Proof of the Collatz Conjecture

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Abstract

This paper presents new insights towards proving the Collatz conjecture.

Introduction

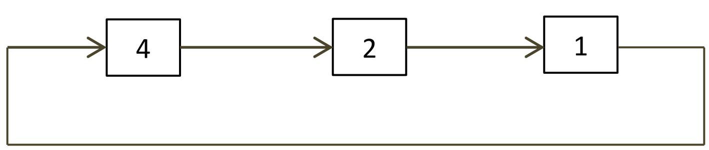
The Collatz function is defined as:

$$C(N) = \begin{cases} 3N + 1\,, & \text{if n is odd} \\ \dfrac{N}{2}, & \text{if n is even} \end{cases}$$

The Collatz conjecture :

Take any positive number *N*. If *N* is odd, multiply it by three and add one. If *N* is even, divide it by two. Repeatedly do this to form a sequence. The Collatz conjecture says that this sequence always ends in 1.

All sequences finally end in the closed loop:



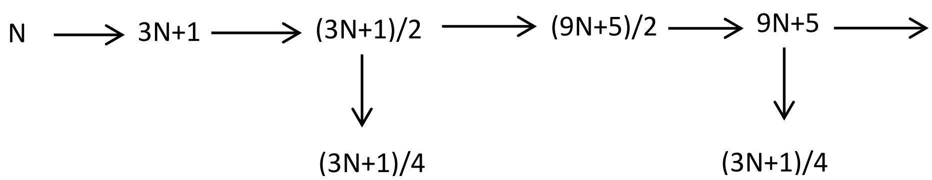
To prove the Collatz conjecture, one has to show that:

1. The sequence will not diverge to infinity.

2. The sequence will not enter some closed loop other than the 4- 2- 1- 4 loop.

Intuitive explanation on why the Collatz sequence cannot diverge to infinity

Let the starting number be *N*.



We can see that no successive Collatz operations can be odd operations, whereas multiple successive even operations can occur before an odd operation. This is because any even number can have 2^m as a factor. Therefore, after every odd Collatz operation there are one or more even operations. This shows that the Collatz number going to infinity is impossible and in fact, in the long run the sequence necessarily converges and descends.

As the starting number is made larger and larger, the probability that many more even operations occur before an odd operation increases. This leads to the conclusion that the probability that the next number being greater than the current number during any Collatz operation necessarily approaches zero as the starting number approaches infinity.

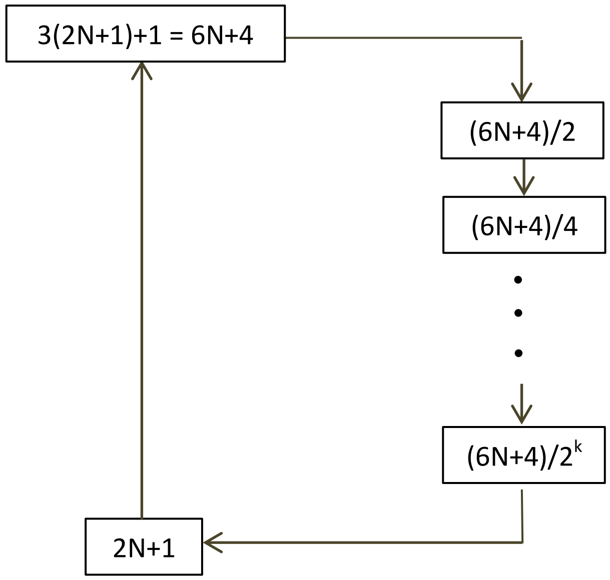
$$\lim_{N\to\infty}\left(Probability\ that\ C(N+1) > C(N)\right) = 0$$

where N is the starting number or any number in the sequence.

Non- existence of closed loop other than 4-2-1-4 loop

Next we present a new approach to disprove non-existence of any other closed loop other than the 4-2-1-4 loop as follows.

We start with an odd integer, *2N + 1*, where *N* is an integer greater than or equal to zero. Since it is odd, we multiply it by 3 and add 1. To form a loop, successive Collatz operations (divide by 2) on the result must give the original integer, i.e.



$$3\,(2N+1)+1 = (2N+1)\,2^{k}\ , \quad k \text{ is positive integer}$$

$$\Longrightarrow\ 6N+4 = (2N+1)\,2^{k}$$

$$\Longrightarrow \frac{6N+4}{2N+1} = 2^{k}$$

$$\Longrightarrow \frac{2(\,3N+2\,)}{2N+1} = 2^{k}$$

$$\Longrightarrow \frac{(\,3N+2\,)}{2N+1} = 2^{k-1}$$

$$\Longrightarrow \frac{(2N+1\,)+(N+1)}{2N+1} = 2^{k-1}$$

$$\Longrightarrow 1+\frac{N+1}{2N+1} = 2^{k-1}$$

Since the right hand side is always a positive integer that is a power of two, the left hand side must also be an integer (not a fraction) and a power of two for both sides to be equal. This is possible only if $N = 0$.

$$\Longrightarrow \; N = 0$$

Since $N = 0$, our starting odd number $2N+1$ will be:

$$\Longrightarrow \; 2N + 1 = 2 * 0 + 1 = 1$$

Thus we have proved that no other closed loop exists other than the 4-2-1-4 loop.

A rigorous proof of non-existence of closed loop other than the 4-2-1-4 loop

The above proof of non-existence of closed loop other than 4-2-1-4 is only meant to show the approach to be used and obviously is far from complete and therefore not rigorous. Next we present a new approach towards a rigorous proof that no other closed loop can exist.

We consider two cases: the Collatz sequence starting number is:

1) Odd
2) Even

<u>Odd starting number</u>

Re-writing the Collatz function:

$$\mathrm{C(N)} = \begin{cases} \mathrm{3N + 1}, & \text{if n is odd} \\ \dfrac{\mathrm{N}}{2}, & if\ n\ is\ even \end{cases}$$

as

$$\mathrm{C(N)} = \begin{cases} \mathrm{aN + 1}, & \text{if n is odd} \\ \mathrm{bN}, & if\ n\ is\ even \end{cases}$$

where

$$a = 3 \quad and \quad b = \frac{1}{2}$$

Case 1: Starting number N is odd

We assume that the initial Collatz number is and odd number that successively results in an odd number after multiplying it by three and adding one, and then dividing the result by two. Note that multiplying an odd number by three and adding one always gives an even number.

$$
\begin{tikzcd}
N \arrow[r] & aN+1 \arrow[r] & abN+b \arrow[r, "\text{odd}"] \arrow[d, "\text{even}"] & a^2bN+ab+1 \arrow[r] & a^2b^2N+ab^2+b \arrow[r, "\text{odd}"] \arrow[d, "\text{even}"] & {} \\
& & ab^2N+b^2 & & a^2b^3N+ab^3+b^2 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
{} \arrow[r] & a^3b^2N+a^2b^2+ab+1 \arrow[r] & a^3b^3N+a^2b^3+ab^2+b \arrow[r, "\text{odd}"] \arrow[d, "\text{even}"] & a^4b^3N+a^3b^3+a^2b^2+ab+1 \arrow[r] & {} \\
& & a^3b^4N+a^2b^4+ab^3+b^2 & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
{} \arrow[r] & a^4b^4N+a^3b^4+a^2b^3+ab^2+b \arrow[r, "\text{odd}"] \arrow[d, "\text{even}"] & a^5b^4N+a^4b^4+a^3b^3+a^2b^2+ab+1 \arrow[r] & {} \\
& a^4b^5N+a^3b^5+a^2b^4+ab^3+b^2 & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
{} \arrow[r] & a^5b^5N+a^4b^5+a^3b^4+a^2b^3+ab^2+b \arrow[r, "\text{odd}"] \arrow[d, "\text{even}"] & {} \\
& a^5b^6N+a^4b^6+a^3b^5+a^2b^4+ab^3+b^2 &
\end{tikzcd}
$$

From the above, the last Collatz number is:

$$a^5b^5N + \ a^4b^5 + a^3b^4 + a^2b^3 + ab^2 + \ b$$

We can generalize this as follows:

$$C(x) = \ a^xb^x\ N \ \ + \ \ a^{x-1}b^x + \ a^{x-2}b^{x-1} \ + \ a^{x-3}b^{x-2} + . \ . \ . + \ a^3b^4 + a^2b^3 + ab^2 + \ b$$

We can see that the part:

$$(\ a^{x-1}b^x + \ a^{x-2}b^{x-1} \ + \ a^{x-3}b^{x-2} + . \ . \ . + \ a^3b^4 + a^2b^3 + ab^2 \ )$$

is a geometric sequence with ratio $r = ab$ and first term ab^2 .

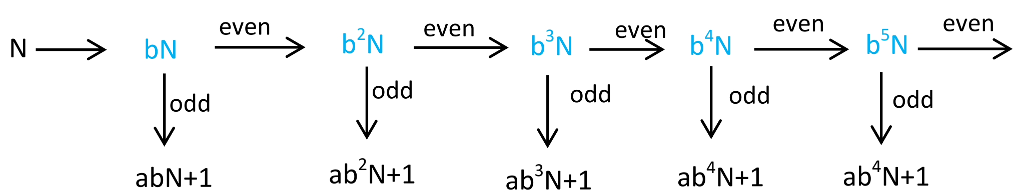
$$(\ a^{x-1}b^x + \ a^{x-2}b^{x-1} \ + \ a^{x-3}b^{x-2} + . \ . \ . + \ a^3b^4 + a^2b^3 + ab^2 \ ) = ab^2\frac{1-(ab\ )^{x-1}}{1-ab}$$

Therefore, the x^{th} number will be:

$$\boxed{a^xb^xN \ + \ ab^2\frac{1-(ab\ )^{x-1}}{1-ab} \ + \ b \qquad \text{............ (1)}}$$

Case 2: Starting number N is even

In this case we assume that the starting number is an even number that successively results in an even number after dividing the result by two.



The y^{th} number will be:

$$\boxed{C(N) = \ b^yN \ldots \ldots \ldots \ldots \ldots (2)}$$

We can reach any number along the Collatz sequence by combining equations (1) and (2).

To check if other loops exist, we shall consider two cases and develop separate general equations for each:

1. Collatz sequence starting with odd number and ending in odd number
2. Collatz sequence starting with even number and ending in even number

Collatz sequence starting with odd number and ending in odd number

Any number in the sequence can be expressed as:

$$C\,(N) =$$

$$(((\,(\,a^{x1}b^{x1}\,N\; + \;ab^2\frac{1-(ab\,)^{x1-1}}{1-ab}\; + \;b\,)\,b^{y1}\,a^{x2}b^{x2}\; + \;ab^2\frac{1-(ab\,)^{x2-1}}{1-ab}\; + \;b)b^{y2}\,a^{x3}b^{x3}$$

$$+\,ab^2\frac{1-(ab\,)^{x3-1}}{1-ab}\; + \;b\;)\;b^{y3}\,a^{x4}b^{x4}\; + \;ab^2\frac{1-(ab\,)^{x4-1}}{1-ab}\; + \;b\;)b^{y4}\,a^{x5}b^{x5}$$

$$+\;ab^2\frac{1-(ab\,)^{x5-1}}{1-ab}\; + \;b \qquad \dots\dots\dots\dots.(3)$$

This can be generalized as:

$$C(N) =$$

$$(((\dots\,(\,a^{x1}b^{x1}\,N\; + \;ab^2\frac{1-(ab\,)^{x1-1}}{1-ab}\; + \;b\,)\,b^{y1}\,a^{x2}b^{x2}\; + \;ab^2\frac{1-(ab\,)^{x2-1}}{1-ab}\; + \;b)b^{y2}\,a^{x3}b^{x3}$$

$$+\,ab^2\frac{1-(ab\,)^{x3-1}}{1-ab}\; + \;b\;)\;b^{y3}\,a^{x4}b^{x4}\; + \;ab^2\frac{1-(ab\,)^{x4-1}}{1-ab}\; + \;b\;)b^{y4}\,a^{x5}b^{x5}\; + \;.\;.\;.$$

$$.\;\;.\;\;.+b\,)b^{ym}\,a^{xn}b^{xn}\; + \;ab^2\frac{1-(ab\,)^{xn-1}}{1-ab}\; + \;b \qquad \dots\dots\dots\quad(4)$$

Rearranging equation (3)

$$C\,(N) = (ab)^{x1+x2+x3+x4+x5}\, b^{y1+y2+y3+y4}\;\; N \;\; +$$

$$ab^2\,\frac{1-(ab\,)^{x1-1}}{1-ab}\,(ab)^{x2+x3+x4+x5} b^{y1+y2+y3+y4} \;\; +$$

$$ab^2\,\frac{1-(ab\,)^{x2-1}}{1-ab}\,(ab)^{x3+x4+x5} b^{y2+y3+y4} \; + \;\; ab^2\,\frac{1-(ab\,)^{x3-1}}{1-ab}\,(ab)^{x4+x5} b^{y3+y4} \; +$$

$$ab^2\,\frac{1-(ab\,)^{x4-1}}{1-ab}\,(ab)^{x5} b^{y4} \;\; + \;\;\; ab^2\,\frac{1-(ab\,)^{x5-1}}{1-ab} \;\; +$$

$$b\,(ab\,)^{x2+x3+.x4+x5}\;\; b^{y1+y2+y3+y4} \;\; + \;\; b\,(ab\,)^{x3+x4+x5}\;\; b^{y2+y3+y4} \;\; +$$

$$b\,(ab\,)^{x4+x5}\, b^{y3+y4} \;\; + \;\; b\,(ab\,)^{x5}\;\, b^{y4} \;\; + \;\; b \quad \ldots\ldots\ldots \;\; (5)$$

This can be generalized as:

$$C\,(N\,) = (ab)^{x1+x2+\,.\,.\,.+xn}\, b^{y1+y2+\,.\,.\,.+ym}\;\; N \;\; +$$

$$ab^2\,\frac{1-(ab\,)^{x1-1}}{1-ab}\,(ab)^{x2+x3+\,.\,.\,.+xn} b^{y1+y2+\,.\,.\,.+ym} \;\; +$$

$$ab^2\,\frac{1-(ab\,)^{x2-1}}{1-ab}\,(ab)^{x3+x4+\,.\,.\,.\,.+xn} b^{y2+y3+.\,.\,.+ym} \;\; +$$

$$ab^2\,\frac{1-(ab\,)^{x3-1}}{1-ab}\,(ab)^{x4+x5+\,.\,.\,.+xn} b^{y3+y4+\,.\,.\,.+ym} \;\; + \;\; . \;\; . \;\; . \; +$$

$$ab^2\,\frac{1-(ab\,)^{(xn-1)-1}}{1-ab}\,(ab)^{xn} b^{ym} \;\; + \;\;\; ab^2\,\frac{1-(ab\,)^{xn-1}}{1-ab} \;\; +$$

$$b\,(ab\,)^{x2+x3+.\,.\,.+xn}\;\; b^{y1+y2+\,.\,.\,.+ym} + b\,(ab\,)^{x3+x4+\,.\,.\,.+xn}\;\; b^{y2+y3+\,.\,.\,.+ym} \;\; + \;\; . \; . \; . \; .$$

$$. \;\; . \;\; . \;\; + \;\; b\,(ab\,)^{xn}\;\; b^{ym} \;\; + \;\; b \quad \ldots\ldots\ldots \;\; (6)$$

Collatz sequence starting with even number and ending in even number

Any number in the sequence can be expressed as:

$$((((\, a^{x1}b^{x1}\, Nb^{y1} \;+\; ab^2\frac{1-(ab\,)^{x1-1}}{1-ab} \;+\; b)b^{y2}a^{x2}b^{x2} \;+\; ab^2\frac{1-(ab\,)^{x2-1}}{1-ab} \;+\; b\,)b^{y3}a^{x3}b^{x3} +$$

$$ab^2\frac{1-(ab\,)^{x3-1}}{1-ab} \;+\; b\;)\, b^{y4}a^{x4}b^{x4} + \; ab^2\frac{1-(ab\,)^{x4-1}}{1-ab} \;+\; b\,)\, b^{y5} \quad \ldots\ldots\ldots \;(7)$$

OR

$$(((\, a^{x1}b^{x1}\, Nb^{y1} \;+\; ab^2\frac{1-(ab\,)^{x1-1}}{1-ab} \;+\; b)b^{y2}a^{x2}b^{x2} \;+\; ab^2\frac{1-(ab\,)^{x2-1}}{1-ab} \;+\; b\,)b^{y3}a^{x3}b^{x3} +$$

$$ab^2\frac{1-(ab\,)^{x3-1}}{1-ab} \;+\; b\;)\, b^{y4}a^{x4}b^{x4} + \; ab^2\frac{1-(ab\,)^{x4-1}}{1-ab} \;+\; b \quad \ldots\ldots\ldots \;(8)$$

Equation (7) can be generalized as:

$$C(N\,) =$$

$$((..((\, a^{x1}b^{x1}\, Nb^{y1} + \; ab^2\frac{1-(ab\,)^{x1-1}}{1-ab} + b)b^{y2}a^{x2}b^{x2} \;+\; ab^2\frac{1-(ab\,)^{x2-1}}{1-ab} \;+\; b\,)b^{y3}a^{x3}b^{x3} +$$

$$ab^2\frac{1-(ab\,)^{x3-1}}{1-ab} \;+\; b\;)\, b^{y4}a^{x4}b^{x4} +.\,.\,.+ b)b^{ym-1}a^{xn}b^{xn} \;+\; ab^2\frac{1-(ab\,)^{xn-1}}{1-ab} \;+\; b\,)\, b^{ym} \quad \ldots\ldots.... \;(9)$$

Equation (7.1) can be generalized as:

$$C(N\,) =$$

$$((..((\, a^{x1}b^{x1}\, Nb^{y1} + \; ab^2\frac{1-(ab\,)^{x1-1}}{1-ab} + b)b^{y2}a^{x2}b^{x2} \;+\; ab^2\frac{1-(ab\,)^{x2-1}}{1-ab} \;+\; b\,)b^{y3}a^{x3}b^{x3} +$$

$$ab^2\frac{1-(ab\,)^{x3-1}}{1-ab} \;+\; b\;)\, b^{y4}a^{x4}b^{x4} +.\,.\,.+ b)b^{ym}a^{xn}b^{xn} \;+\; ab^2\frac{1-(ab\,)^{xn-1}}{1-ab} \;+\; b \quad \ldots\ldots.... \;(10)$$

Rearranging equation (7):

$$C\ (N) = \ (ab)^{x1+x2+x3+x4}\, b^{y1+y2+y3+y4+y5}\ \ N\ \ +$$

$$ab^2\,\frac{1-(ab\ )^{x1-1}}{1-ab}(ab)^{x2+x3+x4}b^{y2+y3+y4+y5}\ \ +$$

$$ab^2\,\frac{1-(ab\ )^{x2-1}}{1-ab}(ab)^{x3+x4}b^{y3+y4+y5}\ +\ \ ab^2\,\frac{1-(ab\ )^{x3-1}}{1-ab}(ab)^{x4}b^{y4+y5}\ \ +$$

$$ab^2\,\frac{1-(ab\ )^{x4-1}}{1-ab}b^{y5}\ \ +$$

$$b\ (ab\ )^{x2+x3+.x4}\ \ b^{y2+y3+y4+y5}\ \ +\ \ b\ (ab\ )^{x3+x4}\ \ b^{y3+y4+y5}\ \ +$$

$$b\ (ab\ )^{x4}\, b^{y4+y5}\ \ +\ \ bb^{y5}\qquad \dots\dots\dots\ \ (11)$$

This can be generalized as:

$$C\ (N) = \ (ab)^{x1+x2+...+xn}\, b^{y1+y2+..+ym}\ \ N\ \ +$$

$$ab^2\,\frac{1-(ab\ )^{x1-1}}{1-ab}(ab)^{x2+x3+..+xn}b^{y2+y3+..+ym}\ \ +$$

$$ab^2\,\frac{1-(ab\ )^{x2-1}}{1-ab}(ab)^{x3+x4+..+xn}b^{y3+y4+..+ym}\ +\ \ .\ \ .\ \ .$$

$$b\ (ab\ )^{xn}\, b^{ym-1+ym}\ \ +$$

$$ab^2\,\frac{1-(ab\ )^{xn-1}}{1-ab}b^{ym}\ \ +$$

$$b\ (ab\ )^{x2+x3+..+xn}\ \ b^{y2+y3+..+ym} +\ b\ (ab\ )^{x3+x4+..+xn}\ \ b^{y3+y4+..+ym}\ \ +\ \ .\ \ .\ \ .$$

$$+\, b\ (ab\ )^{xn}\, b^{ym-1+ym}\ \ +\ \ bb^{ym}\qquad \dots\dots\dots\ \ (12)$$

Test of closed loop

A closed loop is formed if :

$$C(N) = N$$

where C(N) is as expressed in equation (4), (9), or (10), or their rearranged forms and checking the above equality for each.

After substituting a = 3 and b = ½ in the above equations, a valid value of *N* cannot be obtained for any values of *x*'s and *y*'s except for those corresponding to the 1-4-2-1 loop, if there are no loops other than 1-4-2-1 loop.

Re-writing equation (6) below:

$$C\,(N) = (ab)^{x1+x2+\,.\,.\,.+xn}\, b^{y1+y2+\,.\,.\,.+ym}\; N \;+$$

$$ab^2 \frac{1-(ab)^{x1-1}}{1-ab} (ab)^{x2+x3+\,.\,.\,.+xn} b^{y1+y2+\,.\,.\,.+ym} \;+$$

$$ab^2 \frac{1-(ab)^{x2-1}}{1-ab} (ab)^{x3+x4+\,.\,.\,.\,.+xn} b^{y2+y3+.\,.\,.+ym} \;+$$

$$ab^2 \frac{1-(ab)^{x3-1}}{1-ab} (ab)^{x4+x5+\,.\,.\,.+xn} b^{y3+y4+\,.\,.\,.+ym} \;+\; .\;.\;.\;+$$

$$ab^2 \frac{1-(ab)^{(xn-1)-1}}{1-ab} (ab)^{xn} b^{ym} \;+\; ab^2 \frac{1-(ab)^{xn-1}}{1-ab} \;+$$

$$b\,(ab)^{x2+x3+.\,.\,.+xn}\; b^{y1+y2+\,.\,.\,.+ym} + b\,(ab)^{x3+x4+\,.\,.\,.+xn}\; b^{y2+y3+\,.\,.\,.+ym} \;+\; .\;.\;.\;.$$

$$.\;.\;.\;+\; b\,(ab)^{xn}\; b^{ym} \;+\; b$$

Rearranging:

$$C\,(N\,) = \,(ab)^{x1+x2+\,.\,.\,.+xn}\,b^{y1+y2+\,.\,.\,.+ym}\;\;N\;\;+$$

$$(ab^2\,\frac{1-(ab\,)^{x1-1}}{1-ab}+b)\;(ab)^{x2+x3+\,.\,.\,.+xn}b^{y1+y2+\;.\,.\,.+ym}\;\;+$$

$$(ab^2\,\frac{1-(ab\,)^{x2-1}}{1-ab}+b)\;(ab)^{x3+x4+\,.\,.\,.\,.+xn}b^{y2+y3+.\,.\,.+ym}\;+$$

$$(ab^2\,\frac{1-(ab\,)^{x3-1}}{1-ab}+b)\;(ab)^{x4+x5+\,.\,.\,.+xn}b^{y3+y4+\,.\,.\,.+ym}\;\;+\;\;.\;\;.\;\;.\;+$$

$$(ab^2\,\frac{1-(ab\,)^{(xn-1)-1}}{1-ab}+b)\;(ab)^{xn}b^{ym}\;\;+\;\;\;ab^2\,\frac{1-(ab\,)^{xn-1}}{1-ab}\;\;+\;\;b\qquad \ldots\ldots\ldots..\;\;(11)$$

Let us test it for $x_1 = y_1 = 1$, which is the case of the 1-4-2-1 loop:

$$C\,(N\,) = \,(ab)^1\,b^1\;\;N\;\;+\;\;2b$$

$$C(N) = N$$

$$\Rightarrow\quad (ab)^1\,b^1\;\;N\;\;+\;\;2b = N$$

$$\Rightarrow\quad (\frac{3}{2})^1\,(\frac{1}{2})^1\;\;N\;\;+\;\;2(\frac{1}{2}) = N$$

$$\Rightarrow\;\;\frac{3}{4}\,N\;\;+\;\;1 = N$$

$$\Rightarrow\;\;N = 4$$

Thus we have reduced the Collatz conjecture to the problem of whether C(N) = N in the above equations, for some positive integer N. We need to show that C(N) = N only for the 1-4-2-1 loop, i.e. if no other loops exist (which is the more likely case).

$$C(N) = N$$

Inserting C(N) from equation (11) :

$$N =$$

$$\frac{1 - \quad (ab)^{x1+x2+...+xn}\, b^{y1+y2+...+ym}}{(ab^2 \frac{1-(ab)^{x1-1}}{1-ab} + b)\, (ab)^{x2+x3+...+xn} b^{y1+y2+...+ym} + \,.\,.\,.+ \; (ab^2 \frac{1-(ab)^{(xn-1)-1}}{1-ab} + b)\, (ab)^{xn} b^{ym} \;+\; ab^2 \frac{1-(ab)^{xn-1}}{1-ab} \;+\; b}$$

We have thus reduced the problem to whether a positive integer exists that satisfies the above equation. It should be possible to deduce from the above equation that no positive integer N exists that satisfies the above equation. Note that the equation we have used here is for sequences that start with odd number and ‘end’ in odd number. We need to do the same for the two equations (7) and (8) corresponding to sequences that begin with even number and ‘end’ in even number.

Conclusion

Proving the Collatz conjecture requires proving that:

1. The sequence will not diverge to infinity
2. There are no other closed loops other than the 1-4-2-1 loop.

We have presented an intuitive proof of the first. For the second, we have developed equations for C(N) and reduced the Collatz conjecture to a problem of whether C(N) =N, for some positive integer N.

Thanks to Almighty God Jesus Christ and His Mother Our Lady Saint Virgin Mary

References