SPEED OF GRAVITY CAN BE DIFFERENT FROM THE SPEED OF LIGHT.

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Abstract. In Einstein's relativity each massless field should propagate with the speed of light, including the gravitational field. The 3D-brane universe model is an alternative theory of gravity. Unlike Einstein's relativity, it is not so restrictive. In the present paper we build the theory of electromagnetism in a curved 3D-brane universe where the speed of gravity c_{gr} can be different from the speed of electromagnetic waves c_{el}, which is the speed of light.

1. Introduction.

Within the paradigm of a 3D-brane universe the gravitational field is described by a time-dependent 3D metric with the components

$$g_{ij}=g_{ij}(t,x^1,x^2,x^3),\quad 1\leqslant i,j\leqslant 3. \tag{1.1}$$

Through t we denote the cosmological time. As it was shown in [1] (see also [2] and [3]), the three-dimensional metric (1.1) obeys the following differential equations:

$$\begin{gathered}\frac{\dot b_{ij}}{c_{\text{gr}}}-\sum_{k=1}^{3}\frac{\dot b^k_k}{c_{\text{gr}}}\,g_{ij}-\sum_{k=1}^{3}(b_{ki}\,b^k_j+b_{kj}\,b^k_i)-\frac{g_{ij}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_q\,b^q_k-\\-\frac{g_{ij}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q+\sum_{k=1}^{3}b^k_k\,b_{ij}+R_{ij}-\frac{R}{2}\,g_{ij}+\Lambda\,g_{ij}=\frac{8\,\pi\,\gamma}{c_{\text{gr}}^4}\,T_{ij},\end{gathered} \tag{1.2}$$

Here γ is Newton's gravitational constant (see [4]), Λ is the cosmological constant (see [5]), R_{ij} are the components of the three-dimensional Ricci tensor of the metric (1.1), R is the three-dimensional scalar curvature, and b_{ij} are given by the formula

$$b_{ij}=\frac{\dot g_{ij}}{2\,c_{\text{gr}}}. \tag{1.3}$$

Through c_{gr} in (1.2) and (1.3) we denote some constant that replaces the speed of light used in the standard relativity. It presents the speed of gravity that can be different from the speed of light. The right hand side of (1.2) is determined by the variational derivative of $\mathcal{L}_{\text{mat}}$, where $\mathcal{L}_{\text{mat}}$ is the Lagrangian of matter.

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The equations (1.2) were derived in [1] through 3D reduction of the standard Einstein's equations (see [6]). Later on the equations (1.2) were rederived within purely three-dimensional Lagrangian and Hamiltonian approaches (see [8], [9], and [10]). Here we use the following Lagrangian for the gravitational field:

$$\mathcal{L}_{\mathrm{gr}} = -\frac{c_{\mathrm{gr}}^4}{16\,\pi\,\gamma}\,(\rho + 2\,\Lambda), \tag{1.4}$$

where

$$\rho = \sum_{k=1}^{3}\sum_{q=1}^{3} b^k_k\,b^q_q - R - \sum_{k=1}^{3}\sum_{q=1}^{3} b^k_q\,b^q_k. \tag{1.5}$$

The coefficient in (1.4) is slightly different from that used in [8], [9], and [11]. This coefficient here is chosen in such a way that the value of $\mathcal{L}_{\mathrm{gr}}$ is measured in the units of the energy density. The coefficient in the right hand side of (1.2) is also slightly different from that of [8] since c_{gr} can be different from the speed of light.

In the standard four-dimensional relativistic electrodynamics the electromagnetic field is described by the four-dimensional vector potential (see § 9 in Chapter III of [12]). In this paper we return back to the three-dimensional vector potential $\mathbf{A}$ along with the scalar potential φ. The electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ here again are two separate vector fields given by the classical formulas:

$$\mathbf{E} = -\operatorname{grad}\varphi - \frac{1}{c_{\mathrm{el}}}\,\frac{\partial\mathbf{A}}{\partial t}, \qquad\qquad \mathbf{H} = \operatorname{rot}\mathbf{A}. \tag{1.6}$$

Note that c_{el} in (1.6) is the speed of light. The equations (1.6) here are understood in the covectorial form, i. e. the covariant components of $\mathbf{A}$ are used in them:

$$E_i = -\nabla_i\varphi - \frac{1}{c_{\mathrm{el}}}\,\frac{\partial A_i}{\partial t}, \qquad\qquad H^i = \sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\varepsilon^{ijk}}{\sqrt{\det g}}\,\nabla_j A_k. \tag{1.7}$$

The term ε^{ijk} in the right hand side of the second formula (1.7) is the Levi-Civita symbol (see [13] and § 43 of Chapter I in [14]). Choosing the scalar potential φ and the components of the covector $\mathbf{A}$ for dynamic variables of the electromagnetic field, we write the Lagrangian of the electromagnetic field as follows:

$$\mathcal{L}_{\mathrm{el}} = \frac{|\mathbf{E}|^2 - |\mathbf{H}|^2}{8\,\pi}. \tag{1.8}$$

The electromagnetic field plays the role of matter in the present paper. Therefore

$$\mathcal{L}_{\mathrm{mat}} = \mathcal{L}_{\mathrm{el}} \tag{1.9}$$

and the total Lagrangian of the theory below is

$$\mathcal{L} = \mathcal{L}_{\mathrm{gr}} + \mathcal{L}_{\mathrm{el}}. \tag{1.10}$$

The main goal of the present paper is to apply the general results of [8], [9] and [11] to this particular case given by the Lagrangian (1.10).

2. Legendre transformation and conjugate variables.

Typically Lagrangians are functions of dynamic variables and their time derivatives. In the case of the metric components g_{ij} in (1.1) their time derivatives are presented by the quantities b_{ij} in (1.3). Therefore we write

$$\mathcal{L}_{\text{gr}} = \mathcal{L}_{\text{gr}}(\mathbf{g}, \mathbf{b}). \tag{2.1}$$

The Lagrangian $\mathcal{L}_{\text{el}}$ in (1.8) does not depend on b_{ij}. But it depends on g_{ij} since

$$|\mathbf{E}|^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} g^{ij}\, E_i\, E_j, \qquad\qquad |\mathbf{H}|^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} g_{ij}\, H^i\, H^j. \tag{2.2}$$

Due to (1.6), (1.7), and (2.2), the Lagrangian $\mathcal{L}_{\text{el}}$ depends on $\dot{\mathbf{A}}$, but it does not depend on $\dot{\varphi}$. Therefore we can write the formulas similar to (2.1):

$$\mathcal{L}_{\text{el}} = \mathcal{L}_{\text{el}}(\mathbf{g}, \varphi, \mathbf{A}, \dot{\mathbf{A}}), \qquad\qquad \mathcal{L} = \mathcal{L}(\mathbf{g}, \mathbf{b}, \varphi, \mathbf{A}, \dot{\mathbf{A}}). \tag{2.3}$$

The Legendre transformation associated with the Lagrangian $\mathcal{L}(\mathbf{g}, \varphi, \mathbf{A}, \dot{\mathbf{A}})$ in (1.10) is given by the following formulas:

$$\beta^{ij} = \Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\mathbf{g},\varphi,\mathbf{A},\dot{\mathbf{A}}}, \qquad \alpha^i = \Bigl(\frac{\delta\mathcal{L}}{\delta \dot{A}^i}\Bigr)_{\mathbf{g},\mathbf{b},\varphi,\mathbf{A}}, \qquad \psi = \Bigl(\frac{\delta\mathcal{L}}{\delta\dot{\varphi}}\Bigr)_{\mathbf{g},\mathbf{b},\varphi,\mathbf{A},\dot{\mathbf{A}}}. \tag{2.4}$$

The quantities β^{ij}, α^i, and ψ are generalized momenta conjugate to b_{ij}, $\dot{A}^i$, and $\dot{\varphi}$ respectively. The last variational derivative in (2.4) is zero since according to (2.3) the Lagrangian $\mathcal{L}$ does not depend on $\dot{\varphi}$. Therefore

$$\psi = \Bigl(\frac{\delta\mathcal{L}}{\delta\dot{\varphi}}\Bigr)_{\mathbf{g},\mathbf{b},\varphi,\mathbf{A},\dot{\mathbf{A}}} = 0. \tag{2.5}$$

The other two variational derivatives in (2.4) are calculated explicitly:

$$\beta^{ij} = \Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\mathbf{g},\varphi,\mathbf{A},\dot{\mathbf{A}}} = \frac{c_{\text{gr}}^4}{8\pi\gamma}\Bigl(b^{ij} - \sum_{k=1}^{3} b^k_k\, g^{ij}\Bigr), \tag{2.6}$$

$$\alpha^i = \Bigl(\frac{\delta\mathcal{L}}{\delta \dot{A}_i}\Bigr)_{\mathbf{g},\mathbf{b},\varphi,\mathbf{A}} = -\frac{E^i}{4\,\pi\, c_{\text{el}}}. \tag{2.7}$$

Due to (2.5) the Legendre transformation (2.4) is partly degenerate and hence is not invertible as a whole.

3. Euler-Lagrange equations.

The Lagrangian (1.10) is used in order to write the following action integral:

$$S = \iint \mathcal{L}\,\sqrt{\det g}\; d^3x\, dt. \tag{3.1}$$

The stationary action principle (see [15]) applied to action integrals leads to Euler-Lagrange equations. In the case of the action integral (3.1) in [9] two Euler-Lagrange equations were derived. One of them describes gravity:

$$-\frac{1}{2\,c_{\text{gr}}}\,\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\mathbf{g},\varphi,\mathbf{A},\dot{\mathbf{A}}}-\frac{1}{2}\,\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\mathbf{g},\varphi,\mathbf{A},\dot{\mathbf{A}}}\sum_{q=1}^{3}b^q_q+\Bigl(\frac{\delta\mathcal{L}}{\delta g_{ij}}\Bigr)_{\mathbf{b},\varphi,\mathbf{A},\dot{\mathbf{A}}}=0. \tag{3.2}$$

The sort of matter in [9] was not specified. Therefore the second Euler-Lagrange equation in [9] was written with respect to some abstract dynamic variables of matter $Q_1,\,\ldots\,,\,Q_n$ and $W_1,\,\ldots\,,\,W_n$ (see (5.6) in [9]). In the present paper matter is presented by the electromagnetic field (see (1.9)). Its dynamic variables here are the scalar potential φ, the vector potential $\mathbf{A}$, and their time derivatives. Therefore the Euler-Lagrange equations of matter here are written as follows:

$$-\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}}{\delta\dot{\varphi}}\Bigr)_{\mathbf{g},\mathbf{b},\varphi,\mathbf{A},\dot{\mathbf{A}}}-\Bigl(\frac{\delta\mathcal{L}}{\delta\dot{\varphi}}\Bigr)_{\mathbf{g},\mathbf{b},\varphi,\mathbf{A},\dot{\mathbf{A}}}\sum_{q=1}^{3}c_{\text{gr}}\,b^q_q+\Bigl(\frac{\delta\mathcal{L}}{\delta\varphi}\Bigr)_{\mathbf{g},\mathbf{b},\mathbf{A},\dot{\mathbf{A}}}=0, \tag{3.3}$$

$$-\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}}{\delta\dot{A}_i}\Bigr)_{\mathbf{g},\mathbf{b},\varphi,\mathbf{A}}-\Bigl(\frac{\delta\mathcal{L}}{\delta\dot{A}_i}\Bigr)_{\mathbf{g},\mathbf{b},\varphi,\mathbf{A}}\sum_{q=1}^{3}c_{\text{gr}}\,b^q_q+\Bigl(\frac{\delta\mathcal{L}}{\delta A_i}\Bigr)_{\mathbf{g},\mathbf{b},\varphi,\dot{\mathbf{A}}}=0. \tag{3.4}$$

Due to (2.5) the Euler-Lagrange equation (3.3) reduces to

$$\Bigl(\frac{\delta\mathcal{L}}{\delta\varphi}\Bigr)_{\mathbf{g},\mathbf{b},\mathbf{A},\dot{\mathbf{A}}}=0. \tag{3.5}$$

The gravitational part $\mathcal{L}_{\text{gr}}$ of the Lagrangian (1.10) does not depend on φ. Therefore the Euler-Lagrange equation (3.5) reduces to

$$\Bigl(\frac{\delta\mathcal{L}_{\text{el}}}{\delta\varphi}\Bigr)_{\mathbf{g},\mathbf{b},\mathbf{A},\dot{\mathbf{A}}}=0. \tag{3.6}$$

Using (1.6) and (1.8), one can calculate the variational derivative in (3.6) explicitly:

$$\Bigl(\frac{\delta\mathcal{L}_{\text{el}}}{\delta\varphi}\Bigr)_{\mathbf{g},\mathbf{b},\mathbf{A},\dot{\mathbf{A}}}=\frac{1}{4\,\pi}\operatorname{div}\mathbf{E}=\frac{1}{4\,\pi}\sum_{i=1}^{3}\nabla_{\!i}E^i. \tag{3.7}$$

Due to (3.7) the equation (3.6) yields the well-known Maxwell equation for the electric field in the absence of charges (see § 1 in Chapter II of [12]):

$$\operatorname{div}\mathbf{E}=0. \tag{3.8}$$

The gravitational part $\mathcal{L}_{\text{gr}}$ of the Lagrangian (1.10) does not depend on the vector potential $\mathbf{A}$ and on its time derivative. Therefore (3.4) reduces to:

$$-\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}_{\text{el}}}{\delta\dot{A}_i}\Bigr)_{\mathbf{g},\mathbf{b},\varphi,\mathbf{A}}-\Bigl(\frac{\delta\mathcal{L}_{\text{el}}}{\delta\dot{A}_i}\Bigr)_{\mathbf{g},\mathbf{b},\varphi,\mathbf{A}}\sum_{q=1}^{3}c_{\text{gr}}\,b^q_q+\Bigl(\frac{\delta\mathcal{L}_{\text{el}}}{\delta A_i}\Bigr)_{\mathbf{g},\mathbf{b},\varphi,\dot{\mathbf{A}}}=0. \tag{3.9}$$

Using (1.7) and (1.8), one can calculate the variational derivatives in (3.6) explicitly:

$$\begin{aligned}&\Bigl(\frac{\delta\mathcal{L}_{\text{el}}}{\delta\dot{A}_i}\Bigr)_{\mathbf{g},\mathbf{b},\varphi,\mathbf{A}}=-\frac{E^i}{4\,\pi\,c_{\text{el}}},\\&\Bigl(\frac{\delta\mathcal{L}_{\text{el}}}{\delta A_i}\Bigr)_{\mathbf{g},\mathbf{b},\varphi,\dot{\mathbf{A}}}=-\frac{1}{4\,\pi}\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\varepsilon^{ijk}}{\sqrt{\det g}}\,\nabla_{j}H_k.\end{aligned}\tag{3.10}$$

The first formula (3.10) coincides with (2.7). The term ε^{ijk} in the right hand side of the second formula (3.10) is the same Levi-Civita symbol as in (1.7). Substituting (3.10) into (3.9) we derive the following equation:

$$\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\varepsilon^{ijk}}{\sqrt{\det g}}\,\nabla_{j}H_k=\frac{1}{c_{\text{el}}}\,\frac{\partial E^i}{\partial t}+\frac{c_{\text{gr}}}{c_{\text{el}}}\sum_{q=1}^{3}E^i\,b^q_q.\tag{3.11}$$

The equation (3.11) can be written in a vectorial form:

$$\operatorname{rot}\mathbf{H}=\frac{1}{c_{\text{el}}}\,\frac{\partial\mathbf{E}}{\partial t}+\frac{c_{\text{gr}}}{c_{\text{el}}}\sum_{q=1}^{3}\mathbf{E}\,b^q_q.\tag{3.12}$$

The equation (3.10) is very similar to one of the Maxwell equations in the absence of charges and currents (see § 1 of Chapter II in [12]). Unlike the first equation (1.7), the equation (3.12) is written with respect to the contravariant components of $\mathbf{E}$. This is clear from (3.11).

The second term from the right hand side of the equation (3.11) is absent in the standard Maxwell equation. This term in (3.12) is called the Hubble term. Assuming the expansion of the universe to be uniform and isotropic (which is true at large scales), we can relate this term with the Hubble parameter (see [16]):

$$c_{\text{gr}}\sum_{q=1}^{3}b^q_q\approx 3\,H.\tag{3.13}$$

The value of the Hubble parameter H in (3.13) at the current cosmological time is known as the Hubble constant $H_0=H(t_0)$. Wikipedia in [16] provides somewhat conflicting experimental data for the value of H_0 obtained using several different experimental techniques. Roughly averaging these conflicting data, we can write

$$H_0\approx 70\,\frac{\text{km}}{\text{s}\cdot\text{Mpc}}.$$

Now we can proceed to the equation (3.2). Substituting (1.10) into (3.2), we get:

$$\begin{aligned}&-\frac{1}{2\,c_{\text{gr}}}\,\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\Bigr)_{\mathbf{g},\varphi,\mathbf{A},\dot{\mathbf{A}}}-\frac{1}{2}\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\Bigr)_{\mathbf{g},\varphi,\mathbf{A},\dot{\mathbf{A}}}\sum_{q=1}^{3}b^q_q+\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta g_{ij}}\Bigr)_{\mathbf{b},\varphi,\mathbf{A},\dot{\mathbf{A}}}=\\&=\frac{1}{2\,c_{\text{gr}}}\,\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}_{\text{el}}}{\delta b_{ij}}\Bigr)_{\mathbf{g},\varphi,\mathbf{A},\dot{\mathbf{A}}}+\frac{1}{2}\Bigl(\frac{\delta\mathcal{L}_{\text{el}}}{\delta b_{ij}}\Bigr)_{\mathbf{g},\varphi,\mathbf{A},\dot{\mathbf{A}}}\sum_{q=1}^{3}b^q_q-\Bigl(\frac{\delta\mathcal{L}_{\text{el}}}{\delta g_{ij}}\Bigr)_{\mathbf{b},\varphi,\mathbf{A},\dot{\mathbf{A}}}.\end{aligned}\tag{3.14}$$

The variational derivative in the first two terms of (3.14) is already calculated. Indeed, since the Lagrangian $\mathcal{L}_{\text{el}}$ in (1.8) does not depend on b_{ij}, due to (1.10) this variational derivative is actually given by the formula (2.6):

$$\left(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\right)_{\mathbf{g},\varphi,\mathbf{A},\dot{\mathbf{A}}} = \frac{c_{\text{gr}}^4}{8\pi\gamma}\left(b^{ij} - \sum_{k=1}^{3} b^k_k\, g^{ij}\right). \tag{3.15}$$

In order to calculate the variational derivative in the third term of the equation (3.14) we can use the results of [9]. Slightly modifying the formula (6.13) from therein, we can write the following formula:

$$\begin{aligned}\left(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta g_{ij}}\right)_{\mathbf{b},\varphi,\mathbf{A},\dot{\mathbf{A}}} = \frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\Biggl(\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{1}{2}\,b^k_q\,b^q_k\,g^{ij} - \sum_{k=1}^{3}\sum_{q=1}^{3} 2\,b^i_k\,b^k_q\,g^{qj} \,+\\ +\sum_{k=1}^{3} 2\,b^k_k\,b^{ij} - \sum_{k=1}^{3}\sum_{q=1}^{3}\frac{1}{2}\,b^k_k\,b^q_q\,g^{ij} - R^{ij} + \frac{R}{2}\,g^{ij} - \Lambda\,g^{ij}\Biggr).\end{aligned} \tag{3.16}$$

As we noted in (2.3), the Lagrangian of the electromagnetic field (1.8) does not depend on the components of the tensor field $\mathbf{b}$. Therefore

$$\left(\frac{\delta\mathcal{L}_{\text{el}}}{\delta b_{ij}}\right)_{\mathbf{g},\varphi,\mathbf{A},\dot{\mathbf{A}}} = 0. \tag{3.17}$$

The Lagrangian (1.8) depends on the components of the metric (1.1) due to (2.2). In order to calculate the last variational derivative in (3.14) we use the second formula (1.7). The covariant derivative $\nabla_j A_k$ in this formula is calculated as follows:

$$\nabla_j A_k = \frac{\partial A_k}{\partial x^j} - \sum_{q=1}^{3}\Gamma^q_{jk}\,A_q, \tag{3.18}$$

see (5.12) in § 5 of Chapter III in [17]). The components of the metric connection Γ^q_{jk} in (3.18) are symmetric with respect to the indices j and k, while the Levi-Civita symbol ε^{ijk} in (1.7) is skew-symmetric with respect to these indices. Therefore the first and the second formulas (1.7) reduce to the following ones:

$$E_i = -\frac{\partial\varphi}{\partial x^i} - \frac{1}{c_{\text{el}}}\,\frac{\partial A_i}{\partial t}, \qquad\qquad H^i = \sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\varepsilon^{ijk}}{\sqrt{\det g}}\,\frac{\partial A_k}{\partial x^j}. \tag{3.19}$$

The formulas (3.19) mean that the components E_i of the electric field do not depend on the metric at all, while the components H^i of the magnetic field depend on the metric only through the square root $\sqrt{\det g}$ in the denominator of the second formula (3.19). Using this fact, from (1.8), (2.2), and (3.1) we derive

$$\left(\frac{\delta\mathcal{L}_{\text{el}}}{\delta g_{ij}}\right)_{\mathbf{b},\varphi,\mathbf{A},\dot{\mathbf{A}}} = -\frac{E^i\,E^j + H^i\,H^j}{8\,\pi} + \frac{|\mathbf{E}|^2 + |\mathbf{H}|^2}{16\,\pi}\,g^{ij}. \tag{3.20}$$

We can compare (3.20) with the previously obtained results. Taking into account

(1.9) with (3.17) and applying the formulas (3.20) and (3.21) from [9], we get

$$\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{ij}}=\frac{E_i\,E_j+H_i\,H_j}{8\,\pi}-\frac{|\mathbf{E}|^2+|\mathbf{H}|^2}{16\,\pi}\,g_{ij}. \tag{3.21}$$

In the standard four-dimensional formalism the energy-momentum tensor of the electromagnetic field is given by the formula (4.4) from § 4 of Chapter V in [12]. This formula is written as follows:

$$T_{qj}=-\frac{1}{4\pi}\sum_{p=0}^{3}\sum_{i=0}^{3}\left(F_{pq}\,G^{pi}\,F_{ij}-\frac{1}{4}F_{pi}\,F^{pi}\,G_{qj}\right). \tag{3.22}$$

Let's choose an orthonormal frame where the metric (1.1) is given by the identity matrix and the four-dimensional Minkowski metric is diagonal

$$g_{ij}=g^{ij}\begin{Vmatrix}1&0&0\\0&1&0\\0&0&1\end{Vmatrix},\qquad G_{ij}=G^{ij}=\begin{Vmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{Vmatrix}. \tag{3.23}$$

In such a frame the four-dimensional tensor of the electromagnetic field is presented by the following two skew-symmetric matrices:

$$F^{pq}=\begin{Vmatrix}0&-E^1&-E^2&-E^3\\E^1&0&-H^3&H^2\\E^2&H^3&0&-H^1\\E^3&-H^2&H^1&0\end{Vmatrix},\quad F_{pq}=\begin{Vmatrix}0&E_1&E_2&E_3\\-E_1&0&-H_3&H_2\\-E_2&H_3&0&-H_1\\-E_3&-H_2&H_1&0\end{Vmatrix}. \tag{3.24}$$

Applying (3.23) and (3.24) to (3.22) and (3.21), we derive

$$T_{ij}=-2\,\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{ij}}\quad\text{for}\quad 1\leqslant i,j\leqslant 3. \tag{3.25}$$

The formula (3.25) is consistent with the previously derived formulas (4.40) in [8], (1.13) in [9], and (1.11) in [11] since here we have slightly changed the formula for $\mathcal{L}_{\text{gr}}$ in (1.4) magnifying its coefficient by the factor $c=c_{\text{gr}}$ as compared to the previous papers. Hence the Lagrangian $\mathcal{L}_{\text{mat}}$ turned magnified accordingly.

Let's return back to the Euler-Lagrange equation (3.14). Substituting (3.15), (3.16), and (3.17) into (3.14), and keeping in mind (3.20), we derive

$$\begin{aligned}&-\frac{1}{c_{\text{gr}}}\sum_{k=1}^{3}\sum_{q=1}^{3}\dot{b}_{kq}\,g^{ik}\,g^{jq}+\sum_{q=1}^{3}(b^i_q\,b^{\,qj}+b^j_q\,b^{\,qi})+\\&+\frac{1}{c_{\text{gr}}}\sum_{q=1}^{3}\dot{b}^q_q\,g^{ij}-\sum_{q=1}^{3}b^q_q\,b^{ij}+\frac{g^{ij}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q+\frac{g^{ij}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_q\,b^q_k-\\&-R^{ij}+\frac{R}{2}\,g^{ij}-\Lambda\,g^{ij}=-\frac{16\,\pi\,\gamma}{c^4_{\text{gr}}}\left(\frac{\delta\mathcal{L}_{\text{el}}}{\delta g_{ij}}\right)_{\mathbf{b},\varphi,\mathbf{A},\dot{\mathbf{A}}}.\end{aligned} \tag{3.26}$$

Lowering indices i and j in (3.26) and taking into account (3.20), (3.21), and (3.25), we find that (3.26) is equivalent to the previously derived equation (1.2). The variational derivative in the right hand side of the equation (1.2) in our present case is given by the explicit formula (3.21).

4. The energy conservation law.

The energy conservation law in the 3D-brane universe model was already derived in [11]. Here we specify this energy conservation law for the case where matter is presented by the electromagnetic field. The energy density in [11] is given by the formula (4.2) therein. For our present case this formula is written as

$$\mathcal{H}=\sum_{i=1}^{3}\sum_{j=1}^{3}\beta^{ij}\,b_{ij}+\sum_{i=1}^{3}\alpha^{i}\,\dot{A}_i+\psi\,\dot{\varphi}-\mathcal{L}. \tag{4.1}$$

The quantities β^{ij}, α^i, and ψ in (4.1) are given by (2.5), (2.6) and (2.7). Applying these formulas along with (1.9), (1.10), (1.4), (1.5), and (1.8), we derive

$$\begin{aligned}\mathcal{H}=\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\left(\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_q\,b^q_k-\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q-R+2\,\Lambda\right)-\\ -\frac{1}{4\,\pi\,c_{\text{el}}}\sum_{i=1}^{3}E^i\,\dot{A}_i-\frac{|\mathbf{E}|^2-|\mathbf{H}|^2}{8\,\pi}.\end{aligned} \tag{4.2}$$

We apply the first formula (1.7) in order to transform the formula (4.2). This yields

$$\begin{aligned}\mathcal{H}=\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\left(\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_q\,b^q_k-\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q-R+2\,\Lambda\right)+\\ +\frac{|\mathbf{E}|^2+|\mathbf{H}|^2}{8\,\pi}+\frac{1}{4\,\pi}\sum_{i=1}^{3}E^i\,\nabla_i\varphi.\end{aligned} \tag{4.3}$$

Let Ω be a closed three-dimensional domain with the smooth boundary $\partial\Omega$ in a 3D-brane universe. The energy density (4.3) is used in order to determine the amount of total energy enclosed in the domain Ω:

$$\mathcal{E}=\int\limits_{\Omega}\mathcal{H}\,\sqrt{\det g}\;d^3x \tag{4.4}$$

The last term in (4.3) contributes to (4.4) through the integral

$$\mathcal{E}_7=\frac{1}{4\,\pi}\int\limits_{\Omega}\sum_{i=1}^{3}E^i\,\nabla_i\varphi\,\sqrt{\det g}\;d^3x. \tag{4.5}$$

Applying (3.8) to (4.5), we can transform this integral as follows:

$$\mathcal{E}_7=\frac{1}{4\,\pi}\int\limits_{\Omega}\sum_{i=1}^{3}\nabla_i(E^i\,\varphi)\,\sqrt{\det g}\;d^3x=\frac{1}{4\,\pi}\int\limits_{\partial\Omega}(\varphi\,\mathbf{E},\mathbf{n})\,dS. \tag{4.6}$$

Here $\mathbf{n}$ is the unit normal vector on the boundary $\partial\Omega$ and dS is the infinitesimal area element of this boundary. The last integral in (4.6) means that the energy $\mathcal{E}_7$ is attributed not to the bulk of the domain Ω, but to its boundary. For this reason we can omit the last term in (4.3) and write this formula as

$$\mathcal{H}=\frac{c_{\mathrm{gr}}^4}{16\,\pi\,\gamma}\left(\sum_{k=1}^{3}\sum_{q=1}^{3}b_q^k\,b_k^q-\sum_{k=1}^{3}\sum_{q=1}^{3}b_k^k\,b_q^q-R+2\,\Lambda\right)+\frac{|\mathbf{E}|^2+|\mathbf{H}|^2}{8\,\pi}. \tag{4.7}$$

In order to derive the energy conservation law we should calculate the time derivative of the integral (4.4) with the function (4.7) in it:

$$\dot{\mathcal{E}}=\int\limits_{\Omega}\frac{\partial\mathcal{H}}{\partial t}\sqrt{\det g}\,d^3x+\int\limits_{\Omega}\mathcal{H}\,\frac{\partial(\sqrt{\det g}\,)}{\partial t}\,d^3x. \tag{4.8}$$

The partial derivative in the second integral in (4.8) is easily calculated using (1.3) and Jacobi's formula for differentiating determinants (see [18]):

$$\frac{\partial(\sqrt{\det g}\,)}{\partial t}=c_{\mathrm{gr}}\sqrt{\det g}\,\sum_{q=1}^{3}b_q^q. \tag{4.9}$$

Due to (4.9) the formula (4.8) is rewritten as

$$\dot{\mathcal{E}}=\int\limits_{\Omega}\frac{\partial\mathcal{H}}{\partial t}\sqrt{\det g}\,d^3x+\int\limits_{\Omega}\sum_{q=1}^{3}c_{\mathrm{gr}}\,\mathcal{H}\,b_q^q\sqrt{\det g}\,d^3x. \tag{4.10}$$

In order to calculate the first integral in (4.10) we subdivide the energy density $\mathcal{H}$ given by the formula (4.7) into six terms

$$\mathcal{H}=\mathcal{H}_1+\mathcal{H}_2+\mathcal{H}_3+\mathcal{H}_4+\mathcal{H}_5+\mathcal{H}_6, \tag{4.11}$$

where

$$\mathcal{H}_1=\frac{c_{\mathrm{gr}}^4}{16\,\pi\,\gamma}\sum_{k=1}^{3}\sum_{q=1}^{3}b_q^k\,b_k^q, \qquad \mathcal{H}_4=\frac{c_{\mathrm{gr}}^4}{16\,\pi\,\gamma}\,2\,\Lambda, \tag{4.12}$$

$$\mathcal{H}_2=-\frac{c_{\mathrm{gr}}^4}{16\,\pi\,\gamma}\sum_{k=1}^{3}\sum_{q=1}^{3}b_k^k\,b_q^q, \qquad \mathcal{H}_5=\frac{1}{8\,\pi}\,|\mathbf{E}|^2, \tag{4.13}$$

$$\mathcal{H}_3=-\frac{c_{\mathrm{gr}}^4}{16\,\pi\,\gamma}\,R \qquad \mathcal{H}_6=\frac{1}{8\,\pi}\,|\mathbf{H}|^2, \tag{4.14}$$

The term $\mathcal{H}_4$ in (4.11) is the most simple, it is constant. Therefore

$$\frac{\partial\mathcal{H}_4}{\partial t}=0. \tag{4.15}$$

For the term $\mathcal{H}_1$, using (4.12) and (1.3), we derive

$$\frac{\partial\mathcal{H}_1}{\partial t}=\frac{c_{\mathrm{gr}}^4}{16\,\pi\,\gamma}\sum_{k=1}^{3}\sum_{q=1}^{3}\left(2\,\dot{b}_{kq}\,b^{kq}-4\,c_{\mathrm{gr}}\sum_{r=1}^{3}b_q^k\,b_r^q\,b_k^r\right). \tag{4.16}$$

Similar approach can be applied to $\mathcal{H}_2$ in (4.11). Using (4.13) and (1.3), we get

$$\frac{\partial\mathcal{H}_2}{\partial t}=\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\sum_{k=1}^{3}\sum_{q=1}^{3}\sum_{r=1}^{3}\Bigl(4\,c_{\text{gr}}b^k_r\,b^r_k\,b^q_q-2\,\dot b_{kr}\,g^{kr}\,b^q_q\Bigr).\tag{4.17}$$

Time derivatives $\dot b_{kq}$ and $\dot b_{kr}$ in (4.16) and (4.17) can be calculated using the equation (1.2). As a result the formulas (4.16) and (4.17) are written as

$$\begin{aligned}\frac{\partial\mathcal{H}_1}{\partial t}&=\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\sum_{k=1}^{3}\sum_{q=1}^{3}\Bigl(2\,\dot b^k_k\,b^q_q-c_{\text{gr}}\sum_{r=1}^{3}b^k_r\,b^r_k\,b^q_q+c_{\text{gr}}\sum_{r=1}^{3}b^k_k\,b^r_r\,b^q_q-\\&-2\,c_{\text{gr}}\,R_{kq}\,b^{kq}\Bigr)+\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\sum_{q=1}^{3}c_{\text{gr}}\,(R-2\,\Lambda)\,b^q_q-\sum_{k=1}^{3}\sum_{q=1}^{3}2\,c_{\text{gr}}\,\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{kq}}\,b^{kq}.\end{aligned}\tag{4.18}$$

$$\begin{aligned}\frac{\partial\mathcal{H}_2}{\partial t}&=\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\sum_{k=1}^{3}\sum_{q=1}^{3}\Bigl(-6\,\dot b^k_k\,b^q_q-3\,c_{\text{gr}}\sum_{r=1}^{3}b^k_r\,b^r_k\,b^q_q-c_{\text{gr}}\sum_{r=1}^{3}b^k_k\,b^r_r\,b^q_q\Bigr)-\\&-\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\sum_{q=1}^{3}c_{\text{gr}}\,(R-6\,\Lambda)\,b^q_q+\sum_{k=1}^{3}\sum_{q=1}^{3}\sum_{r=1}^{3}2\,c_{\text{gr}}\,\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{kr}}\,g^{kr}\,b^q_q.\end{aligned}\tag{4.19}$$

Both formulas (4.18) and (4.19) comprises terms with time derivatives $\dot b^q_q$. In order to remove these terms we multiply (1.2) by g^{ij} and sum up with respect to indices i and j. As a result we get the following relationship:

$$\begin{aligned}-\sum_{k=1}^{3}2\,\dot b^k_k&=\frac{3}{2}\sum_{k=1}^{3}\sum_{r=1}^{3}c_{\text{gr}}\,b^k_r\,b^r_k+\frac{1}{2}\sum_{k=1}^{3}\sum_{r=1}^{3}c_{\text{gr}}\,b^k_k\,b^r_r+\\&+c_{\text{gr}}\,\frac{R}{2}-c_{\text{gr}}\,3\,\Lambda-\frac{16\,\pi\,\gamma}{c_{\text{gr}}^3}\sum_{k=1}^{3}\sum_{r=1}^{3}\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{kr}}\,g^{kr}.\end{aligned}\tag{4.20}$$

Applying the relationship (4.20) to (4.18) and (4.19), we derive

$$\begin{aligned}\frac{\partial\mathcal{H}_1}{\partial t}+\frac{\partial\mathcal{H}_2}{\partial t}&=\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\sum_{k=1}^{3}\sum_{q=1}^{3}\Bigl(-c_{\text{gr}}\sum_{r=1}^{3}b^k_r\,b^r_k\,b^q_q+\\&+c_{\text{gr}}\sum_{r=1}^{3}b^k_k\,b^r_r\,b^q_q-2\,R_{kq}\,b^{kq}\Bigr)+\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\sum_{q=1}^{3}c_{\text{gr}}\,R\,b^q_q-\\&-\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\sum_{q=1}^{3}c_{\text{gr}}\,2\,\Lambda\,b^q_q-\sum_{k=1}^{3}\sum_{q=1}^{3}2\,c_{\text{gr}}\,\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{kq}}\,b^{kq}.\end{aligned}\tag{4.21}$$

In the next step we consider the time derivative of the fifth term $\mathcal{E}_5$ in the sum (4.11) given by the second formula (4.13):

$$\frac{\partial\mathcal{H}_5}{\partial t}=\frac{1}{4\,\pi}\sum_{i=1}^{3}\frac{\partial E^i}{\partial t}\,E_i+\frac{1}{4\,\pi}\sum_{i=1}^{3}\sum_{j=1}^{3}c_{\text{gr}}\,b_{ij}\,E^i\,E^j.\tag{4.22}$$

We transform the first term in (4.22) using (3.11). This yields

$$\begin{aligned}\frac{\partial\mathcal{H}_5}{\partial t}=\frac{c_{\text{el}}}{4\,\pi}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\varepsilon^{ijk}}{\sqrt{\det g}}\,E_i\,\nabla_{\!j}H_k\,+\\+\,\frac{c_{\text{gr}}}{4\,\pi}\biggl(\,\sum_{i=1}^{3}\sum_{j=1}^{3}b_{ij}\,E^i\,E^j-\sum_{q=1}^{3}b^q_q\,|\mathbf{E}|^2\biggr).\end{aligned}\tag{4.23}$$

The time derivative of the sixth term $\mathcal{E}_6$ in the sum (4.11) given by the second formula (4.14) is handled similarly. In this case we have

$$\frac{\partial\mathcal{H}_6}{\partial t}=\frac{1}{4\,\pi}\sum_{i=1}^{3}\frac{\partial H^i}{\partial t}\,H_i+\frac{1}{4\,\pi}\sum_{i=1}^{3}\sum_{j=1}^{3}c_{\text{gr}}\,b_{ij}\,H^i\,H^j.\tag{4.24}$$

In order to calculate the time derivative of H^i we apply the second formula (1.7):

$$\begin{aligned}\frac{\partial H^i}{\partial t}=\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\partial}{\partial t}\biggl(\frac{\varepsilon^{ijk}\,\nabla_{\!j}A_k}{\sqrt{\det g}}\biggr)=\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\varepsilon^{ijk}}{\sqrt{\det g}}\,\frac{\partial\nabla_{\!j}A_k}{\partial t}-\sum_{q=1}^{3}c_{\text{gr}}\,\cdot\\\cdot\,b^q_q\,H^i=\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\varepsilon^{ijk}}{\sqrt{\det g}}\,\frac{\partial}{\partial t}\biggl(\frac{\partial A_k}{\partial x^j}-\sum_{q=1}^{3}\Gamma^q_{jk}\,A_q\biggr)-\sum_{q=1}^{3}c_{\text{gr}}\,b^q_q\,H^i=\\=\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\varepsilon^{ijk}}{\sqrt{\det g}}\,\biggl(\frac{\partial\dot A_k}{\partial x^j}-\sum_{q=1}^{3}\Gamma^q_{jk}\,\dot A_q-\sum_{q=1}^{3}\dot\Gamma^q_{jk}\dot A_q\biggr)-\sum_{q=1}^{3}c_{\text{gr}}\,b^q_q\,H^i.\end{aligned}\tag{4.25}$$

The time derivatives of the components of metric connection $\dot\Gamma^q_{jk}$ in (4.25) are symmetric with respect to the indices j and k, while the Levi-Civita symbol ε^{ijk} in (4.25) is skew-symmetric with respect to these indices. Therefore the term with $\dot\Gamma^q_{jk}$ in (4.25) does vanish and we can write this formula as

$$\frac{\partial H^i}{\partial t}=\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\varepsilon^{ijk}}{\sqrt{\det g}}\,\nabla_{\!j}\dot A_k-\sum_{q=1}^{3}c_{\text{gr}}\,b^q_q\,H^i.\tag{4.26}$$

Note that the first formula (1.7) can be written as a formula for $\dot A_k$. Indeed, we have $\dot A_k=-c_{\text{el}}\,E_k-c_{\text{el}}\,\nabla_k\varphi$. Applying this formula to (4.26), we derive

$$\frac{\partial H^i}{\partial t}=-\sum_{j=1}^{3}\sum_{k=1}^{3}c_{\text{el}}\,\frac{\varepsilon^{ijk}}{\sqrt{\det g}}\,\bigl(\nabla_{\!j}E_k+\nabla_{\!j}\nabla_k\varphi\bigr)-\sum_{q=1}^{3}c_{\text{gr}}\,b^q_q\,H^i.\tag{4.27}$$

The double covariant derivatives $\nabla_{\!j}\nabla_k\varphi$ are symmetric with respect to to the indices j and k, while the Levi-Civita symbol ε^{ijk} in (4.27) is skew-symmetric with respect to these indices. Therefore the term with $\nabla_{\!j}\nabla_k\varphi$ in (4.27) does vanish:

$$\frac{\partial H^i}{\partial t}=-\sum_{j=1}^{3}\sum_{k=1}^{3}c_{\text{el}}\,\frac{\varepsilon^{ijk}}{\sqrt{\det g}}\,\nabla_{\!j}E_k-\sum_{q=1}^{3}c_{\text{gr}}\,b^q_q\,H^i.\tag{4.28}$$

Now we substitute (4.28) into (4.24). As a result we derive the formula

$$\begin{aligned}\frac{\partial\mathcal{H}_6}{\partial t}=-\frac{c_{\text{el}}}{4\,\pi}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\varepsilon^{ijk}}{\sqrt{\det g}}\,H_i\nabla_{\!j}E_k\,+\\ +\,\frac{c_{\text{gr}}}{4\,\pi}\biggl(\,\sum_{i=1}^{3}\sum_{j=1}^{3}b_{ij}\,H^i\,H^j-\sum_{q=1}^{3}b^q_q\,|\mathbf{H}|^2\biggr).\end{aligned}\tag{4.29}$$

Adding (4.23) and (4.29), we obtain the following equality:

$$\begin{aligned}\frac{\partial\mathcal{H}_5}{\partial t}+\frac{\partial\mathcal{H}_6}{\partial t}=-\frac{c_{\text{el}}}{4\,\pi}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\varepsilon^{ijk}}{\sqrt{\det g}}\,\nabla_{\!i}(E_j\,H_k)\,+\\ +\,c_{\text{gr}}\biggl(\,\sum_{i=1}^{3}\sum_{j=1}^{3}b_{ij}\,\frac{E^i\,E^j+H^i\,H^j}{4\,\pi}-\sum_{q=1}^{3}b^q_q\,\frac{|\mathbf{E}|^2+|\mathbf{E}|^2}{4\,\pi}\biggr).\end{aligned}\tag{4.30}$$

It is known that the Levi-Civita symbol is associated with the volume tensor $\boldsymbol{\omega}$:

$$\omega_{ijk}=\sqrt{\det g}\,\varepsilon_{ijk},\qquad\qquad \omega^{ijk}=\frac{\varepsilon^{ijk}}{\sqrt{\det g}},\tag{4.31}$$

see § 6 of Chapter II in [17]. It is also known that covariant derivatives of the volume tensor $\boldsymbol{\omega}$ are zero, see § 7 of Chapter IV in [17]:

$$\nabla_{\!s}\omega_{ijk}=0,\qquad\qquad \nabla_{\!s}\omega^{ijk}.\tag{4.32}$$

Applying (4.31) and (4.32) to the first term in the right hand side of (4.30), we get

$$\begin{aligned}\frac{\partial\mathcal{H}_5}{\partial t}+\frac{\partial\mathcal{H}_6}{\partial t}=-\sum_{i=1}^{3}\nabla_{\!i}\biggl(\frac{c_{\text{el}}}{4\,\pi}\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\varepsilon^{ijk}}{\sqrt{\det g}}\,E_j\,H_k\biggr)\,+\\ +\,c_{\text{gr}}\biggl(\,\sum_{i=1}^{3}\sum_{j=1}^{3}b_{ij}\,\frac{E^i\,E^j+H^i\,H^j}{4\,\pi}-\sum_{q=1}^{3}b^q_q\,\frac{|\mathbf{E}|^2+|\mathbf{E}|^2}{4\,\pi}\biggr).\end{aligned}\tag{4.33}$$

In (4.33) we see the following quantities:

$$S^i=\frac{c_{\text{el}}}{4\,\pi}\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\varepsilon^{ijk}}{\sqrt{\det g}}\,E_j\,H_k.\tag{4.34}$$

The quantities (4.34) are the components of the vector field $\mathbf{S}$, where

$$\mathbf{S}=\frac{c_{\text{el}}}{4\,\pi}\,[\mathbf{E},\mathbf{H}].\tag{4.35}$$

The vector field $\mathbf{S}$ given by the in the formula (4.35) is known as the Umov-Pointing vector (see § 4 of Chapter II in [12]). In classical electrodynamics this vector field is interpreted as the density vector of the electromagnetic energy flow.

Let's apply the formulas (4.35) and (4.34) to the relationship (4.33). As a result this relationship is rewritten in terms of the divergence of the vector field $\mathbf{S}$:

$$\begin{gathered}\frac{\partial\mathcal{H}_5}{\partial t}+\frac{\partial\mathcal{H}_6}{\partial t}=-\operatorname{div}\mathbf{S}+\\ +c_{\text{gr}}\biggl(\sum_{i=1}^{3}\sum_{j=1}^{3}b_{ij}\,\frac{E^i\,E^j+H^i\,H^j}{4\,\pi}-\sum_{q=1}^{3}b^q_q\,\frac{|\mathbf{E}|^2+|\mathbf{E}|^2}{4\,\pi}\biggr).\end{gathered}\tag{4.36}$$

The third term $\mathcal{H}_3$ in (4.11) is the most complicated. It is given by the first formula (4.14), where R is the three-dimensional scalar curvature associated with the metric (1.1). It is calculated through the Ricci tensor:

$$R=\sum_{i=1}^{3}\sum_{j=1}^{3}R_{ij}\,g^{ij},\tag{4.37}$$

see § 8 of Chapter IV in [17]. Applying (4.37) to (4.14) and then differentiating the result with respect to time variable t, we get

$$\frac{\partial\mathcal{H}_3}{\partial t}=-\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\,\dot{R}=-\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\sum_{i=1}^{3}\sum_{j=1}^{3}\bigl(\dot{R}_{ij}\,g^{ij}-2\,c_{\text{gr}}\,R_{ij}\,b^{ij}\bigr).\tag{4.38}$$

The Ricci tensor is produced from the curvature tensor:

$$R_{ij}=\sum_{k=1}^{3}R^k_{ikj}.\tag{4.39}$$

see § 8 of Chapter IV in [17]. Applying (4.39) to (4.38), we derive

$$\frac{\partial\mathcal{H}_3}{\partial t}=\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\sum_{i=1}^{3}\sum_{j=1}^{3}\biggl(2\,c_{\text{gr}}\,R_{ij}\,b^{ij}-\sum_{k=1}^{3}\dot{R}^k_{ikj}\,g^{ij}\biggr).\tag{4.40}$$

The components of the curvature tensor are given by the formula

$$R^q_{ikj}=\frac{\partial\Gamma^q_{ji}}{\partial x^k}-\frac{\partial\Gamma^q_{ki}}{\partial x^j}+\sum_{s=1}^{3}\Gamma^q_{ks}\Gamma^s_{ji}-\sum_{s=1}^{3}\Gamma^q_{js}\Gamma^s_{ki},\tag{4.41}$$

see § 8 of Chapter IV in [17]. Differentiating (4.41) with respect to t, we get

$$\begin{gathered}\dot{R}^q_{ikj}=\frac{\partial\dot{\Gamma}^q_{ji}}{\partial x^k}-\frac{\partial\dot{\Gamma}^q_{ki}}{\partial x^j}+\sum_{s=1}^{3}\dot{\Gamma}^q_{ks}\Gamma^s_{ji}-\sum_{s=1}^{3}\dot{\Gamma}^q_{js}\Gamma^s_{ki}+\\ +\sum_{s=1}^{3}\Gamma^q_{ks}\dot{\Gamma}^s_{ji}-\sum_{s=1}^{3}\Gamma^q_{js}\dot{\Gamma}^s_{ki}=\nabla_k\dot{\Gamma}^q_{ji}-\nabla_j\dot{\Gamma}^q_{ki}.\end{gathered}\tag{4.42}$$

The time derivatives of the connection components in (4.42) are components of a tensor field. Therefore applying covariant derivatives to them is consistent. Moreover, we can calculate these time derivatives explicitly. For this purpose we need to differentiate the Levi-Civita formula determining the components of the metric connection (see § 7 of Chapter III in [17]):

$$\Gamma^k_{ij}=\frac{1}{2}\sum_{r=1}^{3}g^{kr}\left(\frac{\partial g_{rj}}{\partial x^i}+\frac{\partial g_{ir}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^r}\right).\tag{4.43}$$

Differentiating (4.43) with respect to t and taking into account (1.3), we derive

$$\begin{aligned}\dot\Gamma^k_{ij}&=-\,c_{\text{gr}}\sum_{s=1}^{3}2\,b^k_s\,\Gamma^s_{ij}+c_{\text{gr}}\sum_{r=1}^{3}g^{kr}\left(\frac{\partial b_{rj}}{\partial x^i}+\frac{\partial b_{ir}}{\partial x^j}-\right.\\&\left.-\,\frac{\partial b_{ij}}{\partial x^r}\right)=c_{\text{gr}}\sum_{r=1}^{3}g^{kr}\left(\nabla_i b_{rj}+\nabla_j b_{ri}-\nabla_r b_{ij}\right).\end{aligned}\tag{4.44}$$

Now we can apply (4.44) to (4.42). As a result we get $\dot R^q_{ikj}$ for applying to (4.40):

$$\begin{aligned}\dot R^q_{ikj}&=c_{\text{gr}}\sum_{r=1}^{3}g^{qr}\,(\nabla_k\nabla_j-\nabla_j\nabla_k)\,b_{ri}\,+\\&+c_{\text{gr}}\sum_{r=1}^{3}g^{qr}\left(\nabla_k\nabla_i b_{rj}-\nabla_k\nabla_r b_{ji}+\nabla_j\nabla_r b_{ki}-\nabla_j\nabla_i b_{rk}\right).\end{aligned}\tag{4.45}$$

Using (4.45) and (4.39), we calculate the time derivative of the Ricci tensor:

$$\dot R_{ij}=c_{\text{gr}}\sum_{k=1}^{3}\bigl(\nabla_k\nabla_j\,b^k_i+\nabla_k\nabla_i b^k_j-\nabla_j\nabla_i b^k_k\bigr)-c_{\text{gr}}\sum_{k=1}^{3}\sum_{r=1}^{3}g^{kr}\,\nabla_k\nabla_r b_{ij}.\tag{4.46}$$

The formula (4.46) is ready for applying it to (4.38). This yields

$$\begin{aligned}\frac{\partial\mathcal{H}_3}{\partial t}&=\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\sum_{i=1}^{3}\sum_{j=1}^{3}2\,c_{\text{gr}}\,R_{ij}\,b^{ij}\,-\\&-\sum_{k=1}^{3}\nabla_k\left(\frac{c_{\text{gr}}^5}{8\,\pi\,\gamma}\left(\sum_{i=1}^{3}\nabla_i b^{ik}-\sum_{i=1}^{3}\sum_{q=1}^{3}g^{ik}\,\nabla_i\,b^q_q\right)\right).\end{aligned}\tag{4.47}$$

Note that the second term in the right hand side of the formula (4.47) is similar to the first term in the right hand side of the formula (4.33). Using this similarity, we introduce the vector field $\mathbf{J}$ with the components

$$J^k=\frac{c_{\text{gr}}^5}{8\,\pi\,\gamma}\left(\sum_{i=1}^{3}\nabla_i b^{ik}-\sum_{i=1}^{3}\sum_{q=1}^{3}g^{ik}\,\nabla_i\,b^q_q\right).\tag{4.48}$$

The vector $\mathbf{J}$ with the components (4.48) should be interpreted as the density of the gravitational energy flow.

Let's apply the formula (4.48) to the relationship (4.47). As a result this relationship is rewritten in terms of the divergence of the vector field $\mathbf{J}$:

$$\frac{\partial\mathcal{H}_3}{\partial t}=-\,\operatorname{div}\mathbf{J}+\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\sum_{i=1}^{3}\sum_{j=1}^{3}2\,c_{\text{gr}}\,R_{ij}\,b^{ij}.\tag{4.49}$$

Now we can put the formulas (4.21), (4.15), (4.49), (4.36) together and then, relying upon the formula (4.11), we can write the following formula for $\mathcal{H}$:

$$\begin{aligned}\frac{\partial\mathcal{H}}{\partial t}&=-\biggl(\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\biggl(\sum_{k=1}^{3}\sum_{r=1}^{3}b_r^k\,b_k^r-\sum_{k=1}^{3}\sum_{r=1}^{3}b_k^k\,b_r^r-R+2\,\Lambda\biggr)+\\&+\frac{|\mathbf{E}|^2+|\mathbf{E}|^2}{4\,\pi}\biggr)\sum_{q=1}^{3}c_{\text{gr}}\,b_q^q-\sum_{i=1}^{3}\sum_{j=1}^{3}2\,c_{\text{gr}}\,\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{ij}}\,b^{ij}+\\&+\sum_{i=1}^{3}\sum_{j=1}^{3}c_{\text{gr}}\,b_{ij}\,\frac{E^i\,E^j+H^i\,H^j}{4\,\pi}-\operatorname{div}\mathbf{J}-\operatorname{div}\mathbf{S}.\end{aligned}\tag{4.50}$$

In the present case matter is presented by the electromagnetic field. Therefore we can apply the formulas (1.9) and (3.21) to (4.50). This yields

$$\begin{aligned}\frac{\partial\mathcal{H}}{\partial t}&=-\biggl(\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\biggl(\sum_{k=1}^{3}\sum_{r=1}^{3}b_r^k\,b_k^r-\sum_{k=1}^{3}\sum_{r=1}^{3}b_k^k\,b_r^r-R+2\,\Lambda\biggr)+\\&+\frac{|\mathbf{E}|^2+|\mathbf{E}|^2}{8\,\pi}\biggr)\sum_{q=1}^{3}c_{\text{gr}}\,b_q^q-\operatorname{div}\mathbf{J}-\operatorname{div}\mathbf{S}.\end{aligned}\tag{4.51}$$

Comparing (4.51) with the formula (4.7) for $\mathcal{H}$, we can reduce (4.51) to

$$\frac{\partial\mathcal{H}}{\partial t}+\sum_{q=1}^{3}c_{\text{gr}}\,b_q^q\,\mathcal{H}+\operatorname{div}(\mathbf{J}+\mathbf{S})=0.\tag{4.52}$$

Since $\mathbf{J}$ and $\mathbf{S}$ in (4.52) are interpreted as the densities of the gravitational energy flow and the electromagnetic energy flow respectively, their sum

$$\mathsf{J}=\mathbf{J}+\mathbf{S}\tag{4.53}$$

should be interpreted as the density of the total energy flow. The components of J in [11] are denoted through $\mathcal{J}^1$, $\mathcal{J}^2$, $\mathcal{J}^3$. Applying (4.53) to (4.52), we derive

$$\frac{\partial\mathcal{H}}{\partial t}+\sum_{q=1}^{3}c_{\text{gr}}\,b_q^q\,\mathcal{H}+\operatorname{div}\mathsf{J}=0.\tag{4.54}$$

The relationship (4.54) is the differential presentation of the energy conservation law in the 3D-brane universe model for the case where matter is presented by the electromagnetic field in the absence of charges and currents. In general case the energy conservation law within the three-dimensional paradigm was derived in [11].

There is an integral presentation of the energy conservation law (4.54). It is written using the energy integral (4.4) with the energy density (4.7):

$$\frac{\partial}{\partial t}\int\limits_{\Omega}\mathcal{H}\,\sqrt{\det g}\;d^3x+\int\limits_{\partial\Omega}(\mathsf{J},\mathbf{n})\,dS=0.\tag{4.55}$$

The relationship (4.55) is derived using (4.10). Verbally the relationship (4.55) is expressed as the following theorem.

Theorem 4.1. *The increment of the total energy of the gravitational and electromagnetic fields per unit time in a closed 3D-domain Ω is equal to the energy supplied to the domain per unit time through its boundary $\partial\Omega$.*

Like the density of the energy flow in (4.53), the energy density in (4.7) is obviously subdivided into two parts:

$$\mathcal{H} = \mathcal{H}_{\text{gr}} + \mathcal{H}_{\text{el}}, \tag{4.56}$$

where

$$\mathcal{H}_{\text{el}} = \frac{|\mathbf{E}|^2 + |\mathbf{H}|^2}{8\,\pi}, \tag{4.57}$$

$$\mathcal{H}_{\text{gr}} = \frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\left(\sum_{k=1}^{3}\sum_{q=1}^{3} b^k_q\, b^q_k - \sum_{k=1}^{3}\sum_{q=1}^{3} b^k_k\, b^q_q - R + 2\,\Lambda\right). \tag{4.58}$$

Despite the subdivision of $\mathcal{H}$ in (4.56) and $\mathbf{J}$ in (4.53) into two separate parts, the gravitational and electromagnetic fields do interact with each other. Therefore there are no separate energy conservation laws for these two fields.

5. Concluding remarks.

The main result of the present paper is the theory of electromagnetism embedded into the 3D-brane universe paradigm with two speed constants c_{gr} and c_{el}, where c_{gr} is the speed of gravity, while c_{el} is the regular speed of light. The equality

$$c_{\text{gr}} = c_{\text{el}} \tag{5.1}$$

is mandatory in Einstein's theory of gravity. The 3D-brane universe paradigm is different. In this paradigm the inequality

$$c_{\text{gr}} \neq c_{\text{el}} \tag{5.2}$$

is admissible. Admitting the inequality (5.2), we introduce the three-dimensional vector potential $\mathbf{A}$ and the scalar potential φ as two separate fields. The electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ are again two separate fields like in classical physics. They are introduced through the formulas (1.6) and (1.7). Then the Lagrangian of the electromagnetic field is introduced by means of the classical formula (1.8) and the Maxwell equations (3.8) and (3.12) are derived as Euler-Lagrange equations with minor changes in them as compared to classical ones.

The Maxwell equations (3.8) and (3.12) are complemented with the equations (1.2) for the gravitational field. These equations are also derived as Euler-Lagrange equations for the Lagrangian (1.10). Their right hand sides are expressed through the electric and magnetic fields through the formulas (3.25) and (3.21).

The energy density $\mathcal{H}_{el}$ and the energy flow density $\mathbf{S}$ of the electromagnetic field in the 3D-brane universe paradigm are given by the classical formulas (4.57) and (4.35). The formulas (4.58) and (4.48) for the energy density $\mathcal{H}_{gr}$ and for the components of the energy flow density vector $\mathbf{J}$ of the gravitational field are new results of the present paper.

Registration of gravitational waves in LIGO and Virgo interferometers is a recent advance in physics. The first detection GW150914 was reported in [19]. Later on in [20], based on gravitational waves observations, the following bounds for the speed of gravitational waves were claimed, restricting the inequality (5.2):

$$0.55\, c_{\text{el}} \leqslant c_{\text{gr}} \leqslant 1.42\, c_{\text{el}}.$$

In 2017 the gravitational wave detection GW170817 was associated with the gamma-ray burst GRB170817A. Relying on this association, in [21] it was claimed that the equality (5.1) is fulfilled with the possible relative difference of the order of 10^{-15}. However, the association of GW170817 with GRB170817A is the unique case of such an association thus far. The further study in [22] of 105 gamma-ray bursts during the LIGO-Virgo third run 03A did not reveal any associated gravitational wave detections. Therefore the inequality (5.2) is not yet completely disproved experimentally.

8. Dedicatory.

This paper is dedicated to my sister Svetlana Abdulovna Sharipova.

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