Rutherford cross section in the laboratory frame

Anindya Kumar Biswas*
Department of Physics;
North-Eastern Hill University,
Mawkynroh-Umshing, Shillong-793022.

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In this pedagogical article, we elucidate on the direct derivation of the classical non-relativistic Rutherford scattering cross section, differential, in the laboratory frame of two equal mass particles, ala relativistic quantum mechanics as presented in the book of Bjorken and Drell.

* anindya@nehu.ac.in

I. INTRODUCTION

The classical non-relativistic Rutherford scattering cross section, differential, in the laboratory frame is computed in the classical mechanics in two steps. First the Rutherford scattering cross section, differential, in the center of mass frame is calculated, [1], as $\frac{k^2}{16E^2_{c.o.m}}\frac{1}{\sin^4\frac{\Theta}{2}}$, where, Θ is the c.o.m scattering angle. Then using transformation from the the center of mass to the laboratory frame, [1], the differential scattering cross section is calculated in the laboratory frame, for two particles of equal mass and elastic scattering, as $\frac{k^2}{E^2_{lab}}\frac{cos\theta}{\sin^4\theta}$, where, θ is the laboratory frame scattering angle. On the other hand, in the relativistic quantum mechanics of Dirac, the differential scattering cross section can be calculated in the laboratory frame directly, [2]. Taking non-relativistic limit yields us the classical non-relativistic differential cross-section.

II.

The Dirac equation of a spin half particle, of rest mass m_0 and charge q, in the presence of electromagnetic field, A^μ, is given by

$$(i\hbar\gamma^\mu\partial_\mu - q\gamma^\mu A_\mu - m_0 c)\psi(x) = 0$$

The transition amplitude, S_{fi}, of an electron of charge, e, in the presence of electrmagnetic field, A^μ, is given by

$$S_{fi} = -\frac{ie}{\hbar}\int d^4x\bar{\psi}_f(x)\gamma_\mu\psi_i(x)A^\mu$$

where, $\psi_i(x)$ and $\psi_f(x)$ represent initial and final free states of an electron, away from the electromagnetic field, A^μ. If the electromagnetic field, A^μ is generated by a proton current, then

$$A^\mu(x) = -\mu_0 ce\int d^4x^{'} D(x-x^{'})\bar{\psi}^P_f(x)\gamma^\mu\psi^P_i(x^{'})$$

where, $\psi^P_i(x)$ and $\psi^P_f(x)$ represent initial and final free states of a proton, away from the electron. The free electromagnetic field propagator or the Green's function is given by,

$$D(x-x^{'}) = \frac{1}{(2\pi)^4}\frac{1}{\hbar^2}\int d^4q\frac{-1}{q^2}e^{-iq.(x-x^{'})/\hbar}$$

Hence, the transition amplitude of an electron and a proton getting scattered from each other, is given by

$$S_{fi} = -i\frac{\mu_0 ce^2}{\hbar^3}\frac{1}{(2\pi)^4}\int d^4q\frac{1}{q^2}\int d^4x\bar{\psi}_f(x)\gamma_\mu\psi_i(x)e^{-iq.x/\hbar}\int d^4x^{'}\bar{\psi}^P_f(x^{'})\gamma^\mu\psi^P_i(x^{'})e^{iq.x^{'}/\hbar}$$

Putting in the expressions of $\psi_i(x)$, $\psi_f(x)$ and $\psi^P_i(x)$, $\psi^P_f(x)$ respectively, we get the transition amplitude as

$$\begin{aligned}
S_{fi} &= -i\frac{\mu_0 ce^2}{\hbar^3}\frac{m_0c^2M_0c^2}{\sqrt{E_fE_iE^P_fE^P_i}V^2}\frac{1}{(2\pi)^4}\int d^4q\frac{1}{q^2}\bar{u}_f\gamma_\mu u_i\bar{u}^P_f\gamma^\mu u^P_i\int d^4xe^{i(p_f-p_i-q).x/\hbar}\int d^4x^{'}e^{i(P_f-P_i+q).x^{'}/\hbar} \\
&= -i\frac{\mu_0 ce^2}{\hbar^3}\frac{m_0c^2M_0c^2}{\sqrt{E_fE_iE^P_fE^P_i}V^2}\frac{1}{(2\pi)^4}\int d^4q\frac{1}{q^2}\bar{u}_f\gamma_\mu u_i\bar{u}^P_f\gamma^\mu u^P_i(2\pi)^4\hbar^4\delta^4(p_f-p_i-q)(2\pi)^4\hbar^4\delta^4(P_f-P_i+q) \\
&= -i\mu_0 ce^2\hbar^5\frac{m_0c^2M_0c^2}{\sqrt{E_fE_iE^P_fE^P_i}V^2}\frac{\bar{u}_f\gamma_\mu u_i\bar{u}^P_f\gamma^\mu u^P_i}{(p_f-p_i)^2}(2\pi)^4\delta^4(P_f+p_f-P_i-p_i)
\end{aligned}$$

The transition probability from an initial state i to a final state f, $S_{fi}S^*_{fi}$, is obtained as

$$|S_{fi}|^2 = \mu_0^2c^2e^4\hbar^{10}\frac{m_0^2c^4M_0^2c^4}{E_fE_iE^P_fE^P_iV^4}\frac{Tr[\gamma_\mu u_i\bar{u}_i\gamma_\lambda u_f\bar{u}_f]Tr[\gamma^\mu u^P_i\bar{u}^P_i\gamma^\lambda u^P_f\bar{u}^P_f]}{(p_f-p_i)^4}\frac{cVT}{\hbar^4}(2\pi)^4\delta^4(P_f+p_f-P_i-p_i)$$

Summing over the final spin states and averaging over the initial spin states, one gets,

$$\frac{1}{4}\Sigma_{fi}|S_{fi}|^2 = \mu_0^2c^2e^4\hbar^6cVT\frac{m_0^2c^4M_0^2c^4}{4E_fE_iE^P_fE^P_iV^4}\frac{Tr[\gamma_\mu\frac{\not{p}_i+m_0c}{2m_0c}\gamma_\lambda\frac{\not{p}_f+m_0c}{2m_0c}]Tr[\gamma^\mu\frac{\not{P}_i+M_0c}{2M_0c}\gamma^\lambda\frac{\not{P}_f+M_0c}{2M_0c}]}{(p_f-p_i)^4}(2\pi)^4\delta^4(P_f+p_f-P_i-p_i)$$

By integrating over the full final states phase space and dividing by the incident flux of electron, $\frac{v_i}{V}$ and the the time of travel, T, of the electron from the initial state to the final state, one gets a quantity of the dimension of length square referred to as total scattering cross-section and denoted as σ, as below,

$$\sigma = \int \frac{\frac{1}{4}\Sigma_{fi}|S_{fi}|^2 \frac{Vd^3p_f}{(2\pi)^3\hbar^3}\frac{Vd^3P_f}{(2\pi)^3\hbar^3}}{\frac{v_i}{V}T} = \mu_0^2c^2e^4\hbar^6 cVT \int \frac{m_0^2c^4M_0^2c^4}{4E_fE_iE_f^PE_i^PV^4}\frac{V^2}{(2\pi)^6\hbar^6}\frac{1}{\frac{v_i}{V}T}|\vec{p}_f|^2d|\vec{p}_f|d\Omega_f d^3P_f$$
$$\frac{Tr[\gamma_\mu\frac{\not{p}_i+m_0c}{2m_0c}\gamma_\lambda\frac{\not{p}_f+m_0c}{2m_0c}]Tr[\gamma^\mu\frac{\not{P}_i+M_0c}{2M_0c}\gamma^\lambda\frac{\not{P}_f+M_0c}{2M_0c}]}{(p_f-p_i)^4}(2\pi)^4\delta^4(P_f+p_f-P_i-p_i)$$

Dividing by the differential solid angle swept out by the final state electron, one derives the differential scattering cross-section, denoted as $\frac{d\sigma}{d\Omega_f}$ and given by

$$\frac{d\sigma}{d\Omega_f} = \mu_0^2c^2e^4\hbar^6 cVT \int \frac{m_0^2c^4M_0^2c^4}{4E_fE_iE_f^PE_i^PV^4}\frac{V^2}{(2\pi)^6\hbar^6}\frac{1}{\frac{v_i}{V}T}|\vec{p}_f|^2d|\vec{p}_f|d^3P_f$$
$$\frac{Tr[\gamma_\mu\frac{\not{p}_i+m_0c}{2m_0c}\gamma_\lambda\frac{\not{p}_f+m_0c}{2m_0c}]Tr[\gamma^\mu\frac{\not{P}_i+M_0c}{2M_0c}\gamma^\lambda\frac{\not{P}_f+M_0c}{2M_0c}]}{(p_f-p_i)^4}(2\pi)^4\delta^4(P_f+p_f-P_i-p_i)$$

Introducing the identity $\frac{c}{2E_f^P} = \int \theta(P_f^0)\delta(P_f^2-M_0^2)dP_f^0$,

$$\frac{d\sigma}{d\Omega_f} = \mu_0^2c^2e^4\hbar^6 VT \int \frac{m_0^2c^4M_0^2c^4}{2E_fE_iE_i^PV^4 16m_0^2c^2M_0^2c^2}\frac{V^2}{(2\pi)^6\hbar^6}\frac{1}{\frac{v_i}{V}T}\theta(P_f^0)\delta(P_f^2-M_0^2)|\vec{p}_f|^2d|\vec{p}_f|d^4P_f$$
$$\frac{Tr[\gamma_\mu(\not{p}_i+m_0c)\gamma_\lambda(\not{p}_f+m_0c)]Tr[\gamma^\mu(\not{P}_i+M_0c)\gamma^\lambda(\not{P}_f+M_0c)]}{(p_f-p_i)^4}(2\pi)^4\delta^4(P_f+p_f-P_i-p_i)$$
$$= \mu_0^2c^2e^4c^4 \int \frac{1}{32E_fE_iE_i^Pv_i}\frac{1}{(2\pi)^2}\theta(P_f^0)\delta(P_f^2-M_0^2)|\vec{p}_f|^2d|\vec{p}_f|d^4P_f$$
$$\frac{Tr[\gamma_\mu(\not{p}_i+m_0c)\gamma_\lambda(\not{p}_f+m_0c)]Tr[\gamma^\mu(\not{P}_i+M_0c)\gamma^\lambda(\not{P}_f+M_0c)]}{(p_f-p_i)^4}\delta^4(P_f+p_f-P_i-p_i)$$
$$= \mu_0^2c^2e^4c^4 \int \frac{1}{32E_fE_iE_i^Pv_i}\frac{1}{(2\pi)^2}\theta(P_f^0)\delta(P_f^2-M_0^2)|\vec{p}_f|\frac{E_f}{c}d\frac{E_f}{c}d^4P_f$$
$$\frac{32[(P_i.p_i)(P_f.p_f)+(P_i.p_f)(P_f.p_i)-M_0^2c^2(p_i.p_f)-m_0^2c^2(P_i.P_f)+2m_0^2M_0^2c^4]}{(p_f-p_i)^4}\delta^4(P_f+p_f-P_i-p_i)$$
$$= \frac{1}{\epsilon_0^2}e^4 \int \frac{1}{E_fE_iE_i^Pv_i}\frac{4}{(4\pi)^2}\theta(P_f^0)\delta(P_f^2-M_0^2)|\vec{p}_f|E_fdE_fd^4P_f$$
$$\frac{[(P_i.p_i)(P_f.p_f)+(P_i.p_f)(P_f.p_i)-M_0^2c^2(p_i.p_f)-m_0^2c^2(P_i.P_f)+2m_0^2M_0^2c^4]}{(p_f-p_i)^4}\delta^4(P_f+p_f-P_i-p_i)$$
$$= 4(\frac{e^2}{4\pi\epsilon_0})^2 \int \frac{1}{E_iE_i^Pv_i}\theta(P_f^0)\delta(P_f^2-M_0^2)|\vec{p}_f|dE_fd^4P_f$$
$$\frac{[(P_i.p_i)(P_f.p_f)+(P_i.p_f)(P_f.p_i)-M_0^2c^2(p_i.p_f)-m_0^2c^2(P_i.P_f)+2m_0^2M_0^2c^4]}{(p_f-p_i)^4}\delta^4(P_f+p_f-P_i-p_i)$$
$$= 4(\frac{e^2}{4\pi\epsilon_0})^2\frac{1}{E_iE_i^Pv_i} \int \theta(P_i^0+p_i^0-p_f^0)\delta((P_i+p_i-p_f)^2-M_0^2)|\vec{p}_f|dE_f$$
$$\frac{[(P_i.p_i)(P_f.p_f)+(P_i.p_f)(P_f.p_i)-M_0^2c^2(p_i.p_f)-m_0^2c^2(P_i.P_f)+2m_0^2M_0^2c^4]}{(p_f-p_i)^4}|_{P_f=P_i+p_i-p_f}$$

In the laboratory frame, $\vec{P_i}=0$. The differential scattering cross-section, $\frac{d\sigma}{d\Omega_f}|_{lab}$, is thereby expressed as

$$\frac{d\sigma}{d\Omega_f}|_{lab} = 4(\frac{e^2}{4\pi\epsilon_0})^2\frac{1}{M_0c^2E_iv_i}\int_{m_0c^2}^{E_i+M_0c^2}\frac{|\vec{p_f}|dE_f}{(p_f-p_i)^4}\delta(2m_0^2c^2-2\frac{E_i}{c}\frac{E_f}{c}+2\vec{p_i}.\vec{p_f}+2M_0(E_i-E_f))$$
$$[2M_0^2E_fE_i+M_0(p_i.p_f)(E_i-E_f-M_0c^2)+M_0c^2m_0^2[2(E_f-E_i)+M_0c^2]]$$

As a routine, two kinds of approximations at this point make the calculation proceed easily, [2]. This is acheived by either assuming E_f is far away from the proton mass scale or, from the electron mass scale. Here we take a different route. We assume a hypothetical proton with its mass being equal to that of the electron i.e. we set $m_0 = M_0$ and proceed forward. We get,

$$\frac{d\sigma}{d\Omega_f}|_{lab} = 4(\frac{e^2}{4\pi\epsilon_0})^2 \frac{1}{m_0c^2E_iv_i}\int_{m_0c^2}^{E_i+m_0c^2}\frac{|\vec{p_f}|dE_f}{(p_f-p_i)^4}\delta(2m_0^2c^2-2\frac{E_i}{c}\frac{E_f}{c}+2\vec{p_i}.\vec{p_f}+2m_0(E_i-E_f)) \\ [2m_0^2E_fE_i+m_0(p_i.p_f)(E_i-E_f-m_0c^2)+m_0^3c^2[2(E_f-E_i)+m_0c^2]]$$

resorting to non-relativistic limit tentamounting to,

$$E_i = m_0c^2(1+\frac{v_i^2}{2c^2}),$$
$$\frac{E_i}{c}\frac{E_f}{c} = m_0^2c^2+\frac{1}{2}m_0^2(v_i^2+v_f^2),$$
$$E_i-E_f = \frac{1}{2}m_0(v_i^2-v_f^2),$$
$$\vec{p_i}.\vec{p_f} = m_0^2v_iv_fcos\theta,$$

one achieves,

$$\delta(2m_0^2c^2-2\frac{E_i}{c}\frac{E_f}{c}+2\vec{p_i}.\vec{p_f}+2m_0(E_i-E_f)) = \delta(-m_0^2(v_i^2+v_f^2)+2m_0^2v_iv_fcos\theta+m_0^2(v_i^2-v_f^2))$$
$$= \delta(-2m_0^2v_f^2+2m_0^2v_iv_fcos\theta) = \frac{1}{2m_0^2v_f}\delta(-v_f+v_icos\theta)$$
$$(p_f-p_i)^2 = 2m_0^2c^2-2p_i.p_f = 2[m_0^2c^2-(\frac{E_i}{c}\frac{E_f}{c}-\vec{p_i}.\vec{p_f})]$$
$$= m_0^2[-v_i^2-v_f^2+2v_iv_fcos\theta]$$
$$[2m_0^2E_fE_i+m_0(p_i.p_f)(E_i-E_f-m_0c^2)+m_0^3c^2[2(E_f-E_i)+m_0c^2]]$$
$$= m_0^2[3m_0^2c^4+2m_0^2c^2v_f^2+(\frac{E_i}{c}\frac{E_f}{c}-\vec{p_i}.\vec{p_f})(\frac{1}{2}((v_i^2-v_f^2)-c^2)]$$
$$= 2m_0^4c^4$$

Hence, the differential scattering cross-section, in the NR, for two non-identical particles of equal mass and equal but opposite charges, in the laboratory frame i.e. when one particle is at rest initially is given by,

$$\frac{d\sigma}{d\Omega_f}|_{lab} = 4(\frac{e^2}{4\pi\epsilon_0})^2\frac{1}{m_0c^2m_0c^2v_i}\int_{m_0c}^{2m_0c}\frac{m_0v_fm_0v_fdv_f}{m_0^4v_i^4\sin^4\theta}\frac{1}{2m_0^2v_f}\delta(v_f-v_icos\theta)2m_0^4c^4$$
$$= 4(\frac{e^2}{4\pi\epsilon_0})^2\frac{cos\theta}{m_0^2v_i^4\sin^4\theta}$$
$$= 4(\frac{e^2}{4\pi\epsilon_0})^2\frac{cos\theta}{(2E_{lab})^2\sin^4\theta}$$
$$= (\frac{e^2}{4\pi\epsilon_0})^2\frac{cos\theta}{E_{lab}^2\sin^4\theta}$$
$$= \frac{k^2}{E_{lab}^2}\frac{cos\theta}{\sin^4\theta}$$

III. ACKNOWLEDGEMENT

The reference where the second part is done, has not reached the author. Hopefully, nothing new has been presented in the paper.

[1] Herbert Goldstein, Charles Poole, John Safko, *Classical Mechanics* third edition, p110, p119, Pearson, Inc., New Delhi 110017, India. ISBN 978-81-7758-283-3.
[2] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics*(McGraw-Hill Book Compnay, Inc., USA,1964), p132.