_\varkappa(x)$ with hyperfinite momentum cut-off $\varkappa\in{}^*\mathbb{R}_{+;\infty}$ is

$${}^*\varphi_\varkappa(x)=\frac{1}{(2\pi)^{3/2}}{}^*\!\!\int_{|k|\le\varkappa}{}^*e^{-i(k,x)}[{}^*a^\dagger(k\,)+{}^*a(-k)]\frac{d^3k}{{}^*\left(\sqrt{2\mu(k)}\right)}. \tag{1.5}$$

The spatially cut-off internal interaction Hamiltonian with hyperfinite momentum cut-off $\varkappa\in{}^*\mathbb{R}_{+;\infty}$ is

$${}^*H_{I,\varkappa}(g)=\sum_{j=0}^4\binom{4}{j}{}^*\!\!\int_{|k_1|\le\varkappa}\cdots{}^*\!\!\int_{|k_{j+1}|\le\varkappa}\cdots{}^*\!\!\int_{|k_4|\le\varkappa}{}^*a^\dagger(k_1\,)\cdots{}^*a^\dagger(k_j\,){}^*a(-k_{j+1})\times$$

$$\times\ {}^*a(-k_4)\left({}^*\hat{g}\left(\sum_{i=1}^{4} k_i\right)\right)\prod_{i=1}^{4}{}^*\mu(k)^{1/2}d^3k_i. \tag{1.6}$$

We also need internal number operator with hyperfinite momentum cut-off $\varkappa \in {}^*\mathbb{R}_{+;\infty}$

$$^*N_{\varkappa} = {}^*\!\!\int_{|k|\leq\varkappa} {}^*a^{\dagger}(k\ )\,{}^*a(k)d^3k \tag{1.7}$$

and the domain

$$D_{0,\varkappa} = \bigcap_{n\in{}^*\mathbb{N}} D\left({}^*H_{0,\varkappa}^{n}\right). \tag{1.8}$$

Remark 1.1 Note that the domain $D_{0,\varkappa}$ is a nonstandard *external* set so there is no standard set $\mathcal{D}$ such that $D_{0,\varkappa} = {}^*\mathcal{D}$.

Proposition 1.1 Let W_σ be a standard operator $W_\sigma\colon F \to F$ of the form

$$W_\sigma = \int_{|k_1|\leq\sigma}\cdots\int_{|k_m|\leq\sigma} w(k_1,\ldots,k_m)\ a^{\dagger}(k\ _1)\cdots\ a(-k_m)\prod_{i=1}^{m} d^3k_i \tag{1.9}$$

and let N_σ be a standard operator $N_\sigma\colon F \to F$ of the form

$$N_\sigma = \int_{|k|\leq\varkappa} a^{\dagger}(k\ )\ (k)d^3k. \tag{1.10}$$

Assume that for all σ such that $0 < \sigma < \infty$ the inequality holds

$$\int\cdots\int\ \chi_\sigma(k_1,\ldots,k_m)w^2(k_1,\ldots,k_m)\prod_{i=1}^{m} d^3k_i\ < \infty,$$

where $\chi_\sigma(k_1,\ldots,k_m) = 1$ if $|k_i| \leq \sigma$ for all $1 \leq i \leq m$, and $\chi_\kappa(k_1,\ldots,k_m) = 0$ otherwise. Then for all σ such that $0 < \sigma < \infty$ and for all j such that $|j| \leq m$ the inequality holds

$$\left\|(N_\sigma + I)^{-\frac{j}{2}}W_\sigma(N_\sigma + I)^{\frac{(m-j)}{2}}\right\| \leq$$

$$\leq \left(\int\cdots\int\ \chi_\sigma(k_1,\ldots,k_m)w^2(k_1,\ldots,k_m)\prod_{i=1}^{m} d^3k_i\ \right)^{\frac{1}{2}}. \tag{1.11}$$

Proposition 1.2 Let ${}^*W_\varkappa$ be internal operator ${}^*W_\varkappa\colon {}^*F \to {}^*F$ of the form

$$^*W_{\varkappa} = {}^*\!\!\int_{|k_1|\leq\varkappa}\cdots{}^*\!\!\int_{|k_{\boldsymbol{m}}|\leq\varkappa} {}^*w(k_1,\ldots,k_{\boldsymbol{m}})\,{}^*a^{\dagger}(k\ _1)\cdots{}^*a(-k_{\boldsymbol{m}})\prod_{i=1}^{\boldsymbol{m}} d^3k_i. \tag{1.12}$$

Then for all $\varkappa$ such that $\varkappa \in {}^*\mathbb{R}_{+}$, and for all j such that $|j| \leq m, m \in {}^*\mathbb{N}_\infty$ the inequality holds

$$\left\|(N_\varkappa + I)^{-\frac{j}{2}}W_\varkappa(N_\varkappa + I)^{\frac{(\boldsymbol{m}-j)}{2}}\right\| \leq$$

$$\leq \left({}^*\!\!\int\cdots{}^*\!\!\int\ {}^*\chi_\varkappa(k_1,\ldots,k_{\boldsymbol{m}})^*w^2(k_1,\ldots,k_m)\prod_{i=1}^{\boldsymbol{m}} d^3k_i\ \right)^{\frac{1}{2}}. \tag{1.13}$$

Proof It follows directly from (1.11) by transfer.

Remark 1.2 It follows from (2.11) that:

(1) ${}^*H_{I,\varkappa}(g)$ is well defined on the domain $D_{0,\varkappa}$,

(2) there is a $*$-closure $*\text{-}\overline{{}^*H_{I,\varkappa}(g)}$ with domain $D\left(*\text{-}\overline{{}^*H_{I,\varkappa}(g)}\right) \supset D_{0,\varkappa}$,

(3) external set $D_{0,\varkappa}$ is a $*$-core for ${}^*H_{I,\varkappa}(g)$ i.e., $*\text{-}\left(\overline{{}^*H_{I,\varkappa}(g) \restriction D_{0,\varkappa}}\right) = {}^*H_{I,\varkappa}(g)$

Remark 1.3 The operator $*\text{-}\overline{{}^{*}H_{I,\varkappa}(g)}$ is external mapping $*\text{-}\overline{{}^{*}H_{I,\varkappa}(g)}: {}^{*}F \to {}^{*}F$ i.e., there is no standard operator $T: F \to F$ with domain $D(T)$ such that:

(1) ${}^{*}D(T) = D\left(* \text{-}\overline{{}^{*}H_{I,\varkappa}(g)}\right)$ and (2) ${}^{*}T \upharpoonright {}^{*}D(T) = *\text{-}\overline{{}^{*}H_{I,\varkappa}(g)} \upharpoonright D\left(* \text{-}\overline{{}^{*}H_{I,\varkappa}(g)}\right)$.

Thus we cannot derive the desired properties of the operator $* \text{-}\overline{{}^{*}H_{I,\varkappa}(g)}$ by using Robinson transfer principle [2]-[7].

As has been explained in [8] classical model theoretical nonstandard analysis NSA does not power enough to resolve the stated in [8] problems in constructive quantum field theory related to physical dimension $d = 4$,

In order to avoid any difficultness mentioned above, in this paper as in [8] we deal by using a non-conservative extension of NSA developed in [11].

Remind that Robinson nonstandard analysis (NSA) many developed using set theoretical objects called super-structures [5]-[7]. A superstructure V(S) over a set S is defined in the following way: $\mathrm{V}_0(\mathrm{S}) = \mathrm{S}$, $\mathrm{V}_{n+1}(\mathrm{S}) = \mathrm{V}_n(\mathrm{S}) \cup \mathrm{P}\big(\mathrm{V}_n(\mathrm{S})\big)$, $\mathrm{V}(\mathrm{S}) = \bigcup_{n\in\mathbb{N}} \mathrm{V}_{n+1}(\mathrm{S})$. Making $\mathrm{S} = \mathbb{R}$ will suffice for virtually any construction necessary in analysis. Bounded formulas are formulas where all quantifiers occur in the form: $\forall x\ (x \in y \to \cdots\ ), \exists x\ (x \in y \to \cdots\ )$. A nonstandard embedding is a mapping $*: \mathrm{V}(X) \to \mathrm{V}(Y)$ from a superstructure $\mathrm{V}(X)$ called the standard universe, into another superstructure $\mathrm{V}(Y)$ called nonstandard universe, satisfying the following postulates:
1. $Y = {}^{*}X$
2. ***Transfer Principle*** For every bounded formula $\Phi(x_1, \dots, x_n)$ and elements $a_1, \dots, a_n \in V(X)$ the property $\Phi(a_1, \dots, a_n)$ is true for $a_1, \dots, a_n$ in the standard universe if and only if it is true for ${}^{*}a_1, \dots, {}^{*}a_n$ in the nonstandard universe $V(X) \models \Phi(x_1, \dots, x_n) \leftrightarrow V(Y) \models \Phi({}^{*}a_1, \dots, {}^{*}a_n)$.
3. ***Non-triviality*** For every infinite set A in the standard universe, the set $\{{}^{*}a | a \in A\}$ is a proper subset of ${}^{*}A$.
Definition 1.12 A set x is internal if and only if x is an element of ${}^{*}A$ for some $A \in \mathrm{V}(\mathbb{R})$. Let X be a set and $A = \{A_i\}_{i\in I}$ a family of subsets of X .Then the collection A has the infinite intersection property, if any infinite sub collection $J \subset I$ has non-empty intersection. Nonstandard universe is σ-saturated if whenever $\{A_i\}_{i\in I}$ is a collection of internal sets with the infinite intersection property and the cardinality of I is less than or equal to σ.
Remark 1.4 For each standard universe $U = V(X)$ there exists canonical language L_U and for each nonstandard universe $W = V(Y)$ there exists corresponding canonical nonstandard language ${}^{*}L = L_W$ [5],[7]

4.***The restricted rules of conclusion*** If Let A and B well formed, closed formulas so that $A, B \in {}^{*}L$. If $W \models A$, then $\neg A \nvdash_{RMP} B$. Thus, if a statement A holds in nonstandard universe, we cannot obtain from formula $\neg A$ any formula B whatsoever.
Definition 1.13 [11] A set $S \subset {}^{*}\mathbb{N}$ is a hyper inductive if the following statement holds in $V(Y)$:

$$\bigwedge_{\alpha\in{}^{*}\mathbb{N}}(\alpha \in S \to \alpha^{+} \in S).$$

Here $\alpha^{+} = \alpha + 1$.Obviously a set ${}^{*}\mathbb{N}$ is a hyper inductive.

5. ***Axiom of hyper infinite induction***

$$\forall S(S \subset {}^{*}\mathbb{N})\left\{\forall\beta(\beta \subset {}^{*}\mathbb{N})\left[\bigwedge_{1\leq\alpha<\beta}(\alpha \in S \to \alpha^{+} \in S)\right] \to S = {}^{*}\mathbb{N}\right\}.$$

Example 1.1 Remind the proof of the following statement: structure $(\mathbb{N},<,=)$ is a well-ordered set.
Proof Let X be a nonempty subset of $\mathbb{N}$. Suppose X does not have a <-least element. Then consider the set $\mathbb{N}\backslash X$. Case1. $\mathbb{N}\backslash X=\emptyset$. Then $X=\mathbb{N}$ and so 0 is a $<$ -least element but this is a contradiction. Case2. $\mathbb{N}\backslash X\neq\emptyset$. Then $1\in\mathbb{N}\backslash X$ otherwise 1 is a $<$ -least element but this is a contradiction. Assume now that there exists some $n\in\mathbb{N}\backslash X$ such that $n\neq 1$, but since we have supposed that X does not have a $<$ -least element, thus $n+1\notin X$. Thus we see that for all n the statement $n\in\mathbb{N}\backslash X$ implies that $n+1\in\mathbb{N}\backslash X$. We can conclude by axiom of induction that $n\in\mathbb{N}\backslash X$ for all $n\in\mathbb{N}$. Thus $\mathbb{N}\backslash X=\mathbb{N}$ implies $X=\emptyset$. This is a contradiction to X being a non-empty subset of $\mathbb{N}$. Remind that structure $({}^*\mathbb{N},<,=)$ is not a well-ordered set [5]-[7]. We set now $X_1={}^*\mathbb{N}\backslash\mathbb{N}$ and thus${}^*\mathbb{N}\backslash X_1=\mathbb{N}$. In contrast with a set X mentioned above the assumption $n\in{}^*\mathbb{N}\backslash X_1$ implies that $n+1\in{}^*\mathbb{N}\backslash X_1$ if and only if n is finite, since for any infinite $n\in{}^*\mathbb{N}\backslash\mathbb{N}$ the assumption $n\in{}^*\mathbb{N}\backslash X_1$ contradicts with a true statement $V(Y)\vDash n\notin{}^*\mathbb{N}\backslash X_1$=$\mathbb{N}$ and therefore in accordance with postulate 4 we cannot obtain from $n\in{}^*\mathbb{N}\backslash X_1$ any closed formula B whatsoever.
For further information on non-classical nonstandard analysis namely$NSA^{\#}$, we refer to [8]-[13].
Abbreviation1.2 In this paper we adopt the following notations [8]. For a standard set E we often write E_{st}, let ${}^{\sigma}E_{\mathrm{st}}=\{{}^*x|x\in E_{\mathrm{st}}\}$.We identify z with ${}^{\sigma}z$ i.e., $z\equiv{}^{\sigma}z$ for all $z\in\mathbb{C}$. Hence, ${}^{\sigma}E_{\mathrm{st}}=E_{\mathrm{st}}$ if $E\subseteq\mathbb{C}$, e.g., ${}^{\sigma}\mathbb{C}=\mathbb{C}$, ${}^{\sigma}\mathbb{R}=\mathbb{R}$, etc. Let ${}^*\mathbb{R}_c^{\#}$, ${}^*\mathbb{R}_{c,\approx}^{\#}$, ${}^*\mathbb{R}_{c,\approx+}^{\#}$, ${}^*\mathbb{R}_{c,\mathrm{fin}}^{\#}$, ${}^*\mathbb{R}_{c,\infty}^{\#}$, ${}^*\mathbb{N}_{\infty}$ de-note the sets of Cauchy hyper-real numbers, Cauchy infinitesimal hyper-real numbers, Cauchy positive infinitesimal hyperreal numbers, Cauchy finite hyper-real numbers, Cauchy infinite hyper-real numbers and infinite hypernatural numbers, respectively. Note that ${}^*\mathbb{R}_{c,\mathrm{fin}}^{\#}={}^*\mathbb{R}_c^{\#}\backslash{}^*\mathbb{R}_{c,\infty}^{\#}$.
Definition 1.13 Let H be external hyper infinite dimensional vector space over the complex field ${}^*\mathbb{C}_c^{\#}={}^*\mathbb{R}_c^{\#}+i{}^*\mathbb{R}_c^{\#}$. An inner product on H is a${}^*\mathbb{C}_c^{\#}$-valued function, $\langle\cdot,\cdot\rangle: H\times H\to{}^*\mathbb{C}_c^{\#}$, such that (1) $\langle ax+by,z\rangle=\langle ax,z\rangle+\langle by,z\rangle$, (2) $\overline{\langle x,y\rangle}=\langle y,x\rangle$, (3) $\|x\|^2\equiv\langle x,x\rangle\geq 0$ with equality $\langle x,x\rangle=0$ if and only if $x=0$.
Theorem 1.1 (Generalized Schwarz Inequality) Let $\{H,\langle\cdot,\cdot\rangle\}$be an inner product space, then for all $x,y\in H$: $|\langle x,y\rangle|\leq\|x\|\|y\|$ and equality holds if and only if x and y are linearly dependent.
Theorem 1.2 Let $\{H,\langle\cdot,\cdot\rangle\}$be an inner product space, and $\|x\|_{\#}=\sqrt{\langle x,x\rangle}$. Then $\|\cdot\|_{\#}$ is a ${}^*\mathbb{R}_c^{\#}$ -valued #-norm on a space H. Moreover $\langle x,x\rangle$ is #-continuous on Cartesian product $H\times H$, where H is viewed as the #-normed space $\{H,\|\cdot\|_{\#}\}$.
Definition 1.14 A non-Archimedean Hilbert space H is a #-complete inner product space.
Two elements x and y of non-Archimedean Hilbert space H are called orthogonal if $\langle x,y\rangle=0$.
Definition 1.15 The graph of the linear transformation $T: H\to H$ is the set of pairs $\{\langle\phi,T\phi\rangle|(\phi\in D(T))\}$. The graph of the operator T, denoted by $\Gamma(\mathrm{T})$, is thus a subset of $H\times H$ which is a non-Archimedean Hilbert space with the following inner product $(\langle\phi_1,\psi_1\rangle,\langle\phi_2,\psi_2\rangle)$. Operator T is called a #-closed operator if $\Gamma(\mathrm{T})$ is a #-closed subset of $H\times H$.
Definition 1.16 Let T_1 and T be operators on H. If $\Gamma(T_1)\supset\Gamma(\mathrm{T})$, then T_1 is said to be an extension of T and we write $T_1\supset T$. Equivalently: $T_1\supset T$ if and only if $D(T_1)\supset D(T)$ and $T_1\phi=T\phi$ for all $\phi\in D(T)$.
Definition 1.17 An operator T is #-closable if it has a #-closed extension. Every #-closable operator has a smallest #-closed extension, called its #-closure, which we denote by #-T.
Theorem 1.3 If T is #-closable, then $\Gamma(\#\text{-}\bar{T})=\#\text{-}\overline{\Gamma(T)}$.
Definition 1.18 Let $D(T^*)$ be the set of $\varphi\in H$ for which there is an $\xi\in H$ with $(T\psi,\varphi)=(\psi,\xi)$ for all $\psi\in D(T)$. For each $\varphi\in D(T^*)$, we define $T^*\varphi=\xi$.The operator T^* is called the #-adjoint of T. Note that $\varphi\in D(T^*)$ if and only if $|(T\psi,\varphi)|\leq C\|\psi\|_{\#}$ for all $\psi\in D(T)$. Note that $S\subset T$ implies $T^*\subset S$.
Remark 1.5 Note that for ξ to be uniquely determined by the condition $(T\psi,\varphi)=(\psi,\xi)$ one need

the fact that $D(T)$ is #-dense in H. If the domain $D(T^*)$ is #-dense in H, then we can define $T^{**} = (T^*)^*$.

Theorem 1.4 Let T be a #-densely defined operator on a non-Archimedean Hilbert space H. Then: (a) T^* is #-closed. (b) The operator T is #-closabie if and only if $D(T^*)$ is -dense in which case $T = T^{**}$. (c) If T is #-closable, then $(\#\text{-}\bar{T})^* = T^*$.

Definition 1.19 Let T be a #-closed operator on a non-Archimedean Hilbert space H. A complex number $\lambda \in {}^*\mathbb{C}_c^\#$ is in the resolvent set $\rho(T)$, if $\lambda I - T$ is a bijection of $D(T)$ onto H with a finitely or hyper finitely bounded inverse. If complex number $\lambda \in \rho(T)$, $R_\lambda = (\lambda I - T)^{-1}$ is called the resolvent of T at λ.

Definition 1.20 A #-densely defined operator T on a non-Archimedean Hilbert space is called symmetric or Hermitian if $T \subset T^*$, that is, $D(T) \subset D(T^*)$ and $T\varphi = T^*\varphi$ for all $\varphi \in D(T)$ and equivalently, T is symmetric if and only if $(T\varphi, \psi) = (\varphi, T\psi)$ for all $\varphi, \psi \in D(T)$.

Definition 1.21 A #-densely defined operator T is called self-#-adjoint if $T = T^*$, that is, if and only if T is symmetric and $D(T) = D(T^*)$.

Remark 1.6 A symmetric operator T is always #-closable, since $D(T)$ #-dense in H. If T is symmetric, T^* is a #-closed extension of T so the smallest #-closed extension T^{**} of T must be contained in T^*. Thus for symmetric operators, we have $T \subset T^{**} \subset T^*$, for #-closed symmetric operators we have $T = T^{**} \subset T^*$ and, for self-#-adjoint operators we have $T = T^{**} = T^*$. Thus a #-closed symmetric operator T is self-#-adjoint if and only if T^* is symmetric.

Definition 1.22 A symmetric operator T is called essentially self-#-adjoint if its #-closure $\#\text{-}\bar{T}$ is self-#-adjoint. If T is #-closed, a subset $D \subset D(T)$ is called a core for T if $\#\text{-}\overline{T \restriction D} = T$.

Remark 1.7 If T is essentially self-#-adjoint, then it has one and only one self-#-adjoint extension.

Theorem 1.5 [8] (see [8], sect.15.1) If $g \in S_{\text{fin}}^\#\left({}^*\mathbb{R}_c^{\#\mathbf{3}}\right)$ is real, then

$$H_{I,\varkappa}(g) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} :\varphi_\varkappa^{\#4}(x): g(x)\, d^{\#3}x \tag{1.14}$$

is essentially self #-adjoint on the domain $D_{0,\varkappa}^\# = \bigcap_{n=0}^{{}^*\infty} D\left(H_{0,\varkappa}^n\right)$.

Here $\varphi_\varkappa^\#(x)$ is a nonstandard pointwise-defined operator valued function $\varphi_\varkappa^\#: {}^*\mathbb{R}_c^{\#3} \to L\left(\mathcal{F}^\#\right)$

$$\varphi_\varkappa^\#(x) = \frac{1}{(2\pi)^{3/2}}\, Ext\text{-}\int_{|k|\le\varkappa} (Ext\text{-}\exp[-i(k,x)])\left[a^\dagger(k) + a(-k)\right]\frac{d^{\#3}k}{\sqrt{2\mu(k)}}, \tag{1.15}$$

where $\varkappa \in {}^*\mathbb{R}_{c+,\infty}^\#$.

The main purpose of the present paper is to extend the result of [8] to $\lambda(\varphi^{2n})_4, n > 2$. Our notation and definitions are the same as in [8].

We remind that for every function $f \in C_0^{{}^*\infty}\left({}^*\mathbb{R}_{c,\text{fin}}^{\#4}, {}^*\mathbb{R}_{c,\text{fin}}^\#\right),)$, the averaged free quantum field

$$\varphi_\varkappa^\#(f) = \frac{1}{(2\pi)^{3/2}}\, Ext\text{-}\int_{|k|\le\varkappa} (Ext\text{-}\exp[\,t\mu(k) - i(k,x)])\left[a^\dagger(k) + a(-k)\right] f(x)\frac{d^{\#3}k}{\sqrt{2\mu(k)}}\, d^{\#4}x, \tag{1.16}$$

is a self-#-adjoint operator on a non-Archimedean Fock space $\mathcal{F}^\#$ [8].

A non -Archimedean $C_\#^*$-algebra of local observables $\mathfrak{A}^\#$ is defined as the #-norm #-closure [8]

$$\mathfrak{A}^\# = \#\text{-}\overline{\bigcup_O \mathfrak{A}^\#(O)}\,, \tag{1.17}$$

where the union takes place over bounded regions O of space-time, and $\mathfrak{A}^\#(O)$ is the von Neumann #-algebra generated by [8]

$$\left\{Ext\text{-}\exp\left(i\varphi_{\varkappa}^{\#}(f)+i\pi_{\varkappa}^{\#}(f)\right)|f\in C_0^{^*\infty}\left({}^*\mathbb{R}_{c,\mathrm{fin}}^{\#4},{}^*\mathbb{R}_{c,\mathrm{fin}}^{\#}\right)\right\}.$$

A non –Archimedean near standard $C_{\#}^{*\approx}$-algebra of physical local observables $\mathfrak{A}_{\approx}^{\#}(O)$ is defined as

$$\mathfrak{A}_{\approx}^{\#}(O)=\left\{Q\in\mathfrak{A}^{\#}(O)|\|Q\|_{\#}\in{}^*\mathbb{R}_{c+,\mathrm{fin}}^{\#}\right\}.$$

Let ${}^{\sigma}G$ be the restricted Poincare group of transformations of 4-dimensional Minkowski space-time $\mathcal{M}_4$. Poincare transformations $\left\{a,\Lambda_{\beta_i}^{(i)}\right\}\in{}^{\sigma}G$ generated by a Lorentz boosts along the x^i-direction $i=1,2,3$ and space-time translation $x\to x+a, a=(\alpha^1,\alpha^2,\alpha^3,\tau)$ are

$$\left\{a,\Lambda_{\beta_1}^{(1)}\right\}(x,t)=$$

$$=(\alpha^1+x^1\cosh\beta_1+t\sinh\beta_1,\tau\ +x^1\sinh\beta_1+t\cosh\beta_1,\alpha^2+x^2,\alpha^3+x^3),\tag{1.18}$$

$$\left\{a,\Lambda_{\beta_2}^{(2)}\right\}(x,t)=$$

$$=(\alpha^1+x^1,\alpha^2+x^2\cosh\beta_2+t\sinh\beta_2,\alpha^3+x^3,\tau\ +x^2\sinh\beta_2+t\cosh\beta_2),\tag{1.19}$$

$$\left\{a,\Lambda_{\beta_3}^{(3)}\right\}(x,t)=$$

$$=(\alpha^1+x^1,\alpha^2+x^2,\alpha^3+x^3\cosh\beta_3+t\sinh\beta_3,\tau+x^1\sinh\beta_3+t\cosh\beta_3).\tag{1.20}$$

Theorem 1.6 For every $\left\{a,\Lambda_{\beta_i}^{(i)}\right\}\in{}^{\sigma}G, i=1,2,3$ and for every bounded set $O\subset{}^*\mathbb{R}_{c,\mathrm{fin}}^{\#3}$ there exists a unitary operators $U_O^{(i)}, i=1,2,3$ such that, for all $f\in C_0^{^*\infty}\left({}^*\mathbb{R}_{c,\mathrm{fin}}^{\#4},{}^*\mathbb{R}_{c,\mathrm{fin}}^{\#}\right)$

$$U_O^{(i)}\left[Ext\text{-}\exp\left(i\varphi_{\varkappa}^{\#}(f)\right)\right]\left(U_O^{(i)}\right)^*\approx Ext\text{-}\exp\left(i\varphi_{\varkappa}^{\#}\left(f_{\left\{a,\Lambda_{\beta_i}^{(i)}\right\}}\right)\right), i=1,2,3,\tag{1.21}$$

where $f_{\left\{a,\Lambda_{\beta_i}^{(i)}\right\}}(x,t)=f\left(\left\{a,\Lambda_{\beta_i}^{(i)}\right\}(x,t)\right)$. This mappings extends to a representation $\sigma_{\left\{a,\Lambda_{\beta_i}^{(i)}\right\}}$ of $*$-automorphisms of $\mathfrak{A}^{\#}$such that

$$\sigma_{\left\{a,\Lambda_{\beta_i}^{(i)}\right\}}\left(\mathfrak{A}_{\approx}^{\#}(O)\right)\approx\mathfrak{A}_{\approx}^{\#}\left(\left\{a,\Lambda_{\beta_i}^{(i)}\right\}O\right), i=1,2,3.\tag{1.22}$$

The formal expressions for the Hamiltonian and Lorentz transformation generators are given by [8]

$$H_{\varkappa}=H_{0,\varkappa}+H_{I,\varkappa}=Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}\left(T_{0,\varkappa}(x)+T_{I,\varkappa}(x)\right)d^{\#3}x,\tag{1.23}$$

$$M_{\varkappa}^{0k}=M_{0,\varkappa}+M_{I,\varkappa}=Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}x^k\left(T_{0,\varkappa}(x)+T_{I,\varkappa}(x)\right)d^{\#3}x,=1,2,3,\tag{1.24}$$

where

$$T_{0,\varkappa}(x)=\frac{1}{2}\left[:\pi_{\varkappa}^{\#2}(x):+m^2:\varphi_{\varkappa}^{\#2}(x):+:\left(\partial_{x_1}^{\#}\varphi_{\varkappa}^{\#}(x)\right)^2:+:\left(\partial_{x_2}^{\#}\varphi_{\varkappa}^{\#}(x)\right)^2:+:\left(\partial_{x_3}^{\#}\varphi_{\varkappa}^{\#}(x)\right)^2:\right]\tag{1.25}$$

is the free energy density with hyperfinite cut-off $\varkappa\in{}^*\mathbb{R}_{c+,\infty}^{\#}$, and where the interaction energy density $T_{I,\varkappa}(x)$ reads

$$T_{I,\varkappa}(x) =: \varphi_{\varkappa}^{\#2n}(x):. \tag{1.26}$$

Formally one verifies the commutation relations

$$[iH_{\varkappa}, M_{\varkappa}^{0k}] = P_{\varkappa}^{k}, k = 1,2,3 \tag{1.27}$$

and

$$[iH_{\varkappa}, P_{\varkappa}^{k}] = 0, k = 1,2,3, \tag{1.28}$$

where $P_{\varkappa}^{k}, k = 1,2,3$ are the momentum operators $P_{\varkappa}^{k} = Ext\text{-}\int_{^*\mathbb{R}_c^{\#3}} P_{\varkappa}^{k}(x)\, d^{\#3}x$ with densities defined by

$$P_{\varkappa}^{k}(x) = \frac{1}{2}\left[:\pi_{\varkappa}^{\#}(x)\partial_{x_k}^{\#}\varphi_{\varkappa}^{\#}(x):+:\partial_{x_k}^{\#}\varphi_{\varkappa}^{\#}(x)\pi_{\varkappa}^{\#}(x):\right]. \tag{1.29}$$

We wish to prove that $Ext\text{-}\exp(i\beta)M_{\varkappa}^{0k}$ implements Lorentz rotations on suitable domain

$$\left[Ext\text{-}\exp\left(i\beta M_{\varkappa}^{0k}\right)\right]\varphi_{\varkappa}^{\#}(x,t)\left[Ext\text{-}\exp\left(-i\beta M_{\varkappa}^{0k}\right)\right] \approx \varphi_{\varkappa}^{\#}\left(\Lambda_{\beta}^{(k)}(x,t)\right), k = 1,2,3, \tag{1.30}$$

where

$$\varphi_{\varkappa}^{\#}(x,t) = [Ext\text{-}\exp(itH_{\varkappa})]\varphi_{\varkappa}^{\#}(x)[Ext\text{-}\exp(-itH_{\varkappa})], \tag{1.31}$$

and $\Lambda_{\beta}^{(k)}(x,t) = \left\{0, \Lambda_{\beta}^{(k)}\right\}(x,t)$.

In differential form (1.30) becomes

$$[iM_{\varkappa}^{0k}, \varphi_{\varkappa}^{\#}(x,t)] \approx t\partial_{x_k}^{\#}\varphi_{\varkappa}^{\#}(x,t) + x_k\partial_t^{\#}\varphi_{\varkappa}^{\#}(x,t), k = 1,2,3. \tag{1.32}$$

We define now

$$M_{\varkappa}^{0k}(t) = [Ext\text{-}\exp(-itH_{\varkappa})]M_{\varkappa}^{0k}[Ext\text{-}\exp(itH_{\varkappa})], k = 1,2,3, \tag{1.33}$$

and using the commutation relations (1.27) and (1.28) we obtain

$$M_{\varkappa}^{0k}(t) \equiv Ext\text{-}\frac{\sum_{r=0}^{*\infty}\left(\mathrm{ad}(-itH_{\varkappa})\right)^{r} M_{\varkappa}^{0k}}{r!^{\#}} = M_{\varkappa}^{0k} - tP_{\varkappa}^{k}, \tag{1.34}$$

since second order and higher terms in t vanish identically. Thus we get

$$\begin{aligned}[iM_{\varkappa}^{0k}, \varphi_{\varkappa}^{\#}(x,t)] &= [Ext\text{-}\exp(itH_{\varkappa})][iM_{\varkappa}^{0k}(t), \varphi_{\varkappa}^{\#}(x,0)][Ext\text{-}\exp(-itH_{\varkappa})] = \\ &= [Ext\text{-}\exp(itH_{\varkappa})][iM_{\varkappa}^{0k} - itP_{\varkappa}^{k}, \varphi_{\varkappa}^{\#}(x,0)][Ext\text{-}\exp(-itH_{\varkappa})], k = 1,2,3.\end{aligned} \tag{1.35}$$

Since $\varphi_{\varkappa}^{\#}(x,0)$ commutes with $M_{I,\varkappa}$ by a standard computation we get

$$[iM_{\varkappa}^{0k}, \varphi_{\varkappa}^{\#}(x,0)] = \left[iM_{0,\varkappa}^{0k}, \varphi_{\varkappa}^{\#}(x,0)\right] = x_k\pi_{\varkappa}^{\#}(x,0), k = 1,2,3. \tag{1.36}$$

Also we get

$$[iP_{\varkappa}^{k}, \varphi_{\varkappa}^{\#}(x,0)] = -\partial_{x_k}^{\#}\varphi_{\varkappa}^{\#}(x,0), k = 1,2,3. \tag{1.37}$$

Substituting (1.36) and (1.37) into (1. 35), we obtain the desired commutation relation (1. 32).

The three main steps to convert the above argument into a rigorous proof are (a) to introduce a spatial cut-off into the Lorentz boost generators in such a way that we obtain a self-#-adjoint operators $M^{0k}_{\varkappa,g}$, $k = 1,2,3$; (b) to show that for suitable bounded regions $O \subset {}^*\mathbb{R}^{\#3}_{c,\mathrm{fin}}$, (1.34) holds in the sense that for every $f \in C_0^{*\infty}\left({}^*\mathbb{R}^{\#3}_{c,\mathrm{fin}}, {}^*\mathbb{R}^{\#}_{c,\mathrm{fin}}\right)$,

$$\left[iM^{0k}_{\varkappa,g}(t), \varphi^{\#}_{\varkappa}(f)\right] \approx \left[iM^{0k}_{\varkappa,g} - iP^{k}_{\varkappa,g}, \varphi^{\#}_{\varkappa}(f)\right], \qquad (1.38)$$

where $P^{k}_{\varkappa,g}$, $k = 1,2,3$ are the locally correct momentum operators. Note that (1. 38) states that $M^{0k}_{\varkappa,g}$ are the locally correct Lorentz boost generators for the region O corresponding to the exact cancellation of higher order terms in (1.34) is the fact that second and higher order terms in $M^{0k}_{\varkappa,g}(t)$ are localized $\approx$ -outside region O and hence $\approx$ -commutes with $\varphi^{\#}_{\varkappa}(f)$. From (1. 38) one obtains the relations

$$\left[iM^{0k}_{\varkappa,g}(t), \varphi^{\#}_{\varkappa}(f)\right] \approx -\varphi^{\#}_{\varkappa}\left(t\frac{\partial^{\#} f}{\partial^{\#} x_k} + x_k\frac{\partial^{\#} f}{\partial^{\#} t}\right), k = 1,2,3, \qquad (1.39)$$

and its direct consequence

$$\left[Ext\text{-}\exp\left(i\beta M^{0k}_{\varkappa,g}\right)\right]\varphi^{\#}_{\varkappa}(x,t)\left[Ext\text{-}\exp\left(-i\beta M^{0k}_{\varkappa,g}\right)\right] \approx \varphi^{\#}_{\varkappa}\left(\Lambda^{(k)}_{\beta}(x,t)\right), k = 1,2,3. \qquad (1.40)$$

Definition 1.23 If $I^3 = [a,b]^3 = [a,b] \times [a,b] \times [a,b]$ is a cube in ${}^*\mathbb{R}^{\#3}_{c,\mathrm{fin}}$, where $[a,b]$ is an #-closed interval in ${}^*\mathbb{R}^{\#}_{c,\mathrm{fin}}$. A causal shadow of I^3 is defined to be the diamond

$$O_{I^3} = \{(x_1, x_2, x_3, t) | a + |t| < x_k < b - |t|; k = 1,2,3\}. \qquad (1.41)$$

Remark 1.8 Note that because we can always translate in the positive $\boldsymbol{x_k}, \boldsymbol{k = 1,2,3}$ directions, it is sufficient to prove Theorem 1.6 for sets $\boldsymbol{O}$ such that both $\boldsymbol{O}$ and $\boldsymbol{\Lambda^{(k)}_{\beta} O, k = 1,2,3}$ are contained in $\boldsymbol{O_{I^3}}$ for some #-closed interval $\boldsymbol{I} \subset {}^*\mathbb{R}^{\#}_{\boldsymbol{c,\mathrm{fin}+}}$. The advantage of working over ${}^*\mathbb{R}^{\#\boldsymbol{3}}_{\boldsymbol{c,\mathrm{fin}+}}$ is that the locally correct Lorentz boost generators $\boldsymbol{M^{0k}_{\varkappa,g}, k = 1,2,3}$ are bounded be1ow.

2. Properties of the Lorentz boost generators $M^{0k}_{\varkappa,g}, k = 1,2,3$

In this section we consider the basic properties of $H_{\varkappa,g}$ and $M^{0k}_{\varkappa,g}, k = 1,2,3$ -in particular, the first order estimates they satisfy. Note that $H_{\varkappa,g}$ and $M^{0k}_{\varkappa,g}, k = 1,2,3$ are well defined operators on a non-Archimedean Fock space $\mathcal{F}^{\#}$. We take the definition of $\mathcal{F}^{\#}$ and the definition of the pointwise-defined time-zero field operators on $\mathcal{F}^{\#}$ as in [8] (see [8, Section 9]). The spatially cut-off Hamiltonian is defined as self-#-adjoint operator on a non-Archimedean Fock space $\mathcal{F}^{\#}$ [8].
Let $g = \{g_0, g_1\}$, where $g_0 = \left\{g_0^{(k)}\right\}, k = 1,2,3$, $g_0^{(k)}, g_1 \in C_0^{*\infty}\left({}^*\mathbb{R}^{\#3}_{c,\mathrm{fin}+}, {}^*\mathbb{R}^{\#}_{\mathrm{c,fin}}\right)$ and $g_0^{(k)}, g_1 \geq 0, k = 1,2,3$. The spatially cut-off Hamiltonian reads

$$H_{\varkappa,g} = H_{\varkappa}(g) = H_{0,\varkappa} + T_{I,\varkappa}(g_1), \qquad (2.1)$$

where $T_{I,\varkappa}(f) = Ext\text{-}\int_{{}^*\mathbb{R}^{\#3}_{c}} f(x) T_{I,\varkappa}(x) d^{\#3}x$ and

$$T_{I,\varkappa}(x) =: \varphi^{\#2n}_{\varkappa}(x): \qquad (2.2)$$

is the interaction energy density. The operator $H_{\varkappa}(g)$ has been studied in [8] and is known to be a self-#-adjoint semibounded operator on $\mathcal{F}^{\#}$. For the region O_{I^3}, defined above in section 1 we set now

$$M_{\varkappa,g}^{0k} = \alpha H_{0,\varkappa} + T_{0,\varkappa}\left(x_k g_0^{(k)}\right) + T_{I,\varkappa}(x_k g_1) \tag{2.3}$$

with $\alpha > 0$, and

$$T_{0,\varkappa}(f) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} f(x) T_{0,\varkappa}(x) d^{\#3}x.$$

We assume now that

$$\alpha + x_k g_0^{(k)}(x) = x_k g_1(x) = x_k, k = 1,2,3 \text{ on } I^3 = [a,b]^3 \subset {}^*\mathbb{R}_{c,\text{fin}+}^{\#3} \tag{2.4}$$

and two additional technical conditions on the $g = \{g_0, g_1\}$

$$x_k g_0^{(k)}(x) = h_k^2(x) \geq 0, h_k \in {}^*C_0^{\infty}\left({}^*\mathbb{R}_{c,\text{fin}+}^{\#3}, {}^*\mathbb{R}_{c,\text{fin}}^{\#}\right), k = 1,2,3 \tag{2.5}$$

and

$$x_k g_1(x) = \left[\alpha + x_k g_0^{(k)}(x)\right] g_1(x). \tag{2.6}$$

We rewrite now the operator $T_{0,\varkappa}(f)$ as

$$T_{0,\varkappa}(f) = T_{0,\varkappa}^{(1)}(f) + T_{0,\varkappa}^{(2)}(f) = Ext\text{-}\int_{|k_1|\leq\varkappa} Ext\text{-}\int_{|k_2|\leq\varkappa} t^{(1)}(k_1,k_2) a^*(k_1) a(k_2)\, d^{\#3}k_1\, d^{\#3}k_2 \tag{2.7}$$

$$+Ext\text{-}\int_{|k_1|\leq\varkappa} Ext\text{-}\int_{|k_2|\leq\varkappa} t^{(2)}(k_1,k_2)[a^*(k_1)a^*(-k_2) + a(-k_1)a(k_2)]\, d^{\#3}k_1\, d^{\#3}k_2 =$$

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \Theta(k_1,\varkappa)\Theta(k_2,\varkappa) t^{(1)}(k_1,k_2) a^*(k_1) a(k_2)\, d^{\#3}k_1\, d^{\#3}k_2$$

$$+Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \Theta(k_1,\varkappa)\Theta(k_2,\varkappa) t^{(2)}(k_1,k_2)[a^*(k_1)a^*(-k_2) + a(-k_1)a(k_2)]\, d^{\#3}k_1\, d^{\#3}k_2,$$

$$t^{(1)}(k_1,k_2) = \text{const}\cdot\Theta(k_1,\varkappa)\Theta(k_2,\varkappa)\left[Ext\text{-}\hat{f}(k_1-k_2)\right] \times [\mu(k_1)+\mu(k_2)+\langle k_1,k_2\rangle + m^2] \times$$

$$\times [\mu(k_1)\mu(k_2)]^{-1/2}, \tag{2.8}$$

$$t^{(2)}(k_1,k_2) = \text{const}\cdot\Theta(k_1,\varkappa)\Theta(k_2,\varkappa)\left[Ext\text{-}\hat{f}(k_1-k_2)\right][-\mu(k_1)+\mu(k_2)+\langle k_1,k_2\rangle + m^2] \times$$

$$\times [\mu(k_1)\mu(k_2)]^{-1/2}, \tag{2.9}$$

where

$$\Theta(k,\varkappa) = \begin{cases} 1 \text{ if } |k| \leq \varkappa, \\ 0 \text{ if } |k| > \varkappa. \end{cases} \tag{2.10}$$

Note that $t^{(1)}, t^{(2)} \in L_2^{\#}\left({}^*\mathbb{R}_c^{\#6}\right)$.

It follows that $T_{0,\varkappa}^{(i)}(f)(N_{\varkappa}+I)^{-1}, i = 1,2$ are bounded,

$$\left\|T_{0,\varkappa}^{(i)}(f)(N_{\varkappa}+I)^{-1}\right\|_{\#} \leq \text{const}\cdot\left\|t^{(i)}\right\|_{L_2^{\#}}.$$

Let $P_{\varkappa}^k(f)$

$$P_{\varkappa}^{k}(f) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} f(x) P_{\varkappa}^{k}(x) d^{\#3}x, \qquad (2.11)$$

Where $P_{\varkappa}^{k}(x)$ is given by (1.29) and $f \in C_0^{*\infty}\left({}^*\mathbb{R}_{c,\text{fin}+}^{\#3}, {}^*\mathbb{R}_{c,\text{fin}}^{\#}\right)$.

Here $N_{\varkappa}$ is the number operator with hyperfinite cut-off $\varkappa$ and we have used the $N_{\varkappa}$-estimate [8]: Let W be a Wick monomial

$$W_{\varkappa} = Ext\text{-}\int_{|k_1|\leq\varkappa} d^{\#3}k_1 \ldots Ext\text{-}\int_{|k_r|\leq\varkappa} d^{\#3}k_r w(k_1, \ldots, k_r) a^{\dagger}(k_1) \cdots a(k_r) \qquad (2.12)$$

with kernel $w \in L_2^{\#}\left({}^*\mathbb{R}_c^{\#3r}\right)$, then

$$\left\| (N_{\varkappa} + I)^{-a/2} W (N_{\varkappa} + I)^{-b/2} \right\|_{\#} \leq \text{const} \cdot \|w\|_{L_2^{\#}}, \qquad (2.13)$$

where $a + b \geq r$. A similar decomposition holds for $P_{\varkappa}^{k}(f), k = 1,2,3$. The result reads:
Proposition 2.1[13] Let $A = T_{0,\varkappa}^{(i)}(f), i = 1,2$ or $P_{\varkappa}^{k}(f)$ with $f \in C_0^{*\infty}\left({}^*\mathbb{R}_{c,\text{fin}+}^{\#3}, {}^*\mathbb{R}_{c,\text{fin}}^{\#}\right)$. Then,

$$\left\| \left(H_{0,\varkappa} + I\right)^{-i/2} A \left(H_{0,\varkappa} + I\right)^{-j/2} \right\|_{\#} < {}^*\infty. \qquad (2.14)$$

That is convenient to approximate the operators $M_{\varkappa,g}^{0k}, k = 1,3,3$ by the operators $M_{\varkappa,\kappa,g}^{0k}, k = 1,3,3$ with an additional momentum cut-off

$$M_{\varkappa,\kappa,g}^{0k} = \alpha H_{0,\varkappa,\kappa} + T_{0,\varkappa,\kappa}\left(x_k g_0^{(k)}\right) + T_{I,\varkappa,\kappa}(x_k g_1),$$

where $T_{0,\varkappa,\kappa}$ and $T_{I,\varkappa,\kappa}$ are defined by cutting off all the momentum integrals at $|k| > \kappa$. That is, $T_{0,\varkappa}$ and $T_{I,\varkappa}$, are expressed as a sum of Wick monomials (2.12) each of which is replaced in the definition of $T_{0,\varkappa,\kappa}$ and $T_{I,\varkappa,\kappa}$ by

$$W_{\varkappa,\kappa} = Ext\text{-}\int_{|k_1|\leq\varkappa} d^{\#3}k_1 \ldots Ext\text{-}\int_{|k_r|\leq\varkappa} d^{\#3}k_r \chi_{\kappa}(k_1, \ldots, k_r) w(k_1, \ldots, k_r) a^{\dagger}(k_1) \cdots a(k_r).$$

Here $\chi_{\kappa}(k_1, \ldots, k_r) = 1$ if $|k_i| \leq \kappa \leq \varkappa$ for all $1 \leq i \leq r$, and $\chi_{\kappa}(k_1, \ldots, k_r) = 0$ otherwise. We abbreviate also

$$M_{0,\varkappa,\kappa,g}^{0k} = \alpha H_{0,\varkappa,\kappa} + T_{0,\varkappa,\kappa}\left(x_k g_0^{(k)}\right), k = 1,2,3.$$

Note that as a rule, estimates that hold for $M_{\varkappa,g}^{0k}$ also hold for $M_{\varkappa,\kappa,g}^{0k}$, uniformly in κ. For example, for all $\kappa \in {}^*\mathbb{R}_{c+,\infty}^{\#}, \kappa \leq \varkappa$:

$$\left\| \left(H_{0,\varkappa,\kappa} + I\right)^{-l_1/2} T_{0,\varkappa,\kappa}^{(i)}(f) \left(H_{0,\varkappa,\kappa} + I\right)^{-l_2/2} \right\|_{\#} \leq \text{const.}, i = 1,2 \qquad (2.15)$$

and

$$\left\| \left(N_{\varkappa,\kappa} + I\right)^{-l_1/2} T_{0,\varkappa,\kappa}^{(i)}(f) \left(N_{\varkappa,\kappa} + I\right)^{-l_2/2} \right\|_{\#} \leq \text{const.}, i = 1,2 \qquad (2.16)$$

for $l_1 + l_2 \geq 2$, where the constants are independent of κ. As a domain of admissible vectors in $\mathcal{F}^{\#}$

$$\mathcal{D}_{\text{fin}}^{\#} = \left\{\psi | \psi = (\psi_0, \psi_1, \ldots) \in \mathcal{F}^{\#}, \psi_n \in C_0^{*\infty}\left({}^*\mathbb{R}_{c,\text{fin}+}^{\#3n}, {}^*\mathbb{R}_{c,\text{fin}}^{\#}\right), \psi_n \equiv 0 \text{ for large } n \in {}^*\mathbb{N} \right\}. \qquad (2.17)$$

Remark 2.1 The operators $M^{0k}_{\varkappa,g}$, $k = 1,2,3$ as constructed above, enjoys the property of being semibounded.

Theorem 2.2 Let $g = \{g_0, g_1\}$ satisfy the condition (2.4). Then there are constants a and b such that for all $\kappa < \varkappa$

$$H_{0,\varkappa} \le a\left(M^{0k}_{\varkappa,\kappa,g} + b\right), k = 1,2,3 \tag{2.18}$$

on the domain $\mathcal{D}^{\#}_{\text{fin}} \times \mathcal{D}^{\#}_{\text{fin}}$.

Proof For $\varepsilon > 0$, there is a constant d such that [8]

$$0 \le H_{0,\varkappa} + T_{I,\varkappa,\kappa}\left(x_k g_1(x)\right) + d, k = 1,2,3 \tag{2.19}$$

on the domain $\mathcal{D}^{\#}_{\text{fin}} \times \mathcal{D}^{\#}_{\text{fin}}$. For $\varepsilon > 0$, there is a constant c such that [8]

$$0 \le H_{0,\varkappa} + T_{0,\varkappa,\kappa}\left(x_k g_0^{(k)}(x)\right) + c, k = 1,2,3 \tag{2.20}$$

on the domain $\mathcal{D}^{\#}_{\text{fin}} \times \mathcal{D}^{\#}_{\text{fin}}$. The inequalities (2.18) follows from adding (2.19) and (2.20).

Proposition 2.3 There are positive constants a, b, c such that

$$M^{0k}_{\varkappa} \le a(H_{\varkappa} + b) \le c\left(M^{0k}_{\varkappa} + b\right), k = 1,2,3 \tag{2.21}$$

on the domain $\mathcal{D}^{\#}_{\text{fin}} \times \mathcal{D}^{\#}_{\text{fin}}$.

Proof Note that for $k = 1,2,3$

$$a(H_{\varkappa} + b) - M^{0k}_{\varkappa} = (a - \alpha)H_{0,\varkappa} - T_{0,\varkappa}\left(x_k g_0^{(k)}(x)\right) + T_{I,\varkappa}\left((a - x_k)g_1(x)\right) + ab.$$

By choosing constant a larger than $\max_k[\sup\{x_k | g_1(x) \neq 0\}]$, we have $(a - x_k)g_1(x) > 0$ and therefore as in (2.19)

$$H_{0,\varkappa} + T_{I,\varkappa}\left((a - x_k)g_1(x)\right) \ge 0.$$

Moreover, by (2.14) we can choose a so that

$$(a - \alpha - 1)\left(H_{0,\varkappa} + I\right) - T_{0,\varkappa}\left(x_k g_0^{(k)}(x)\right) \ge 0.$$

The second part of (2.21) follows by a similar consideration,

3. Quadratic estimates

In this section we prove the self-#-adjointness of the operators $M^{0k}_{\varkappa,\kappa}$, $k = 1,2$, by interpreting the operator $T_{0,\varkappa,\kappa}$ as generalized Kato perturbation [8]. Thus we need proving quadratic inequalities such as

$$\left(H_{0,\varkappa} + I\right)^2 \le a_\kappa\left(H_{0,\varkappa} + \lambda T_{0,\varkappa,\kappa}\left(f_{0,k}\right) + T_{I,\varkappa,\kappa}(f_1) + b\right)^2, \tag{3.1}$$

where a_κ and b are constants with a_κ depending on κ. Here λ is finite constant and $f_{0,k} = \alpha^{-1} x_k g_0^{(k)}(x)$ where $g_0^{(k)}(x)$ satisfies conditions (2.5).

Theorem 3.1 $M^{ik}_{\varkappa,\kappa}$ is essentially self-#-adjoint on $D^{\#}$. There are constants a and b independent of κ, such that for $\kappa < \varkappa$ and $k = 1,2,3$

$$\left(H_{0,\varkappa} + I\right)^2 \le a\left(M^{0k}_{0,\varkappa,\kappa} + b\right). \tag{3.2}$$

Remark 3.1 For φ_4^{2n} we use the "pull through formula" [161. Let $T_\varkappa = \#\text{-}\overline{\left(H_{0,\varkappa} + V_\varkappa\right)}$ and $R(z) = (T_\varkappa - z)^{-1}$. Then

$$a(\boldsymbol{k})R(z) = R\big(z - \mu(\boldsymbol{k})\big)a(\boldsymbol{k}) - R\big(z - \mu(\boldsymbol{k})\big)[a(\boldsymbol{k}), V]R(z). \tag{3.3}$$

We shall always be concerned with operators T that are essentially self-#-adjoint on domain $\mathcal{D}^{\#}_{\text{fin}}$ defined in (2.17), and whose perturbation V is a finite sum of Wick monomials with #-smooth kernels. It follows that $a(\boldsymbol{k})$ is defined on the #-dense domain

$$\mathcal{D}^{\#\prime}_{\text{fin}} = (T_\varkappa - z)\mathcal{D}^{\#}_{\text{fin}} \tag{3.4}$$

and that (3.3) holds on this domain.

Lemma 3.2 Suppose that $T_\varkappa = \#\text{-}\overline{\left(H_{0,\varkappa} + V_\varkappa\right)}$ satisfies the above conditions. Let $\psi \in \mathcal{D}^{\#\prime}_{\text{fin}}$, where $(z - c)$ is in the resolvent set of $T_\varkappa$ for all $c \ge 0$. Then for $r \in \mathbb{N}$ a positive integer

$$a_{(1,r)}R\psi = Ext\text{-}\textstyle\sum_{\text{part.}}(-1)^j\, R_{J_1}V_\varkappa^{I_1}R_{J_2}\cdots R_{J_j}V_\varkappa^{I_j}R_{J_{j+1}}a_{I_{j+1}}\psi, \tag{3.5}$$

where $I = \{i_1, \dots, i_s\}$ be a set of distinct ordered positive integers, $(1,r) = \{1,2,\dots,r\}$, $a_I = Ext\text{-}\prod_{l=1}^{s} a\left(k_{i_l}\right)$ for $s > 0$, $a_I = \mathrm{I}$ for $s = 0$. The sum in (3.5) takes place over all partitions of $\{1,2,\dots,r\}$ into disjoint subsets $I_1, \dots, I_{j+1}$ (including permutations among the subsets) for $j = 0, 1, \dots, r$. The elements of each I_i are taken in natural order. Let $R_{J_l} = R(\zeta), R(z) = (T_\varkappa - z)^{-1}$, where $\zeta = z - Ext\text{-}\sum_{i \in J_l}\mu(k_i)$ and $J_l = I_l \cup I_{j+1} \cup \dots \cup I_{j+1}$. Let $V^I = \left[a\left(k_{i_1}\right), \dots, \left[a\left(k_{i_s}\right), V\right] \dots\right]$ for $s > 0$ and $V^I = 0$ for $s = 0$. Note that the sum (3.5) includes terms where J_{j+1} is empty but not $I_1, \dots, I_j$; this convention adjusts the sign $(-1)^j$ correctly. The $j = 0$ term is simply $R_1 a_{(1,r)}\psi$.

Proof In order to apply (3.5) to the proof of (3.1) we must be able to estimate the commutators

$$X^{(i)}_{\varkappa,\kappa}(k) = \left[a(k), T_{0,\varkappa,\kappa}(f)\right] \tag{3.6}$$

$i = 1, 2$, for sufficiently large k, where $f \in {C_0}^{*\infty}\left({}^*\mathbb{R}^{\#3}_{c,\text{fin}+}, {}^*\mathbb{R}^{\#}_{c,\text{fin}}\right)$.

Lemma 3.3

$$\left\| X^{(2)}_\varkappa(k)(N_\varkappa + I)^{-1/2} \right\|_{\#} = O([\mu(k)]^{-1}). \tag{3.7}$$

Proof $X^{(2)}_\varkappa(k))$ is certainly #-densely defined, say on domain D; it is sufficient to prove (3.7) on D and then $X^{(2)}_\varkappa(k)(N_\varkappa + I)^{-1/2}$ extends to a bounded operator on all vectors of $\mathcal{F}^{\#}$. Now we set

$$X^{(2)}_\varkappa(k) = Ext\text{-}\textstyle\int_{|k| \le \varkappa} w(k,p)^* a(-p)d^{\#3}p,$$

where by (2.9) the kernel $w(k,p)$ can be estimated by

$$|w(k,p)| = |h(k-p)|[\mu(k)]^{-1/2}[\mu(p)]^{-1/2}$$

where $h \in S^{\#}_{\text{fin}}({}^{*}\mathbb{R}^{\#3}_{c})$ is rapidly decreasing. According to (2.13), by a simple calculation one obtains

$$\left\| X^{(2)}_{\varkappa}(k)(N_{\varkappa}+I)^{-1/2} \right\|_{\#} \leq \text{const.} \times \|w(k,\cdot)\|_{\#2} = O([\mu(k)]^{-1}).$$

Lemma 3.4 For arbitrary $\psi \in \mathcal{F}^{\#}$ and $c > 0$

$$A = Ext\text{-}\int_{|k|\leq\varkappa} d^{\#3}k \left\| \left(H_{0,\varkappa}+c+\mu(k)\right)^{-\frac{1}{2}} X^{(1)}_{\varkappa}(k)\left(H_{0,\varkappa}+c\right)^{-\frac{1}{2}}\psi \right\|^{2}_{\#} \leq \text{const.} \times \|\psi\|^{2}_{\#}. \quad (3.8)$$

Proof Let $\mathcal{F}^{\#}_{n}, n \in {}^{*}\mathbb{N}$ be the n-particle Fock space. Now $X^{(1)}_{\varkappa}(k)$ is defined on D for all k and since $X^{(1)}_{\varkappa}(k)$ maps $\mathcal{F}^{\#}_{r}$ into $\mathcal{F}^{\#}_{r-1}$, it is sufficient to prove that (3.8) holds for $\psi \in D \cap \mathcal{F}^{\#}_{n}$ with the constant independent of n. We remark that by the methods of the previous lemma it is easy to show that the integrand in (3.8) is uniformly bounded in k, but different methods are necessary to prove it integrable. Now we define

$$X^{(1)}_{\varkappa}(k) = Ext\text{-}\int_{|k|\leq\varkappa} t^{(1)}(k,p)a(p)d^{\#3}p,$$

where $t^{(1)}(k,p)$ is given by (2.9); therefore we obtain

$$A_{\varkappa} \leq Ext\text{-}\int_{|k|\leq\varkappa} d^{\#3}k\, Ext\text{-}\int_{|p_1|\leq\varkappa} d^{\#3}p_1 \cdots Ext\text{-}\int_{|p_{n-1}|\leq\varkappa} d^{\#3}p_{n-1} \times$$

$$\times \left[\left(Ext\text{-}\sum_{i=1}^{n-1}\mu(p_i)+\mu(k)+c\right)^{-1/2} n^{1/2} \times \right.$$

$$\left. \times\, Ext\text{-}\int_{|p|\leq\varkappa} d^{\#3}p \left|t^{(1)}(k,p)\right| \left(Ext\text{-}\sum_{i=1}^{n-1}\mu(p_i)+\mu(k)+c\right)^{-1/2} |\psi(p_1,\ldots,p_{n-1},p)| \right]^{2}, \quad (3.9)$$

where $a(p)$ has destroyed a particle by

$$(a(p)\psi)(p_1,\ldots,p_{n-1},p) = n^{1/2}\psi(p_1,\ldots,p_{n-1},p). \quad (3.10)$$

By the definition (2.9) we obtain

$$\left|t^{(1)}(k,p)\right| (Ext\text{-}\sum_{i=1}^{n}\mu(p_i)+\mu(k)+c)^{-1/2} \leq \text{const.} \times [\mu(k)]^{1/2}\left(\left|Ext\text{-}\hat{f}(k-p)\right|\right).$$

Replacing now k by p_n in (3.9) we get

$$A_{\varkappa} \leq a \times n \times Ext\text{-}\int_{|p_1|\leq\varkappa} d^{\#3}p_1 \cdots Ext\text{-}\int_{|p_{n-1}|\leq\varkappa} d^{\#3}p_n \times$$

$$\left[[\mu(p_n)]^{1/2} (Ext\text{-}\sum_{i=1}^{n}\mu(p_i)+c)^{-1/2} Ext\text{-}\int_{|p|\leq\varkappa} d^{\#3}p \left(\left|Ext\text{-}\hat{f}(p_n-p)\right|\right) |\psi(p_1,\ldots,p_{n-1},p)| \right]^{2} =$$

$$= a \times Ext\text{-}\sum_{j=1}^{n} Ext\text{-}\int_{|p_1|\leq\varkappa} d^{\#3}p_1 \cdots Ext\text{-}\int_{|p_n|\leq\varkappa} d^{\#3}p_n \times$$

$$\times \left[Ext\text{-}\int_{|p|\leq\varkappa} d^{\#3}p E_j(p_1,\ldots,p_n)\left(\left|Ext\text{-}\hat{f}\left(p_j-p\right)\right|\right) \left|\psi\left(p_1,\ldots,p_{j-1},p,p_{j+1},\ldots,p_n\right)\right| \right]^{2}, \quad (3.11)$$

where a is a constant and

$$E_j(p_1,\ldots,p_n) = \left[\mu\left(p_j\right)/(Ext\text{-}\sum_{i=1}^{n}\mu(p_i)+c)\right]^{1/2}$$

We shall write this symbolically as $E_j(p_j)$, suppressing the other variables. In obtaining (3.11) we have interchanged p_j and p_n, and exploited the symmetry of ψ. In (3.1 I) we wish to replace $E_j(p_j)$ by $E_j(p)$ to get

$$A'_{\varkappa} = a \times Ext\text{-}\sum_{j=1}^{n} Ext\text{-}\int_{|p_1|\leq\varkappa} d^{\#3}p_1 \cdots Ext\text{-}\int_{|p_n|\leq\varkappa} d^{\#3}p_n \times$$

$$\times \left[Ext\text{-}\int_{|p|\leq\varkappa} d^{\#3}pE_j(p)\left(\left|Ext\text{-}\hat{f}(p_j - p)\right|\right)\left|\psi(p_1,\dots,p_{j-1},p,p_{j+1},\dots,p_n)\right|\right]^2$$

For then the integral over p is a convolution between

$$\phi_j(p) = E_j(p)\left|\psi(p_1,\dots,p_{j-1},p,p_{j+1},\dots,p_n)\right|$$

and $h(p) = \left|Ext\text{-}\hat{f}(p)\right|$, and the integral over p_j is the square of the $L_2^{\#}$ #-norm of this convolution. Now we get

$$Ext\text{-}\int_{|p_j|\leq\varkappa} d^{\#3}p_j \left[Ext\text{-}\int_{|p\,|\leq\varkappa} h(p_j - p)\phi_j(p)d^{\#3}p\right]^2 = \left\|\left(Ext\text{-}\hat{h}\right) \times \left(Ext\text{-}\widehat{\phi_j}\right)\right\|_{\#2}^2 \leq$$

$$\leq \left\|Ext\text{-}\hat{h}\right\|_{^*\infty}^2 \times \left\|\phi_j\right\|_{\#2}^2$$

and

$$\left\|Ext\text{-}\hat{h}\right\|_{^*\infty}^2 = Ext\text{-}\int_{|p\,|\leq\varkappa} \left(Ext\text{-}\hat{f}(p)\right) d^{\#3}p < \infty.$$

Therefore,

$$A'_{\varkappa} \leq \text{const.}\times\, Ext\text{-}\sum_{j=1}^{n}\left\|E_j(p_j)\psi(p_1,\dots,p_n)\right\|_{\#2}^2 = \text{const.}\times \left\|\left(Ext\text{-}\sum_{j=1}^{n} E_j^2\right)^{1/2}\psi\right\|_{\#2}^2 \leq$$

$$\leq \text{const.}\times \|\psi\|_{\#2}^2.$$

In order to justify the replacement of $E_j(p_j)$ by $E_j(p)$, we set

$$E_j(p_j) = E_j(p) + \left(E_j(p_j) - E_j(p)\right)$$

and therefore we obtain

$$\left[Ext\text{-}\int_{|p\,|\leq\varkappa} d^{\#3}pE_j(p_j)\left|\left(Ext\text{-}\hat{f}\right)\psi\right|\right]^2 = \left[Ext\text{-}\int_{|p\,|\leq\varkappa} d^{\#3}pE_j(p)\left|\left(Ext\text{-}\hat{f}\right)\psi\right|\right]^2 +$$

$$+\left[Ext\text{-}\int_{|p\,|\leq\varkappa} d^{\#3}p\left(E_j(p_j) - E_j(p)\right)\left|\left(Ext\text{-}\hat{f}\right)\psi\right|\right]^2 + 2\left[Ext\text{-}\int_{|p\,|\leq\varkappa} d^{\#3}pE_j(p)\left|\left(Ext\text{-}\hat{f}\right)\psi\right|\right] \times$$

$$\times \left[Ext\text{-}\int_{|p\,|\leq\varkappa} d^{\#3}p\left(E_j(p_j) - E_j(p)\right)\left|\left(Ext\text{-}\hat{f}\right)\psi\right|\right]. \qquad (3.12)$$

Applying the operation $a \times Ext\text{-}\sum_{j=1}^{n} Ext\text{-}\int_{|p_1|\leq\varkappa} d^{\#3}p_1 \cdots Ext\text{-}\int_{|p_n|\leq\varkappa} d^{\#3}p_n$ to (3.12), we obviously get $A_{\varkappa}$ on the left and $A'_{\varkappa}$ from the first term on the right. To estimate the second term, we note that

$$\left|E_j(p_j) - E_j(p)\right| \leq \left|E_j(p_j)^2 - E_j(p)^2\right|^{\frac{1}{2}} =$$

$$\left|\left(Ext\text{-}\sum_{i\neq j}\mu(p_i)+c\right)\left(\mu(p_j)-\mu(p)\right)\right|^{1/2}\left|\left(Ext\text{-}\sum\mu(p_i)+c\right)\left(Ext\text{-}\sum_{i\neq j}\mu(p_i)+\mu(p)+c\right)\right|^{-1/2}$$

$$\leq \text{const.}\times n^{-\frac{1}{2}}\left|\mu(p_j)-\mu(p)\right|^{\frac{1}{2}}\leq \text{const.}\times n^{-\frac{1}{2}}\left|\left\|p_j\right\|_{\#}-\|p\|_{\#}\right|^{\frac{1}{2}}\leq \text{const.}\times n^{-\frac{1}{2}}\left\|p_j-p\right\|_{\#}^{1/2},$$

where $\|\cdot\|_{\#}$is Euclidian #- norm in ${}^{*}\mathbb{R}_{c}^{\#3}$.Therefore the integral of the second term in (3.12) can be estimated by

$$\text{const.}\times n^{-1}\times Ext\text{-}\sum_j Ext\text{-}\int_{|p_1|\leq\varkappa}d^{\#3}p_1\cdots Ext\text{-}\int_{|p_n|\leq\varkappa}d^{\#3}p_n\times$$

$$\left[Ext\text{-}\int_{|p\,|\leq\varkappa}d^{\#3}p\left\|p_j-p\right\|_{\#}^{1/2}\left|\left(Ext\text{-}\hat{f}(p_j-p)\right)\psi(p_1,\dots,p_{j-1},p,p_{j+1},\dots,p_n)\right|\right].$$

But, as before, this is the square of the $L_2^{\#}$- #-norm of the convolution of the function ψ with a rapidly decreasing function and so it can be estimated by

$$\text{const.}\times n\times Ext\text{-}\sum_j\|\psi\|_{\#}^2\leq \text{const.}\times\|\psi\|_{\#}^2,$$

where the constant is independent of n. The third term resulting from (3.12) can then be estimated by the generalized Schwarz inequality applied to $Ext\text{-}\sum_{j=1}^{n}Ext\text{-}\int_{|p_1|\leq\varkappa}d^{\#3}p_1\cdots Ext\text{-}\int_{|p_n|\leq\varkappa}d^{\#3}p_n$. Hence $A_{\varkappa}$ is bounded as claimed. The single commutators (3.6) are all that we need estimate. For let $I=\{i_1,\dots,i_s\}$; then $\left(T_{0,\varkappa}^{(1)}(f)\right)^{I}=0$ if $\left(T_{0,\varkappa}^{(2)}(f)\right)^{I}$ and $\left(T_{0,\varkappa}^{(2)}(f)\right)^{I}=0$ when $s>2$. When $s=2$, $\left(T_{0,\varkappa}^{(2)}(f)\right)^{I}$ reduces to the constant $2\Theta(k\,,\varkappa)t^{(2)}(k_1-k_2)$; thus for all s, $T_{0,\varkappa}^{(2)}(f)$ satisfies

$$\left\|\left(T_{0,\varkappa}^{(2)}(f)\right)^{I}(N_{\varkappa}+I)^{-1/2}\right\|_{\#}\leq \text{const.}\times Ext\text{-}\prod_{i\in I}[\mu(k_i)]^{-1/2} \tag{3.13}$$

by virtue of (3.7) and (2.11).

Remark 3.2 We now go to prove (3.1) by using the formula (3.5). For convenience, we work now with

$$T_{\varkappa,\kappa}^{0k}(\lambda)=\#\text{-}\overline{\left[\left(H_{0,\varkappa}+\lambda T_{0,\varkappa,\kappa}(f_{0,k})+T_{I,\varkappa,\kappa}(f_1)\right)\upharpoonright D\right]}\,. \tag{3.14}$$

which is $M_{\varkappa,\kappa}^{0k}$ up to constants. To apply the pull-through formula (3.5) it is necessary to know that the operators $T_{\varkappa,\kappa}^{0k}, k=1,2,3$ are self-#-adjoint. For the moment we assume this, postponing the proof until Theorem 3.8. We remark though that in the case $\lambda=0$ $T_{\varkappa,\kappa}^{0k}$ reduces to $H_{\varkappa,\kappa}(f_1)$ which is known to be self-#-adjoint. The next lemma gives an estimate on commutators such that

$$X_{\varkappa,\kappa}^{(3)}(k)=\left[a(k),T_{I,\varkappa,\kappa}(f_1)\right] \tag{3.15}$$

which is finite or hyperfinite polynomial of degree $(2n\,-\,1)$ in the field $\varphi_{\varkappa}^{\#}(x)$. Since $T_{\varkappa,\kappa}^{0k}$ remains semibounded (Theorem 2.2) when perturbed by a polynomial in the field of degree less than $2n$, we have the following estimate in terms of the resolvent $R_{\varkappa,\kappa}(z)=\left(T_{\varkappa,\kappa}-z\right)^{-1}$:

Lemma 3.5 Let $r\in{}^{*}\mathbb{N}$ be a positive integer. There is a $z_0<0$ independent of κ and r such that, for $z_1\leq z_0, z_2\leq z_0$

$$\left\| R_{\varkappa,\kappa}^{1/2}(z_2) T_{\varkappa,\kappa}^{(1,r)} R_{\varkappa,\kappa}^{1/2}(z_1) \right\|_{\#} \leq \text{const.} \times \prod_{i=1}^{r} [\mu(k)]^{-\frac{1}{2}}, \tag{3.16}$$

where the constant is independent of κ, z_1, z_2. Here, in the notation of Lemma 3.2,

$$T_{I,\varkappa,\kappa}^{(1,r)0k} = \left[a(k_1), \left[\cdots \left[a(k_r), T_{I,\varkappa,\kappa}(f_1)\right] \cdots\right]\right].$$

Theorem 3.6 Assume that the operators $T_{\varkappa,\kappa}^{0k}$ are given by (3.14) is self-#-adjoint, where $k \leq \varkappa$. Then there are positive constants $b, c(k)$, and $d(k)$ all independent of λ such that

$$\left(H_{0,\varkappa} + I\right)^2 \leq \left(c(k) + \lambda^2 d(k)\right)\left(T_{\varkappa,\kappa}^{0k} + b\right)^2. \tag{3.17}$$

Proof Obviously it is sufficient to prove that

$$\left\| \left(H_{0,\varkappa} + I\right) R_{\varkappa,\kappa}(-b)\psi \right\|_{\#}^2 \leq \left(c(k) + \lambda^2 d(k)\right) \|\psi\|_{\#}^2 \tag{3.18}$$

for ψ in the dense set $D_{1,k} = \left(T_{\varkappa,\kappa}^{0k} + b\right)D$ as in (3.4). This choice of ψ ensures that $R_{\varkappa,\kappa}(-b)\psi \in D_{1,k}$ is in the domain of all the operators we wish to apply to it. Here b is chosen so large that

$$\left\| \left(H_{0,\varkappa} + I\right)^{1/2} R_{\varkappa,\kappa}(-b)^{1/2} \right\|_{\#}^2 \leq \text{const.}, \tag{3.19}$$

(see 2.18) and so that (3.16) holds with $r = 1$,

$$\left\| R_{\varkappa,\kappa}^{1/2}(z_2) X_{\varkappa,\kappa}^{(3)}(k) R_{\varkappa,\kappa}^{1/2}(z_1) \right\|_{\#} \leq \text{const.} \times \Theta(k, \kappa)[\mu(k)]^{-\frac{1}{2}} \tag{3.20}$$

for $z_i < -b$. Now we get

$$\left\| \left(H_{0,\varkappa} + I\right) R_{\varkappa,\kappa}(-b)\psi \right\|_{\#}^2 = \tag{3.21}$$

But by the pull-through formula (3.3) we get

$$a(k) R_{\varkappa,\kappa}(-b)\psi =$$

where $X_{\varkappa,\kappa}^{(i)}(k), i = 1, 2$, are defined by (3.6) with a momentum cut-off κ. Substituting this into (3.21), we obtain by generalized Schwarz' inequality,

(3.22)

Remark 3.3 We now prove the self-#-adjointness of $M_{g,\varkappa,\kappa}^{0k}, k = 1,2,3$ by treating $T_{0,\varkappa,\kappa}^{0k}$ as a Kato perturbation. Generalized Kato's criterion is [8]:

Proposition 3.7 Let T is a self-#-adjoint operator and let D be a #-core for T. Suppose that A is symmetric and that there are positive constants a and b with $a < 1$ such that

$$\|A\psi\|_{\#} \leq a\|(T+b)\psi\|_{\#}$$

for all $\psi \in D(T)$. Then $T + A$ is self-#-adjoint on $D(T)$ and essentially self-#-adjoint on D.

Theorem 3.8 For $\kappa \leq \varkappa$ and g satiating (2.4), $M^{0k}_{g,\varkappa,\kappa}, k = 1,2,3$ are essentially self-#-adjoint on D.

Proof We show that $T^{0k}_{\varkappa,\kappa}$ given by (3.14) is self-#-adjoint where $f_{0,k} = \left\{x_k g_0^{(k)}/\alpha\right\}, f_1 = x_k g_1/\alpha$, $k = 1,2,3$ and $\lambda = 1$; this is equivalent to the statement of the theorem. We use Theorem 3.6 to prove Theorem 3.8 in spite of the fact that the conclusion of the second theorem appears as a hypothesis of the first. By Lemma 2.1 we know that there is a constant c_1 such that

$$\left\|T^{0k}_{\varkappa,\kappa}(f)\psi\right\|_{\#} \leq c_1\left\|\left(H_{0,\varkappa}+I\right)\psi\right\|_{\#} \tag{3.24}$$

for all $\psi \in D\left(H_{0,\varkappa}\right)$. We choose J to be a sufficiently large integer such that $c_1\left(c(k)+d(k)\right)^{1/2} < J$, where $c(k)$ and $d(k)$ are the constants in (3.17). Let us consider the sequence of values $\lambda = j/J, j = 0,\ldots,J$. Let $P_{j,k}$ be the statement that $T^{0k}_{\varkappa,\kappa}(j/J)$ is self-#-adjoint and $Q_{j,k}$ the statement that $J^{-1}T^{0k}_{0,\varkappa,\kappa}\left(f_{0,k}\right)$ is a Kato perturbation of $T^{0k}_{\varkappa,\kappa}(j/J)$, i.e., $\left\|J^{-1}T^{0k}_{0,\varkappa,\kappa}\left(f_{0,k}\right)\psi\right\|_{\#} \leq a\left\|\left(T^{0k}_{\varkappa,\kappa}(j/J)+b\right)\psi\right\|_{\#}$ for constants a and b with a < 1. As we have already observed, $P_{0,k}$ holds since $T^{0k}_{\varkappa,\kappa}(0)$ reduces to the Hamiltonian $H_{f_1,\varkappa,\kappa}$. Note that $P_{j,k}$ implies $Q_{j,k}, k = 1,2,3$ since, for $\psi \in D\left(T^{0k}_{\varkappa,\kappa}(j/J)\right)$,

$$\left\|J^{-1}T^{0k}_{0,\varkappa,\kappa}\left(f_{0,k}\right)\psi\right\|_{\#} \leq c_1 J^{-1}\left\|\left(H_{0,\varkappa}+I\right)\psi\right\|_{\#} \leq c_1 J^{-1} c_1\left(c(k)+d(k)\right)^{1/2}\left\|\left(T^{0k}_{\varkappa,\kappa}(j/J)+b\right)\psi\right\|_{\#}$$

by the inequality (3.24) and (3.17). However, by Proposition 3.7, the statement $Q_{j,k}$ implies $P_{j+1,k}, k = 1,2,3$.

4. Higher order estimates

In this section we derive higher order estimates of the following form

$$H^{j}_{0,\varkappa} \leq a_\kappa\left(M^{0k}_{0,\varkappa,\kappa}+b\right) \leq c_\kappa\left(H_{0,\varkappa}+I\right)^{nj} \tag{4.1}$$

and

$$H^2_{0,\varkappa} + N^{2n}_{\varkappa} \leq a\left(M^{0k}_{0,\varkappa,\kappa}+b\right)^{2n}, \tag{4.2}$$

where a_κ and c_κ are constants depending on κ. The estimates (4.1) are used to prove that the powers $\left(M^{0k}_{0,\varkappa,\kappa}\right)^j$ are essentially self-#-adjoint on $\mathcal{D}^{\#}_{\text{fin}}$ and do not survive in the #-limit: $\kappa \to_{\#} \varkappa$; on the other hand, the estimate (4.2) does transfer to the #-limit $\kappa = \varkappa$ and, in fact, enables us to prove that this #-limit exists. For real $\tau \in {}^*\mathbb{R}^{\#}_{c}$ we define the generalized number operator with hyperfinite momentum cut-off $\varkappa \in {}^*\mathbb{R}^{\#}_{c,\infty}$

$$N_{\varkappa,\tau} = \textit{Ext-}\int_{|k|\leq\varkappa} a^{\dagger}(k)[\mu(k)]^{\tau}\, a(k)d^{\#3}k. \tag{4.3}$$

Note that $N_{\varkappa,0} = N_\varkappa$ and $N_{\varkappa,1} = H_{0,\varkappa}$.

Lemma 4.1 (1) If $\tau \leq \nu$, then

$$N_{\varkappa,\tau} \leq \text{const.}\cdot N_{\varkappa,\tau}. \tag{4.4}$$

(2) If $\tau > 0, r > 0$, then

$$N_{\varkappa}^{r(1+\tau)} \leq H_{0,\varkappa}^{\tau r} N_{\varkappa,\tau}^{r}. \quad (4.5)$$

(3) Let $\tau \in {}^*\mathbb{R}_{\boldsymbol{c}}^{\#}$ and $r \in {}^*\mathbb{N}$ a positive integer, then for any vector $\psi \in D\left(N_{\varkappa,\tau}^{r/2}\right)$,

$$\left\|N_{\varkappa,\tau}^{\frac{r}{2}}\psi\right\|_{\#} =$$

$$Ext\text{-}\sum_{j=1}^{r}\left[Ext\text{-}\int d^{\#3}k_1 \cdots d^{\#3}k_j p_{rj}\left(\mu_1^{\tau},\dots,\mu_j^{\tau}\right)\left(Ext\text{-}\prod_{j=1}^{r}\Theta\left(k_j\,,\varkappa\right)\right)\left\|a_{(1,j)}\psi\right\|_{\#}^{2}\right], \quad (4.6)$$

where $\Theta(k\,,\varkappa)$ is defined by (2.10), $a_{(1,j)}$ is defined in Lemma 3.2, and p_{rj} is a homogeneous polynomial of degree $r \in {}^*\mathbb{N}$ with positive coeficients that satisfies, for $x_i > 0$,

$$\left(Ext\text{-}\prod_{l=1}^{j}x_l\right)\left(Ext\text{-}\sum_{l=1}^{j}x_l\right)^{r-j} \leq p_{rj}\left(x_1,\dots,x_j\right) \leq \text{const.}\cdot\left(Ext\text{-}\prod_{l=1}^{j}x_l\right)\left(Ext\text{-}\sum_{l=1}^{j}x_l\right)^{r-j}. \quad (4.7)$$

In this section we set

$$M_{0,\varkappa,\kappa}^{0k} = \#\text{-}\overline{\left[\left(H_{0,\varkappa}+V_{\varkappa,\kappa}\right)\upharpoonright D\right]},$$

where $V_{\varkappa,\kappa}^{(k)} = T_{0,\varkappa,\kappa}\left(f_{0,k}\right)+T_{I,\varkappa,\kappa}\left(f_1\right), k = 1,2,3$. Let $R_k(-b) = \left(M_{0,\varkappa,\kappa}^{0k}+b\right)^{-1}$.

Lemma 4.2 Let $r \in {}^*\mathbb{N}$ be a positive integer. Then there are constants a_κ and b where a_κ depends on $\kappa < \varkappa$, such that

$$\left\|\left(H_{0,\varkappa,\kappa}+I\right)^{r/2}\psi\right\|_{\#} \leq a_\kappa \left\|\left(M_{0,\varkappa,\kappa}^{0k}+b\right)^{\frac{r}{2}}\psi\right\|_{\#}, k = 1,2,3 \quad (4.8)$$

for all $\psi \in D\left(\left(M_{0,\varkappa,\kappa}^{0k}+b\right)^{\frac{r}{2}}\right)$.

Proof (4.8) is proved by hyper infinite induction on $r \in {}^*\mathbb{N}$: the cases $r = 1,2$ are already known by Theorem 2.2 and 3.6. Let $\psi \in D_{1,k} = \left(M_{0,\varkappa,\kappa}^{0k}+b\right), k = 1,2,3$, where $b = -z_0$ is chosen sufficiently large that (3.16) and (3.19) hold. By (4.6),

$$A_{r+1,,\varkappa,\kappa} = \left\|\left(H_{0,\varkappa,\kappa}+I\right)^{(r+1)/2}R_k(-b)\psi\right\|_{\#}^{2} =$$

$$Ext\text{-}\sum_{j=1}^{r}Ext\text{-}\int_{|k_1|\leq\kappa}d^{\#3}k_1 \cdots Ext\text{-}\int_{|k_n|\leq\kappa}d^{\#3}k_j p_{rj}\left(\mu_1,\dots,\mu_j\right)\times$$

$$\times\left\|\left(H_{0,\varkappa,\kappa}+Ext\text{-}\sum_{i=1}^{j}\mu(k_i)\;+I\right)^{1/2}a_{(1,j)}R_k\psi\right\|_{\#}^{2}, \quad (4.9)$$

where me have converted all but one $\left(H_{0,\varkappa,\kappa}+I\right)^{1/2}$ into an integral of products of annihilation operators. We apply the pull through formula (3.5) to pull the $a_{(1,j)}$ through the R_k, and we dominate the factor $\left(H_{0,\varkappa,\kappa}+Ext\text{-}\sum_{i=1}^{j}\mu(k_i)\;+I\right)^{1/2}$ by

$$R_{k,J_1}^{1/2} = R_k\left(-b-Ext\text{-}\sum\mu(k_i)\right)$$

by using (3.19). This gives

$$A_{r+1,,\varkappa,\kappa} \leq Ext\text{-}\sum_{j=1}^{r}Ext\text{-}\int_{|k_1|\leq\kappa}d^{\#3}k_1 \cdots Ext\text{-}\int_{|k_n|\leq\kappa}d^{\#3}k_j p_{rj}\left(\mu_1,\dots,\mu_j\right)\times$$

$$\times \left(Ext\text{-}\sum_{\text{part.of }(1,j)} \left\| R_{J_1} V_{\varkappa}^{I_1} R_{J_2} \cdots R_i^{1/2} V_{\varkappa}^{I_i} R_{J_{i+1}}^{1/2} a_{I_{i+1}} \psi \right\|_{\#}^{2} \right). \quad (4.10)$$

Let us consider a typical factor $R_{J_l} V_{\varkappa}^{I_l} R_{J_{l+1}}$, regarded as a function of the variables kil >***, kit where i, E I, , v = l,..., ‘+I t. Because of the momentum cut-off, the estimates (3.16) and (3.23) hold:

$$\left\| R_{J_l}^{1/2} V_{\varkappa}^{I_l} R_{J_{l+1}}^{1/2} \right\|_{\#} \leq \text{const.}\times$$

Note that when t 3 2, (Tc,K(f,,))l 1 is a multiple of the identity. Therefore, from (4.10) and (3.19),

$$A_{r+1,,\varkappa,\kappa} \leq \quad (4.11)$$

where we have set

Lemma 4.3 Let $j \in {}^{*}\mathbb{N}$ be a positive integer. Then there are positive constants b and c_κ, where c_κ depends on κ such that

$$\left\| \left(M_{0,\varkappa,\kappa}^{0k} \right)^{j} \psi \right\|_{\#} \leq c_\kappa \left\| \left(H_{0,\varkappa} + b \right)^{nj} \psi \right\|_{\#}, \ k = 1,2,3. \quad (4.15)$$

Here $2n$ is the order of the interaction.

Theorem 4.4 Let $j \in {}^{*}\mathbb{N}$ be a positive integer. Then $\left(M_{0,\varkappa,\kappa}^{0k} \right)^{j}$ is essentially self-#-adjoint on D.

Theorem 4.5 Let $\tau > 0$ and $r \in {}^{*}\mathbb{N}$ be a positive integer. Then there are constants a and b independent of κ such that

$$\left\| H_{0,\varkappa}^{1/2} N_{\varkappa,-\tau}^{(r-1)/2} \right\|_{\#} \leq a \left\| \left(M_{0,\varkappa,\kappa}^{0k} + b \right)^{\frac{r}{2}} \psi \right\|_{\#} \quad (4.10)$$

for all $\psi \in D\left(\left(M_{0,\varkappa,\kappa}^{0k} + b \right)^{\frac{r}{2}} \right)$.

${}^{*}\mathbb{R}_{\boldsymbol{c}}^{\#3}$ ${}^{*}\mathbb{R}_{\boldsymbol{c}+,\mathbf{fin}}^{\#}$ $\varkappa, \kappa$ ${}^{*}\mathbb{R}_{c,\infty}^{\#}$, ${}^{*}\mathbb{N}_\infty$ ${}^{*}\mathbb{R}_{\boldsymbol{c}}^{\#}$ $\Theta(k, \varkappa)$ (2.10)

5. Essential self-#-adjointness of $M_{\varkappa,g}^{0k}$

In the previous two sections we established a number of properties of the ultraviolet cut-off lorentzian $M_{\varkappa,\kappa}^{0k}, k = 1,2,3$ by methods that depended on $\kappa < \varkappa$ being hyperfinite. Now we take the #-limit $\kappa \to_{\#} \varkappa$ and find that many of the properties of $M_{\varkappa,\kappa}^{0k}$ transfer to the limiting operators $M_{0,\varkappa}^{0k}, k = 1,2,3$. As the next lemma states, $M_{\varkappa,\kappa}^{0k}, k = 1,2,3$ #-converges to $M_{\varkappa}^{0k}, k = 1,2,3$ on the #-dense domain

$$D_n = D\left(H_{0,\varkappa} \right) \cap D\left(N_{\varkappa}^{n} \right), n \in {}^{*}\mathbb{N}. \quad (5.1)$$

Note that #-convergence in this sense is not strong enough to control the #-limiting operator and in Theorem 5.3 we prove that the resolvents $R_{\varkappa,\kappa}^{(k)}(z) = \left(M_{\varkappa,\kappa}^{0k} - z\right)^{-1}, k = 1,2,3$ #-converge in #-norm. From this it follows that the operators $M_{\varkappa}^{0k}, k = 1,2,3$ are essentially self-#-adjoint on D.

Lemma 5.1 Let $\psi \in D_n$, then $M_{\varkappa,\kappa}^{0k}\psi \to_{\#} M_{0,\varkappa}^{0k}\psi, k = 1,2,3$ as $\kappa \to_{\#} \varkappa$.

Proof We write now $M_{\varkappa,\kappa,g}^{0k} = H_{0,\varkappa,\kappa} + T_{0,\varkappa,\kappa}\left(x_k g_0^{(k)}\right) + T_{I,\varkappa,\kappa}(x_k g_1), k = 1,2,3$ of the form

$$M_{\varkappa,\kappa}^{0k} = H_{0,\varkappa,\kappa} + T_{0,\varkappa,\kappa}(f_{0,k}) + T_{I,\varkappa,\kappa}(f_1), k = 1,2,3.$$

By the estimates (2.15), (2.16), and (4.26), $T_{0,\varkappa,\kappa}(f_{0,k})$ and $T_{I,\varkappa,\kappa}$ are defined on D_n, for $\kappa \leq \varkappa$. In fact, precisely these estimates prove #-convergence. For consider the difference

$$A_{\varkappa,\kappa} = T_{I,\varkappa}(f_1) - T_{I,\varkappa,\kappa}(f_1).$$

$A_{\varkappa,\kappa}$ can be written as a sum of Wick monomials whose kernels are the tails of $L_2^{\#}$ kernels. Therefore, by (2.13), $\|A_{\varkappa,\kappa}(N_{\varkappa} + I)^{-n}\|_{\#}$ bounded by the $L_2^{\#}$-#-norms of these tails which go to zero as $\kappa \to_{\#} \varkappa$. Since a similar argument can be made for $T_{0,\varkappa,\kappa}^{(2)}(f)$ it follows that on D_n

$$T_{0,\varkappa,\kappa}^{(2)} + T_{I,\varkappa,\kappa} \to_{\#} T_{0,\varkappa}^{(2)} + T_{I,\varkappa}. \tag{5.2}$$

The strong #-convergence of the differences

$$B_{\varkappa,\kappa}^{(k)} = T_{0,\varkappa}(f_{0,k}) - T_{0,\varkappa,\kappa}(f_{0,k}), k = 1,2,3$$

to zero on $D(H_{0,\varkappa})$ does not follow from a corresponding statement of #-norm #-convergence, since

$$\left\|B_{\varkappa,\kappa}^{(k)}\left(H_{0,\varkappa} + I\right)^{-1}\right\|_{\#} \nrightarrow_{\#} 0 \tag{5.3}$$

as $\kappa \to_{\#} \varkappa$. However, by (2.15) $\left\|B_{\varkappa,\kappa}^{(k)}\left(H_{0,\varkappa} + I\right)^{-1}\right\|_{\#}$ is uniformly bounded in κ. It is thus sufficient to show that $B_{\varkappa,\kappa}^{(k)}\psi_r \to_{\#} 0$ for $r \in {}^*\mathbb{N}$ particle vector $\psi_r = \psi(p_1, \dots, p_r) \in D$. By (2.8) one obtains

$$\left(B_{\varkappa,\kappa}^{(k)}\,\psi_r\right)(p_1, \dots, p_1) = Ext\text{-}\sum_{j=1}^{r} Ext\text{-}\int d^{\#3}k w_{\varkappa,\kappa}(k, p_j)\psi(p_1, \dots, p_{j-1}, k, p_{j+1}, \dots, p_r), \tag{5.4}$$

where

$$w_{\varkappa,\kappa}(k,p) = t^{(1)}(k,p)\left(\Theta(k\,,\varkappa)\Theta(p\,,\varkappa) - \Theta(k\,,\kappa)\Theta(p\,,\kappa)\right), \tag{5.5}$$

where $\Theta(k\,,\kappa)$ is defined by (2.10) with $\varkappa = \kappa$. Therefore,

$$\left|B_{\varkappa,\kappa}^{(k)}\psi\right| \leq 2Ext\text{-}\sum_{j=1}^{r} Ext\text{-}\int_{|k|>\kappa} d^{\#3}k t^{(1)}(k, p_j)\psi(p_1, \dots, p_{j-1}, k, p_{j+1}, \dots, p_r), \tag{5.6}$$

where by (2.15) the right side is an $L_2^{\#}$ function in variables $(p_1, \dots, p_r)$ whose #-norm is bounded by const. $\left\|\left(H_{0,\varkappa} + I\right)^{-1}\psi_r\right\|_{\#}$. Moreover, as $\kappa \to_{\#} \varkappa$, $\left(B_{\varkappa,\kappa}^{(k)}\,\psi_r\right)(p_1, \dots, p_r) \to_{\#} 0$ pointwise so that by the dominated #-convergence theorem $\left\|B_{\varkappa,\kappa}^{(k)}\,\psi_r\right\|_{\#} \to_{\#} 0$. For the proof of resolvent #- convergence we require a #-norm #-convergent statement for $T^{(1)}_{0,\varkappa,\kappa}(f_{0,k})$. The failure in (5.3) is to be expected, for, roughly speaking; we can regard $T^{(1)}_{0,\varkappa,\kappa}(f_{0,k})$ as $H_{0,\varkappa,\kappa}$ and obviously $C_{\varkappa,\kappa} = (H_{0,\varkappa} - H_{0,\varkappa,\kappa})(H_{0,\varkappa} + I)^{-1}$ does not #-converge to zero in #-norm. However, this argument indicates that $\left\| B^{(k)}_{\varkappa,\kappa}(H_{0,\varkappa} + I)^{-\tau} \right\|_{\#} \to_{\#} 0$ for $\tau > 1$.

Lemma 5.2 Let $i, j \in {}^*\mathbb{N}$ be nonnegative integers, and $f \in C_0^{*\infty}({}^*\mathbb{R}^{\#3n}_{c,\mathrm{fin}+}, {}^*\mathbb{R}^{\#}_{c,\mathrm{fin}})$.

(1) For $i + j > 2$,

$$\left\| (H_{0,\varkappa} + I)^{-i/2} \left(T^{(1)}_{0,\varkappa}(f) - T^{(1)}_{0,\varkappa,\kappa}(f) \right) (H_{0,\varkappa} + I)^{-j/2} \right\|_{\#} \to_{\#} 0 \ \text{ as } \kappa \to_{\#} \varkappa \tag{5.7}$$

(2) For $i + j \geq 2$,

$$\left\| (H_{0,\varkappa} + I)^{-\boldsymbol{i/2}} \left(T^{(2)}_{0,\varkappa}(f) - T^{(2)}_{0,\varkappa,\kappa}(f) \right) (H_{0,\varkappa} + I)^{-\boldsymbol{j/2}} \right\|_{\#} \to_{\#} 0 \ \text{ as } \kappa \to_{\#} \varkappa \tag{5.8}$$

(3) For $i + j \geq 2n$,

$$\left\| (H_{0,\varkappa} + I)^{-i/2} \left(T_{I,\varkappa}(f) - T_{I,\varkappa,\kappa}(f) \right) (H_{0,\varkappa} + I)^{-j/2} \right\|_{\#} \to_{\#} 0 \ \text{ as } \kappa \to_{\#} \varkappa \tag{5.9}$$

Proof Equation (5.7) is a consequence of estimates developed in [8] for Wick monomials with one creating and one annihilating leg. These estimates involve $L_1^{\#}$ - $L^{\#}_{*\infty}$ #-norms on the kernels such that

$$\|w\|_{\#1,r} =\approx \text{-essup}_{[k]} \left([\mu(k)]^{-\tau} \left(Ext\text{-}\int_{|p| \leq \varkappa} |w(k,p)|\, d^{\#3}p \right) \right). \tag{5.10}$$

As an example of (5.7), we consider the case $i = 1$ and $j = 2$. As in (5.4),

$$B_{\varkappa,\kappa} = T^{(1)}_{0,\varkappa}(f) - T^{(1)}_{0,\varkappa,\kappa}(f) = Ext\text{-}\int w_{\varkappa,\kappa}(k,p) a^*(k)\, a(p) d^{\#3}k d^{\#3}p.$$

We see that for r particle vector $\psi_r = \psi(p_1, \dots, p_r)$ the inequality holds

$$\left| B_{\varkappa,\kappa}(H_{0,\varkappa} + I)^{-\frac{1}{2}} \psi(p_1, \dots, p_r) \right| \leq$$

$$Ext\text{-}\left\{ \sum_{j=1}^{r} Ext\text{-}\int d^{\#3}k \frac{|w_{\varkappa,\kappa}(k,p_j)|}{[\mu(p_j)]^{1/2}} |\psi(p_1, \dots, p_{j-1}, k, p_{j+1}, \dots, p_r)| \right\}.$$

Therefore $\left\| B_{\varkappa,\kappa}(H_{0,\varkappa} + I)^{-\boldsymbol{1/2}} \psi_r \right\|_{\#}$ is bounded by the #-norm of

$$A_{\varkappa,\kappa}|\psi_r| = Ext\text{-}\int |w_{\varkappa,\kappa}(k,p)| [\mu(p)]^{-1/2} \Theta(p, \varkappa)\, a^*(k) a(p) d^{\#3}k d^{\#3}p |\psi_r|$$

and

$$\left\| (H_{0,\varkappa} + I)^{-\frac{1}{2}} B_{\varkappa,\kappa} (H_{0,\varkappa} + I)^{-1} \right\|_{\#} \leq \left\| (H_{0,\varkappa} + I)^{-\frac{1}{2}} A_{\varkappa,\kappa} (H_{0,\varkappa} + I)^{-\frac{1}{2}} \right\|_{\#} \leq$$

$$\leq \left\| w_{\varkappa,\kappa}(k,p) [\mu(p)]^{-1/2} \right\|_{\#1,1}.$$

by **[l, Lemma 3.1.11]**. According to the definition (5.10) by (5.5) and (2.9) we obtain

$$\left\|w_{\varkappa,\kappa}(k,p)[\mu(p)]^{-1/2}\right\|_{\#1,1} = \sup_k\left\{[\mu(p)]^{-1}Ext\text{-}\int\left|w_{\varkappa,\kappa}(k,p)[\mu(p)]^{-1/2}\right| d^{\#3}p\right\}$$

$$\leq \text{const.}\times$$

$$\left(\approx \text{-esssup}_k\left\{[\mu(k)]^{-\frac{1}{2}}Ext\text{-}\int\left|Ext\text{-}\hat{f}(k-p)\right|\left(\Theta(k,\varkappa)\Theta(p,\varkappa)-\Theta(k,\kappa)\Theta(p,\kappa)\right)d^{\#3}p\right\}\right)$$

$$= \delta(\varkappa,\kappa) \to_{\#} 0 \ \text{ as } \ \kappa \to_{\#} \varkappa. \tag{5.11}$$

Theorem 5.3 There is a semibounded self-#-adjoint operator $\boldsymbol{T_{\varkappa}}$ such that for $\boldsymbol{z}$ sufficiently negative

$$\left\|\left(\left(M^{0k}_{\varkappa,\kappa}-z\right)^{-1}\right)-(T_{\varkappa}-z)^{-1}\right\|_{\#} \to_{\#} 0 \quad \text{as } \kappa \to_{\#} \varkappa. \tag{5.12}$$

Proof We first establish the #-norm #-convergence of the $2n$-th powers $[R_\kappa(-b)]^{2n}$ of the resolvents for all b sufficiently large. Then the #-norm #-convergence of $R_\kappa(-b)$ follows by taking $2n$-th roots and applying the generalized **Stone-Weierstrass Theorem** []. Let $\kappa \leq \varkappa$ be two values of the ultraviolet cut-off. We use the following formula

(5.13)

The differences $M^{0k}_{\varkappa,\varkappa} - M^{0k}_{\varkappa,\kappa}, k = 1,2,3$ contain of three terms

By (**4.22**) we get

where the constant is independent of κ. Thus by (5.8) and (5.9) when $j \ = \ 2$ or 3,

As for $B^{(1)}$, at least one of i or $2n+1\ -i$ is greater than n. Hence by (**4.24**) and (**3.19**),

by (5.7). This establishes the #-convergence of RF. Let $R_{\varkappa}(z) \ = \#\text{-}\lim_{\kappa\to_{\#}\varkappa}R_\kappa(z)$. As a #-limit of resolvents, $R_\kappa(z)$ is itself the resolvent of an operator if and only if the null space N(R(z)) = 0 for some z [l 1, p. 4281. But Kleinstein has observed [12] that this is a direct consequence of Lemma 5.1: Suppose that YE N(R(--b)) where 6 is sufficiently large so that RK(- b) #-converges. Take θ arbitrary in D_n . Then

Theorem 5.4 $M^{0k}_{\varkappa}, k = 1,2,3$ are essentially self-#-adjoint on D.
Proof From the strong #-convergence of $M^{0k}_{\varkappa,\kappa}$ to $M^{0k}_{\varkappa}$ on D_n it follows by a simple argument that

$$M^{0k}_{\varkappa} \upharpoonright D_n \subset T_{\varkappa}. \tag{5.14}$$

Note that by the independence of κ- cut-off, the estimate (4.2) transfers to $T_{\varkappa}$, i.e.,

$$H_{0,\varkappa}^{2}+N_{\varkappa}^{2n}\leq a\left(M_{0,\varkappa,\kappa}^{0k}+b\right)^{2n} \tag{5.15}$$

and therefore $C=D(T_{\varkappa}^{2n})\subset D_{n}$, and from (5.14) one obtains $T_{\varkappa}\upharpoonright C\subset M_{\varkappa}^{0k}\upharpoonright D_{n}$. Now the domain C is a #-core for $T_{\varkappa}$, hence

$$T_{\varkappa}=\#\text{-}\overline{T_{\varkappa}\upharpoonright C}\subset\#\text{-}\overline{M_{\varkappa}^{0k}\upharpoonright D_{n}}$$

a symmetric extension of a self-#-adjoint operator and therefore we conclude that

$$T_{\varkappa}=\#\text{-}\overline{M_{\varkappa}^{0k}\upharpoonright D_{n}}.$$

Essential self-#-adjointness of $M_{\varkappa}^{0k},k=1,2,3$ on the domain D follows from self-#-adjointness on the domain D_{n} by a standard argument.

Corollary 5.5 For suitable constants a,b,c and $k=1,2,3$

$$H_{\varkappa}\leq a\left(M_{\varkappa}^{0k}+b\right), \tag{5.16}$$

$$H_{\varkappa}^{2}\leq c\left(H_{0,\varkappa}^{2}+N_{\varkappa}^{2n}+I\right)\leq a\left(M_{\varkappa}^{0k}+b\right)^{2n}. \tag{5.17}$$

The same inequalities hold with the roles of $H_{\varkappa}$ and $M_{\varkappa}^{0k}$ interchanged so that

$$D\left((H_{\varkappa}+b)^{1/2}\right)=D\left(\left(M_{\varkappa}^{0k}+b\right)^{1/2}\right), \tag{5.18}$$

$$D(H_{\varkappa}^{n})\subset D\left(M_{\varkappa}^{0k}\right), \tag{5.19}$$

$$D\left(\left(M_{\varkappa}^{0k}\right)^{n}\right)\subset D(H_{\varkappa}). \tag{5.20}$$

Proof Since D is a #-core for $M_{\varkappa}^{0k},k=1,2,3$, it is a #-core for $\left(M_{\varkappa}^{0k}+b\right)^{1/2}$ and (5.16) follows from closing (2.2). (5.17) is just a restatement of (5.15). Since $H_{\varkappa}$ is a special case of $M_{\varkappa}^{0k}$ obtained by setting, $g_{0}^{(k)}(x)=0$, it is clear that the higher order estimates (5.15) hold for $T_{\varkappa}=H_{\varkappa}$; hence the roles of $H_{\varkappa}$ and $M_{\varkappa}^{0k},k=1,2,3$ can be interchanged in (5.16) and (5.17).

6. Lorentz covariance

According to the discussion in Section 1 this amounts to showing that if $I^{3}=[a,b]^{3}\subset{}^{*}\mathbb{R}_{\boldsymbol{c},\text{fin}+}^{\#\mathbf{3}}$ and if f is a $C_{0}^{*\infty}\left({}^{*}\mathbb{R}_{c,\text{fin}+}^{\#3},{}^{*}\mathbb{R}_{\text{c,fin}}^{\#}\right)$ function with $\operatorname{supp}(f)\cup\operatorname{supp}\left(f_{\Lambda_{\beta}}\right)$, then

$$\left[Ext\text{-}\exp\left(iM_{\varkappa}^{0k}\beta\right)\right]\varphi_{\varkappa}(f)\left[Ext\text{-}\exp\left(-iM_{\varkappa}^{0k}\beta\right)\right]=\varphi_{\varkappa}\left(f_{\Lambda_{\beta}}\right). \tag{6.1}$$

Notice that (6.1) is operator equality**,** since for ${}^{*}\mathbb{R}_{\text{c,fin}}^{\#}$ valued function f, $\varphi_{\varkappa}(f)$ is a self-#-adjoint operator whose domain includes $D\left(\left(M_{\varkappa}^{ik}+b\right)^{1/2}\right)$. In addition, we prove on domain $D\left(\left(M_{\varkappa}^{ik}+b\right)^{1/2}\right)\times D\left(\left(M_{\varkappa}^{ik}+b\right)^{1/2}\right)$ that

$$\left[Ext\text{-}\exp\left(iM_{\varkappa}^{ik}\beta\right)\right]\varphi_{\varkappa}(x,t)\left[Ext\text{-}\exp\left(-iM_{\varkappa}^{ik}\beta\right)\right]=\varphi_{\varkappa}\left(\Lambda_{\beta}(x,t)\right). \tag{6.2}$$

Here (x,t) and $\Lambda_\beta(x,t)$ are in B_{I^3} , and the forms in (6.2) are #-continuous in x and t by the first-order estimate **(5.16)** and **Lemma 3.2.1 of [4]**.

Notice that the main part in the proof of (6.1) is to verify the commutation relation (1.15) for $f \in C_0^{*\infty}(O_{I^3}, {}^*\mathbb{R}^\#_{c,\mathrm{fin}})$ and g a cut-off function for the region O_{I^3} . For convenience, we assume that a function f with support contained in the region O_ε defined by

$$O_\varepsilon = \{(x_1,x_2,x_3,t) | a+\varepsilon+|t| < x_k < b-\varepsilon-|t|, k=1,2,3; |t|<\varepsilon\}, \qquad (6.3)$$

and where $\varepsilon > 0$ is some small number. This represents no loss of generality since any f in $C_0^{*\infty}(O_{I^3}, {}^*\mathbb{R}^\#_{c,\mathrm{fin}})$ can be presented as a sum of such f. It follows from this assumption that if $|s| < \varepsilon$, then external integral

$$[Ext\text{-}\exp(iH_\varkappa(t+s))]\left\{Ext\text{-}\int_{^*\mathbb{R}_c^{\#3}} \varphi_\varkappa(x) f(x,t) d^{\#3}x\right\}[Ext\text{-}\exp(-iH_\varkappa(t+s))] \qquad (6.4)$$

is related to a non-Archimedean von Neumann algebra $\mathfrak{R}(I^3)$ generated by the set

$$\left\{Ext\text{-}\exp(i\varphi_\varkappa(h_1)) + Ext\text{-}\exp(i\pi_\varkappa(h_2))\} | h_i \in C_0^{*\infty}({}^*\mathbb{R}^{\#3}_{c,\mathrm{fin}+}, {}^*\mathbb{R}^\#_{c,\mathrm{fin}}), \mathrm{supp}(h_i) \subset I^3, i=1,2\right\}.$$

The main parts of the proof are as follows:
Part1. For $\psi \in D(H_\varkappa^{n+3})$ we define

$$F_{ik}(t) = \langle \psi, [iM_\varkappa^{ik}(t), \varphi_\varkappa(f)]\psi\rangle_\# \qquad (6.5)$$

where $M_\varkappa^{ik}(t) = [Ext\text{-}\exp(-itH_\varkappa)]M_\varkappa^{ik}[Ext\text{-}\exp(itH_\varkappa)]$. Note that $F_{ik}(t)$ is well-defined and three times #-continuously #-differentiable by **(5.19) and the [4]:**

$$\left\|(H_\varkappa+b)^{j/2}\varphi_\varkappa(f)(H_\varkappa+b)^{-(j+1)/2}\right\|_\# < {}^*\infty, j = 0,1,2,\ldots. \qquad (6.6)$$

for $j = 0,1,2,\ldots$. Obviously one obtains,

$$\frac{d^\# F_{ik}(t)}{d^\# t} = \langle \psi, \left[[H_\varkappa, M_\varkappa^{ik}(t)], \varphi_\varkappa(f)\right]\psi\rangle_\#, \qquad (6.7)$$

$$\frac{d^{\#2} F_{ik}(t)}{d^\# t^2} = -i\langle \psi, \left[\left[H_\varkappa, [H_\varkappa, M_\varkappa^{ik}(t)]\right], \varphi_\varkappa(f)\right]\psi\rangle_\#. \qquad (6.8)$$

Part2. The commutators in (6.7)-(6.8) can be evaluated. On $D_n^\# \times D_n^\#$ one obtains, in the sense of bilinear forms,

$$[iH_\varkappa, M_\varkappa^{ik}] = P_\varkappa^k + Ext\text{-}\int_{^*\mathbb{R}_c^{\#3}} 2n{:}\varphi_\varkappa^{2n-1}(x)\pi_\varkappa(x){:}\, g_1(x)\left(x_k - \alpha - x_k g_0^{(k)}(x)\right) d^{\#3}x \qquad (6.9)$$

where $P_\varkappa^k, k = 1,2,3$ is a locally correct momentum operators

$$P_\varkappa^k \equiv P_\varkappa^k\left(\frac{d^\#}{d^\# x_k}\left(x_k g_0^{(k)}(x)\right)\right). \qquad (6.10)$$

By (2.5) the integral in (6.9) vanishes, and in analogy **to (1.6),**

$$[iH_\varkappa, M_\varkappa^{0k}] = P_\varkappa^k \qquad (6.11)$$

on $D\left(H_{\varkappa}^{n}\right)\times D\left(H_{\varkappa}^{n}\right)\subset D_{n}^{\#}\times D_{n}^{\#}$. Since the operators $P_{\varkappa}^{k}$ and $M_{\varkappa}^{ik}$ are defined on $D\left(H_{\varkappa}^{n}\right)$, extends to an operator equality on $D\left(H_{\varkappa}^{n+1}\right)$. Therefore, we obtain on the domain $D\left(H_{\varkappa}^{n+2}\right)\times D\left(H_{\varkappa}^{n+2}\right)$ that

$$\left[iH_{\varkappa},[iH_{\varkappa},M_{\varkappa}^{0k}]\right]=[iH_{\varkappa},P_{\varkappa}^{k}]=S^{k},$$

(6.12)

where

$$S^{k}=$$

$$T_{0,\varkappa}\left(\frac{d^{\#2}}{d^{\#}x_{k}^{2}}\left(x_{k}g_{0}^{(k)}(x)\right)\right)-m^{2}Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#3}}:\varphi_{\varkappa}^{2}(x):\frac{d^{\#2}}{d^{\#}x_{k}^{2}}\left(x_{k}g_{0}^{(k)}(x)\right)d^{\#3}x-T_{I,\varkappa}\left(\frac{d^{\#}(g_{1})}{d^{\#}x_{k}}\right).\quad(6.13)$$

Part3. Since $S^{k},k=1,2,3$ are local operators whose kernels vanishes on I^{3} we expect that $S^{k},k=1,2,3$ commutes with $\mathfrak{R}(I^{3})$. The exact statement is $[S^{k},\mathfrak{R}(I^{3})]=0,k=1,2,3$ on domain $D_{n}^{\#}\times D_{n}^{\#}$. It follows from (6.4) and (6.6) on domain $D_{n}^{\#}\times D_{n}^{\#}$ that

$$\left[S^{k},[Ext\text{-}\exp(isH_{\varkappa})\varphi_{\varkappa}(f)Ext\text{-}\exp(-isH_{\varkappa})]\right]=0\quad(6.14)$$

for $|s|<\varepsilon$ and $\operatorname{supp}(f)\subset O_{\varepsilon}$

Part4. The rigorous counterpart of the formal expansion **(1.11)** is to write $F_{ik}(t)$ in terms of its generalized Taylor **series [].** For some s, $|s|\leq|t|$

$$F_{ik}(t)=F_{ik}(0)+tF_{ik}^{\#\prime}(0)+\frac{t^{2}}{2}F_{ik}^{\#\prime}(s).\quad(6.15)$$

For $|t|\leq\varepsilon$ (6.15) on domain $D\left(H_{\varkappa}^{n+3}\right)\times D\left(H_{\varkappa}^{n+3}\right)$ reads

$$[iM_{\varkappa}^{0k}(t),\varphi_{\varkappa}(f)]=\left[iM_{\varkappa}^{ik},\varphi_{\varkappa}(f)\right]-i[iP_{\varkappa}^{k},\varphi_{\varkappa}(f)].$$

(6.16)

Part5. The commutators on the right of (6.16) can be evaluated by passing to the sharp time fields,

$$\varphi_{\varkappa}(f_{s},t)=Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#3}}f(x,s)\,\varphi_{\varkappa}(x,t)d^{\#3}x.$$

where the subscript s indicates the time dependence of a function f. The result for $|t|\leq\varepsilon$ reads

$$[iM_{\varkappa}^{0k}(t),\varphi_{\varkappa}(f_{t},0)]=\pi_{\varkappa}(x_{k}f_{t},0)-t\varphi_{\varkappa}\left(\frac{\partial^{\#}f_{t}}{\partial^{\#}x_{k}},0\right)$$

on domain $D\left(H_{\varkappa}^{n+3}\right)\times D\left(H_{\varkappa}^{n+3}\right)$. That is, for $|t|\leq\varepsilon$ we get

$$[iM_{\varkappa}^{0k}(t),\varphi_{\varkappa}(f_{t},0)]=\pi_{\varkappa}(x_{k}f_{t},t)-\varphi_{\varkappa}\left(t\frac{\partial^{\#}f_{t}}{\partial^{\#}x_{k}},t\right).\quad(6.17)$$

Since $\operatorname{supp}(f)\subset O_{\varepsilon}$, we can integrate (6.17) with respect to t and thus on domain $D\left(H_{\varkappa}^{n+3}\right)\times D\left(H_{\varkappa}^{n+3}\right)$ we obtain

$$[iM_{\varkappa}^{0k}(t),\varphi_{\varkappa}(f_{t},0)]=\pi_{\varkappa}(x_{k}f,t)-\varphi_{\varkappa}\left(t\frac{\partial^{\#}f}{\partial^{\#}x_{k}},t\right)=-\varphi_{\varkappa}\left(x_{k}\frac{\partial^{\#}f}{\partial^{\#}t}+t\frac{\partial^{\#}f_{t}}{\partial^{\#}x_{k}}\right).\quad(6.18)$$

Part6. In order to deduce (6.1) from (6.18) we must show that the equality (6.18) holds on a domain of the form $D\left(\left(M_{\varkappa}^{0k}\right)^{j}\right)\times D\left(\left(M_{\varkappa}^{0k}\right)^{j}\right)$. Note that if $\psi\in D\left(\left(M_{\varkappa}^{0k}\right)^{j}\right)$, then $Ext\text{-}\exp\left(-iM_{\varkappa}^{0k}\beta\right)\psi\in D\left(\left(M_{\varkappa}^{0k}\right)^{j}\right)$ and

$$\mathcal{G}_k(x,t,\beta)=\langle Ext\text{-}\exp\left(-iM_{\varkappa}^{0k}\beta\right)\psi,\varphi_{\varkappa}(x,t)Ext\text{-}\exp\left(-iM_{\varkappa}^{0k}\beta\right)\rangle_{\#}$$

is a #-continuous function of x and t **[4, Lemma 3.2.11** with a distribution #-derivative in β,

$$\langle Ext\text{-}\exp\left(-iM_{\varkappa}^{0k}\beta\right)\psi,\left\{x_k\frac{\partial^{\#}\varphi_{\varkappa}(x,t)}{\partial^{\#}t}+t\frac{\partial^{\#}\varphi_{\varkappa}(x,t)}{\partial^{\#}x_k}\right\}Ext\text{-}\exp\left(-iM_{\varkappa}^{0k}\beta\right)\rangle_{\#}$$

by the equality (6.18).Thus $\mathcal{G}_k(x,t,\beta)$ satisfies the distribution differential equation in partial #-derivatives

$$\frac{\partial^{\#}\mathcal{G}_k(x,t,\beta)}{\partial^{\#}\beta}=x_k\frac{\partial^{\#}\mathcal{G}_k(x,t,\beta)}{\partial^{\#}t}+t\frac{\partial^{\#}\mathcal{G}_k(x,t,\beta)}{\partial^{\#}x_k}. \tag{6.19}$$

The distribution differential equation (6.19) has a unique solution with initial condition $\mathcal{G}_k(x,t,0)$,

$$\mathcal{G}_k(x,t,0)=\langle\psi,\varphi_{\varkappa}(x,t)\psi\rangle_{\#}.$$

This proves (6.2) on $D\left(\left(M_{\varkappa}^{0k}\right)^{j}\right)\times D\left(\left(M_{\varkappa}^{0k}\right)^{j}\right)$ and, by extension, on $D\left(\left(M_{\varkappa}^{0k}+b\right)^{1/2}\right)\times D\left(\left(M_{\varkappa}^{0k}+b\right)^{1/2}\right)$. Obviously the operator statement (6.1) is immediate. It remains only to prove

Lemma 6.1 Let $I^3\subset{}^*\mathbb{R}^{\#3}_{c,\text{fin}+}$, g satisfy (2.3)-(2.5), $\varepsilon>0$, and $f\in in\ C_0^{^*\infty}\left(O_\varepsilon,{}^*\mathbb{R}^{\#}_{c,fin}\right)$. Then, in the sense of bilinear forms

$$\left[iM_{\varkappa}^{0k}(t),\varphi_{\varkappa}(f)\right]=-\varphi_{\varkappa}\left(x_k\frac{\partial^{\#}f}{\partial^{\#}t}+t\frac{\partial^{\#}f}{\partial^{\#}x_k}\right) \tag{6.20}$$

on $D\left(H_{\varkappa}\right)\times D\left(H_{\varkappa}\right)$ or on $D\left(M_{\varkappa}^{0k}\right)\times D\left(M_{\varkappa}^{0k}\right)$.

Proof As we know that (6.20) holds on $D\left(H_{\varkappa}^{n+3}\right)\times D\left(H_{\varkappa}^{n+3}\right)$. Let $\psi\in D\left(H_{\varkappa}\right)$; since $D\left(H_{\varkappa}^{n+3}\right)$ is a #-core for $H_{\varkappa}$, there exists a hyper infinite sequence $\psi_l, l\in{}^*\mathbb{N}$ in $D\left(H_{\varkappa}^{n+3}\right)$ such that $\psi_l\to_{\#}\psi$ and $H_{\varkappa}\psi_l\to_{\#}H_{\varkappa}\psi$ as $l\to{}^*\infty$. By the first order estimate, we have for some constants a and b

$$\left\|\left(M_{\varkappa}^{0k}+a\right)^{1/2}\left(H_{\varkappa}+b\right)^{-1/2}\right\|_{\#}<{}^*\infty. \tag{6.21}$$

and by (6.6) we get

$$\left\|\varphi_{\varkappa}(u_k)\left(H_{\varkappa}+b\right)^{-1/2}\right\|_{\#}<{}^*\infty. \tag{6.22}$$

where $u_k=x_k\frac{\partial^{\#}f}{\partial^{\#}t}+t\frac{\partial^{\#}f}{\partial^{\#}x_k}$ is in $C_0^{^*\infty}\left({}^*\mathbb{R}^{\#4}_{c,\text{fin}+},{}^*\mathbb{R}^{\#}_{c,\text{fin}}\right)$. Therefore,

$$\left(M_{\varkappa}^{0k}+a\right)^{1/2}\psi_l\to_{\#}\left(M_{\varkappa}^{0k}+a\right)^{1/2}\psi \tag{6.23}$$

and

$$\varphi_{\varkappa}(u_k)\psi_l\to_{\#}\varphi_{\varkappa}(u_k)\psi \tag{6.24}$$

Moreover, by (6.6) we obtain

$$\left\|(H_\varkappa + b)^{1/2}\varphi_\varkappa(f)(H_\varkappa + b)^{-1/2}\right\|_\# < {}^*\infty. \tag{6.25}$$

From (6.21) and (6.25) one obtains $D(H_\varkappa) \subset D\left((H_\varkappa + b)^{1/2}\varphi_\varkappa(f)\right)$ and that

$$\left(M_\varkappa^{0k} + a\right)^{1/2}\varphi_\varkappa(f)\psi_l \to_\# \left(M_\varkappa^{0k} + a\right)^{1/2}\varphi_\varkappa(f)\psi. \tag{6.26}$$

Note that

$$\langle\psi_l, [iM_\varkappa^{0k}(t), \varphi_\varkappa(f)]\psi_l\rangle_\# = i\left\langle\left(M_\varkappa^{0k} + a\right)^{1/2}\psi_l, \left(M_\varkappa^{0k} + a\right)^{1/2}\varphi_\varkappa(f)\psi_l\right\rangle_\# -$$

$$-i\left\langle\left(M_\varkappa^{0k} + a\right)^{1/2}\varphi_\varkappa(f)\psi_l, \left(M_\varkappa^{0k} + a\right)^{1/2}\psi_l\right\rangle_\#.$$

And therefore from (6.23) (6.24), and (6.26) we conclude that (6.20) extends by #-continuity to domain $D(H_\varkappa) \times D(H_\varkappa)$. By **(5.20)**, (6.20) is then exactly valid when restricted to $D\left(\left(M_\varkappa^{0k}\right)^n\right) \times D\left(\left(M_\varkappa^{0k}\right)^n\right)$. Finally, the extension to domain $D\left(M_\varkappa^{0k}\right) \times D\left(M_\varkappa^{0k}\right)$ follows directly as above from the inequality

$$\left\|\varphi_\varkappa(f)\left(M_\varkappa^{0k} + b\right)^{-1/2}\right\|_\# < {}^*\infty.$$

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