Quantizing ϕ^6 Oscillon

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I. THEORY

In flat Minkowski background the Lagrangian is

$$S = \frac{1}{2}\int dxdt\left[\dot{\phi}^2 - (\phi')^2 - m^2\left(\phi^2 - \frac{g}{2}\phi^4 + \frac{g^2}{3}\phi^6\right)\right]. \tag{1}$$

The Hamiltonian of the theory is

$$H = \int dx\frac{1}{2}\left[\dot{\phi}^2 + (\phi')^2 + m^2\left(\phi^2 - \frac{g}{2}\phi^4 + \frac{g^2}{3}\phi^6\right)\right]. \tag{2}$$

The equation of motion is therefore

$$\ddot{\phi} - \phi'' + m^2\left(\phi - g\phi^3 + g^2\phi^5\right) = 0 \tag{3}$$

to which the oscillon solution is

$$\phi(t,x) = \epsilon\sqrt{\frac{8}{3g}}\cos\left(\frac{2\pi t}{\tau}\right)\text{sech}(mx\epsilon) \tag{4}$$

with period $\tau = \frac{2\pi}{m\sqrt{1-\epsilon^2}}$.

II. QUANTIZATION PROCEDURE

To quantize the periodic field we shall proceed as Dashen et al. in their quantization of the breather doublet solution of the Sine-Gordon model.

The derivation is done in detail in Dashen et al. paper and is summarized by Rajaraman in "Solitons and Instantons in Qunatum Field Theory".

Let us define $\tilde{Q}(E) \equiv Tr\left(\frac{1}{E-H}\right)$ and $G(T) \equiv Tr[exp(-iHT/\hbar)] = \int \mathcal{D}[\phi(\boldsymbol{x},t)]exp\left\{\frac{i}{\hbar}S[\phi(\boldsymbol{x},t)]\right\}$ as in Rajaraman.

We then use the Stationary Phase Approximation to pick out periodic classical solutions $\phi_{cl}(x,t)$ from $G(T)$ as described in Rajaraman chapter 6 "Functional integrals and the WKB method" to get

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the allowed quantized energy states analogous to deriving Bohr-Sommerfeld quantization condition in Quantum Mechanics.

A necessary ingredient in quantizing a periodic quantum field are the stability angles ν_i where i labels different solutions to equation (5). They are obtained by solving the following stability equation:

$$\left[-\frac{\partial^2}{\partial t^2} + \nabla^2 - \left(\frac{\partial^2 U}{\partial \phi^2}\right)\bigg|_{\phi_{cl}}\right]\xi(x,t) = 0. \tag{5}$$

Since ϕ_{cl} is periodic the solutions to equation (5) will also be periodic. The stability angle is then given by

$$\xi_i(x, t+\tau) = e^{i\nu_i}\xi_i(x,t) \tag{6}$$

Since the energy will in general diverge we must introduce counter terms to the field Lagrangian in order to renormalize the energy (mass) spectrum.

The energy is shown to be given by expression

$$E = E_{cl}[\phi_{cl}] + E_{ct}[\phi_{cl}] + \sum_{i,p_i}(p_i + \frac{1}{2})\hbar\frac{\partial \nu_i}{\partial \tau}, \qquad \text{where} \;\; p_i = 0,1,2,...,\infty, \tag{7}$$

where the following quantization imposing conditions must be satisfied:

$$W_{\{p_i\}}(E) = 2m\pi\hbar \tag{8}$$

$$W_{\{p_i\}}(E) = S_{cl}[\phi_{cl}] + S_{ct}[\phi_{cl}] + E\tau[\phi_{cl}] - \sum_{i,p_i=0}^{\infty}(p_i + \frac{1}{2})\hbar\nu_i \tag{9}$$

III. IDENTIFYING ALL THE PIECES - CLASSICAL ENERGY AND ACTION

The classical energy of the oscillon field is

$$\begin{aligned} E_{cl}[\phi_{cl}] &= \int dx \frac{1}{2}\left[\dot{\phi}^2 + (\phi')^2 + m^2\left(\phi^2 - \frac{g}{2}\phi^4 + \frac{g^2}{3}\phi^6\right)\right] \\ &= \frac{8m\epsilon}{1215g}\left[-360\epsilon^2\cos^4(mt\sqrt{1-\epsilon^2}) + 512\epsilon^4\cos^6(mt\sqrt{1-\epsilon^2}) + 135\left(3-\epsilon^2+2\epsilon^2\cos(2mt\sqrt{1-\epsilon^2})\right)\right] \\ &= \frac{8m}{3g}\epsilon - \frac{8m}{27g}\left(3 - 6\cos(2mt) + 8\cos^4(mt)\right)\epsilon^3 + O(\epsilon^5) \end{aligned} \tag{10}$$

or in terms of period, if we expand out only the epsilon in the spatial part of the field and only then tau in the first term we get

$$
\begin{aligned}
E_{cl}[\phi_{cl}] &= \int dx \frac{1}{2}\left[\dot{\phi}^2 + (\phi')^2 + m^2\left(\phi^2 - \frac{g}{2}\phi^4 + \frac{g^2}{3}\phi^6\right)\right] = \\
&= \frac{8\epsilon}{3gm\tau^2}\left[m^2\tau^2\cos^2\left(\frac{2\pi t}{\tau}\right) + 4\pi^2\sin^2\left(\frac{2\pi t}{\tau}\right)\right] - \frac{8m\epsilon^3}{27g}\cos^2\left(\frac{2\pi t}{\tau}\right)\left[1 + 4\cos\left(\frac{4\pi t}{\tau}\right)\right] + \frac{4096m\epsilon^5}{1215g}\mathrm{co} \\
&= \frac{8m\epsilon}{3g}\left[\cos^2\left(\frac{2\pi t}{\tau}\right) + (1-\epsilon^2)\sin^2\left(\frac{2\pi t}{\tau}\right)\right] - \frac{8m\epsilon^3}{27g}\cos^2\left(\frac{2\pi t}{\tau}\right)\left[1 + 4\cos\left(\frac{4\pi t}{\tau}\right)\right] + \frac{4096m\epsilon^5}{1215g}\cos^6\Big(\\
&= \frac{8m\epsilon}{3g}\left[1 - \epsilon^2\sin^2\left(\frac{2\pi t}{\tau}\right)\right] - \frac{8m\epsilon^3}{27g}\cos^2\left(\frac{2\pi t}{\tau}\right)\left[1 + 4\cos\left(\frac{4\pi t}{\tau}\right)\right] + \frac{4096m\epsilon^5}{1215g}\cos^6\left(\frac{2\pi t}{\tau}\right) \\
&= \frac{8m}{3g}\epsilon - \frac{8m\epsilon^3}{3g}\sin^2\left(\frac{2\pi t}{\tau}\right) - \frac{8m}{27g}\cos^2\left(\frac{2\pi t}{\tau}\right)\left[1 + 4\cos\left(\frac{4\pi t}{\tau}\right)\right]\epsilon^3 + O(\epsilon^5) \\
&= \frac{8m}{3g}\epsilon - \frac{8m\epsilon^3}{27g}\left(6 - 2\cos\left(\frac{4\pi t}{\tau}\right) + \cos\left(\frac{8\pi t}{\tau}\right)\right)
\end{aligned}
$$

The expression is time dependent because our solution to the equation of motion is only approximate.

The classical action over one period is

$$
\begin{aligned}
S_{cl}[\phi_{cl}] &= \frac{1}{2}\int_0^\tau dt \int_{-\infty}^{\infty} dx \left[\dot{\phi}^2 - (\phi')^2 - m^2\left(\phi^2 - \frac{g}{2}\phi^4 + \frac{g^2}{3}\phi^6\right)\right] = \\
&= \frac{4\epsilon}{3gm\tau}\left(4\pi^2 - m^2\tau^2\right) + \frac{4m\tau\epsilon^3}{9g} - \frac{256m\tau\epsilon^5}{243g} = \\
&= -\frac{8m\tau\epsilon^3}{9g} - \frac{256m\tau\epsilon^5}{243g}
\end{aligned} \tag{12}
$$

We can check that $E_{cl} = -\frac{dS_{cl}}{d\tau}$ and get the part that is not time dependent. The results are consistent up to time dependence of equation (11), so

$$
E_{cl} = -\frac{dS_{cl}}{d\tau} = \frac{8m\epsilon}{3g} - \frac{16m\epsilon^3}{9g}. \tag{13}
$$

Note that we will treat τ and ϵ as independent in future analysis, and therefore no relation connecting them will be used. For example, the relevant expression for classical action that we shall use will be the second line of (12), where the identity $\tau = \frac{2\pi}{m\sqrt{1-\epsilon^2}}$ had not yet been employed.

IV. STABILITY ANGLES

The stability equation is the same taking $\phi = \phi_{cl} + \xi$ and expanding equation of motion in ξ to first order.

Now we need to solve the stability equation to get stability angles.

Assuming $\xi(t,x)$ is separable we can write $\xi = \xi_T(t)\xi_X(x)$. Then the stability equation up to the third order in ϵ keeping period non-expanded is

$$\begin{aligned}
\left[-\partial_t^2 + \partial_x^2 - m^2(1 - 3g\phi_{cl}^2 + 5g^2\phi_{cl}^4\right]\xi(t,x) &= 0 \\
\left[-\partial_t^2 + \partial_x^2 - m^2(1 - 8\cos^2(2\pi t/\tau)\mathrm{sech}^2(mx\epsilon)\epsilon^2\right]\xi(t,x) &= 0 \\
\left[-\partial_t^2 + \partial_x^2 - m^2(1 - 8\cos^2(2\pi t/\tau)\epsilon^2\right]\xi_T\xi_X &= 0 \\
&.
\end{aligned} \tag{14}$$

The x dependence only comes into the equation at the 4th order in ϵ therefore we are essentially solving a wave equation with an oscillating spatially flat potential. After calculating the mass spectrum up to ϵ^3 order, higher order contributions shall also be considered in order to include the x dependence into stability equation.

We get two ODEs after using the method of variable separation:

$$\frac{d^2}{dx^2}\xi_X + C^2\xi_X = 0 \tag{15}$$

$$\frac{d^2}{dt^2}\xi_T + m^2\left(1 + \frac{C^2}{m^2} - 8\epsilon^2\cos^2\left(\frac{2\pi t}{\tau}\right)\right)\xi_T = 0. \tag{16}$$

Following the theory of Mathieu functions in Abramowitz et al.'s "Handbook of Mathematical Functions" the Mathieu equation is of form $y'' + (a - 2q\cos 2z)y = 0$ or in our case

$$\begin{aligned}
&\frac{d^2\xi_T}{dt^2} + \left(m^2 + C^2 - 4\epsilon^2 m^2\left[1 + \cos\left(2\frac{2\pi t}{\tau}\right)\right]\right)\xi_T = \ddot{\xi}_T + \left(\left[C^2 + m^2 - 4\epsilon^2 m^2\right] - 4\epsilon^2 m^2\cos\left(2\frac{2\pi t}{\tau}\right)\right)\xi_T = \\
&\frac{d^2\xi_T}{dz^2}\left(\left[C^2 + m^2 - 4\epsilon^2 m^2\right]\frac{\tau^2}{4\pi^2} - \frac{\tau^2}{\pi^2}\epsilon^2 m^2\cos\left(2\frac{2\pi t}{\tau}\right)\right)\xi_T = 0
\end{aligned} \tag{17}$$

Hence we can identify

$$\begin{aligned}
a &= \frac{(C^2 + m^2 - 4m^2\epsilon^2)\tau^2}{4\pi^2} \\
q &= \frac{m^2\epsilon^2\tau^2}{2\pi^2} \\
z &= \frac{2\pi t}{\tau}
\end{aligned} \tag{18}$$

We can write a general analytic solution in terms of Mathieu functions, the Mathieu cosine function $\mathrm{Ce}[a,q,z]$ and the Mathieu sine function $\mathrm{Se}[a,q,z]$.

$$\xi_T(t) = A_1\mathrm{Ce}\left[\frac{(C^2 + m^2 - 4m^2\epsilon^2)\tau^2}{4\pi^2}, \frac{m^2\epsilon^2\tau^2}{2\pi^2}, \frac{2\pi t}{\tau}\right] + A_2\mathrm{Se}\left[\frac{(C^2 + m^2 - 4m^2\epsilon^2)\tau^2}{4\pi^2}, \frac{m^2\epsilon^2\tau^2}{2\pi^2}, \frac{2\pi t}{\tau}\right] \tag{19}$$

We shall use notation $\mathrm{Se}_{a,q}(z)$ and $\mathrm{Ce}_{a,q}(z)$ and introduce the Mathieu cosine and sine function derivatives with respect to z. These function are called Mathieu cosine or sine prime functions. They will be denoted as $\mathrm{Se}'_{a,q}(z)$ and $\mathrm{Ce}'_{a,q}(z)$.

Using Floquet's Theorem (Bloch Theorem) which states that periodic solutions must exist for wave-like differential equations with periodic potential, there exist solutions to the Mathieu equation of form

$$F_\mu(z) = e^{i\mu z} P(z), \tag{20}$$

where μ depends on a and q, and $P(z)$ is a periodic function with the same period as the potential in the Mathieu equation - namely π. μ is called the characteristic exponent. We can therefore write the general solution to a Mathieu equation in terms of periodic solutions rather than just any Mathieu functions as

$$y = AF_\mu(z) + BF_\mu(-z) \tag{21}$$

We can see that upon transformation $t \to t + \tau$,

$$\begin{aligned} F(z) \to F_\mu\left(\frac{2\pi(t+\tau)}{\tau}\right) &= e^{i\mu\frac{2\pi(t+\tau)}{\tau}} P\left(\frac{2\pi(t+\tau)}{\tau}\right) \\ &= e^{2\pi i\mu} e^{i\mu z} P(z) = e^{2\pi i\mu} F(z) \end{aligned} \tag{22}$$

The way to extract stability angles from stability equation is

$$\xi_i(x, t+\tau) = e^{i\nu_i} \xi_i(x, t) \tag{23}$$

Since ξ_T is an explicit solution to the Mathieu equation we can set $\xi_T = y$ with an appropriate identification of a, q, z and get the stability angles

$$\nu = 2\pi\mu. \tag{24}$$

To solve for μ consider a solution for our fixed a and q to the same Mathieu equation with boundary conditions

$$y_1(0) = 1; \quad y_1'(0) = 0. \tag{25}$$

For such solutions we can see that

$$\cos(\pi\mu) - y_1(\pi) = 0 \tag{26}$$

Solving Mathieu equation with boundary conditions gives an analytic solution

$$y_1(z) = \frac{\mathrm{Ce}'_{a,q}(0)\mathrm{Se}_{a,q}(z) - \mathrm{Ce}_{a,q}(z)\mathrm{Se}'_{a,q}(0)}{\mathrm{Ce}'_{a,q}(0)\mathrm{Se}_{a,q}(0) - \mathrm{Ce}_{a,q}(0)\mathrm{Se}'_{a,q}(0)} \tag{27}$$

with the same variables a, q and z as before.

Hence

$$\mu = \pm\frac{\arccos y_1(\pi)}{\pi} + 2k; \quad k \in \mathbb{Z} \tag{28}$$

and so all possible candidates for stability angles are

$$\begin{aligned} \nu_i &= \pm 2\arccos y_1(\pi) + 4\pi i; \quad i \in \mathbb{Z} \equiv \\ \equiv \nu_i^{\pm} &= \pm 2\arccos\left(\frac{\mathrm{Ce}'_{a,q}(0)\mathrm{Se}_{a,q}(\pi) - \mathrm{Ce}_{a,q}(\pi)\mathrm{Se}'_{a,q}(0)}{\mathrm{Ce}'_{a,q}(0)\mathrm{Se}_{a,q}(0) - \mathrm{Ce}_{a,q}(0)\mathrm{Se}'_{a,q}(0)}\right) + 4\pi i; \quad i \in \mathbb{Z} \end{aligned} \tag{29}$$

Any solution for characteristic exponent μ will have the discrete term $2k$ because of the periodicity of cosine and sine. These do not correspond to distinct stability angles so we do not need to take them into account. The first two positive angles (as we only need to sum over positive stability angles (See Dashen et al.) are

$$\begin{aligned} \nu_1 &= 2\mathrm{arccos} y_1(\pi) \\ \nu_2 &= -2\mathrm{arccos} y_1(\pi) + 4\pi \end{aligned} \tag{30}$$

However, the angle ν_2 is just a result corresponding to the complex conjugate case of the exponential, hence not a separate stability angle (same as Dashen et al. treatment). Therefore the only stability angle is ν_1. Otherwise we would always get cancellation between these two solutions as one is a complex conjugate pair of the complex phase in $e^{i\nu}$. Also note that our stability angles must be real in order to have stable, non-decaying solutions to the stability equation.

As in Rajaraman and Dashen we are only interested in the basic oscillon state so set for all i, $p_1 = 0$. We have a continuous set of stability angles depending on the value of continuous parameter C which comes into solutions as a separation constant, assuming that we can indeed treat τ and ϵ as independent and that ϵ is a fixed small parameter.

The only stability angle is given by

$$\begin{aligned} \cos\frac{\nu}{2} &= \frac{\mathrm{Ce}'_{a,q}(0)\mathrm{Se}_{a,q}(\pi) - \mathrm{Ce}_{a,q}(\pi)\mathrm{Se}'_{a,q}(0)}{\mathrm{Ce}'_{a,q}(0)\mathrm{Se}_{a,q}(0) - \mathrm{Ce}_{a,q}(0)\mathrm{Se}'_{a,q}(0)} = \\ &= \frac{\mathrm{Ce}_{a,q}(\pi)}{\mathrm{Ce}_{a,q}(0)}, \end{aligned} \tag{31}$$

since $\mathrm{Ce}'_{a,q}(0) = 0$.

Therefore we find a simple formula for Mathieu characteristic exponent which we re-expressed as the stability angle. The formula for characteristic exponent is

$$\cos(\mu\pi) = \frac{\mathrm{Ce}_{a,q}(\pi)}{\mathrm{Ce}_{a,q}(0)} \tag{32}$$

which should be correct for all real characteristic exponents for any a, q.

Mathieu Cosine function can be expanded around $q = 0$ using a well known formula

$$\mathrm{Ce}[a_\mu(q), q, z] = \cos(\mu z) + \frac{1}{4}\left(\frac{\cos((\mu-2)z)}{\mu-1} - \frac{\cos((\mu+2)z)}{\mu+1}\right) q + ... \tag{33}$$

where μ is the characteristic exponent, and $a_\mu(q)$ is the characteristic value, or parameter a, which can in general be uniquely determined from μ and q. So the knowledge of a and q determines μ, and similarly knowing μ and q we can determine a. We can therefore use this expression to verify our derived result for characteristic exponent directly from a known expansion and see that they in fact agree.

$$\begin{aligned}
\cos(\mu\pi) &= \frac{\cos(\mu\pi) + \frac{1}{4}\left(\frac{\cos((\mu-2)\pi)}{\mu-1} - \frac{\cos((\mu+2)\pi)}{\mu+1}\right)\frac{m^2\tau^2\epsilon^2}{2\pi^2}}{1 + \frac{1}{4}\left(\frac{1}{\mu-1} - \frac{1}{\mu+1}\right)\frac{m^2\tau^2\epsilon^2}{2\pi^2}} \\
\frac{\cos(\mu\pi)}{\mu-1} - \frac{\cos(\mu\pi)}{\mu+1} &= \frac{\cos((\mu-2)\pi)}{\mu-1} - \frac{\cos((\mu+2)\pi)}{\mu+1} \\
0 &= 0
\end{aligned} \tag{34}$$

For small q we can use the identity given by Abramowitz and Stegun 20.3.15 which is

$$a = \mu^2 + \frac{q^2}{2(\mu^2-1)} + \frac{\left(5\mu^2+7\right)q^4}{32(\mu^2-1)^3(\mu^2-4)} + ... \tag{35}$$

Using terms up to q^2, and solving the quartic equation for μ gives two solutions $+1$, -1 and $\pm\left(\frac{\tau\sqrt{C^2+m^2}}{2\pi} - \frac{m^2\tau\epsilon^2}{\pi\sqrt{C^2+m^2}}\right)$. Taking the positive solution we have the expression for the stability angles up to order ϵ^2:

$$\nu = \tau\sqrt{C^2+m^2} - \frac{2m^2\tau\epsilon^2}{\sqrt{C^2+m^2}} \tag{36}$$

We get the same result as we would taking $q \to 0$ and expanding $\sqrt{a}$ for characteristic exponent.

We could impose periodic boundary conditions on the stability equation which would give as in the Sine-Gordon case

$$LC_n = 2n\pi, \tag{37}$$

with L the box size. When taking $L \to \infty$ the sum over all stability angles becomes an integral over all values of C.

This integral will have a quadratically and a logarithmically divergent term.

V. SUMMING THE STABILITY ANGLES

Let us insert the expression for ϵ in terms of τ into the equation for stability angles and action. When expanding in terms of ϵ the insertion does not matter since τ is of order 1. Solving for ϵ from the identity for τ there are two solutions. However, since we have to square epsilon, it does not matter which one we take as they are + and - the same expression.

The stability angle is then

$$\begin{aligned} \nu &= \tau\sqrt{C^2+m^2} - \frac{2\left(m^2\tau^2-4\pi^2\right)}{\tau\sqrt{C^2+m^2}} \\ &= \frac{2\pi\sqrt{C^2+m^2}}{m} + \frac{\left(C^2-3m^2\right)\pi\epsilon^2}{m\sqrt{C^2+m^2}} \end{aligned} \tag{38}$$

Its derivative with respect to the period is

$$\frac{d\nu}{d\tau} = \sqrt{C^2+m^2} - \frac{2\left(m^2\tau^2+4\pi^2\right)}{\tau^2\sqrt{C^2+m^2}} \tag{39}$$

The sum over all stability angles is now an integral over all values of C. Since the integral will diverge we can introduce a cut-off which we later take to infinity.

$$\begin{aligned} \frac{d}{d\tau}\sum\nu &= \int_{-\infty}^{\infty}\frac{d\nu}{d\tau}dC \to \\ &\to \int_{-\Lambda}^{\Lambda}\frac{d\nu}{d\tau}dC \end{aligned} \tag{40}$$

This gives:

$$\begin{aligned} \int_{-\Lambda}^{\Lambda}\frac{d\nu}{d\tau}dC &= \int_{-\Lambda}^{\Lambda}\left[\sqrt{C^2+m^2} - \frac{2\left(m^2\tau^2+4\pi^2\right)}{\tau^2\sqrt{C^2+m^2}}\right]dC \\ &= \Lambda\sqrt{\Lambda^2+m^2} - \frac{16\pi^2}{\tau^2}\ln\left(\frac{\Lambda+\sqrt{\Lambda^2+m^2}}{m}\right) - m^2\left[\text{arcsinh}\frac{\Lambda}{m} + 2\ln\left(\frac{\Lambda+\sqrt{\Lambda^2+m^2}}{m}\right)\right] \end{aligned} \tag{41}$$

Now notice that

$$\text{arcsinh}\frac{\Lambda}{m} = \ln\left(\frac{\Lambda+\sqrt{\Lambda^2+m^2}}{m}\right). \tag{42}$$

Hence

$$\int_{-\Lambda}^{\Lambda} \frac{d\nu}{d\tau} dC = \Lambda\sqrt{\Lambda^2+m^2} - \frac{16\pi^2 + 3m^2\tau^2}{\tau^2} \ln\left(\frac{\Lambda + \sqrt{\Lambda^2+m^2}}{m}\right), \tag{43}$$

and

$$\frac{1}{2}\frac{d}{d\tau}\sum \nu = \lim_{\Lambda\to\infty}\left[\frac{1}{2}\Lambda\sqrt{\Lambda^2+m^2} - \left(\frac{8\pi^2}{\tau^2} + \frac{3m^2}{2}\right)\ln\left(\frac{\Lambda+\sqrt{\Lambda^2+m^2}}{m}\right)\right]. \tag{44}$$

Or without the derivative:

$$\frac{1}{2}\sum \nu = \lim_{\Lambda\to\infty}\left[\frac{1}{2}\Lambda\sqrt{\Lambda^2+m^2}\tau + \left(\frac{8\pi^2}{\tau} - \frac{3m^2\tau}{2}\right)\ln\left(\frac{\Lambda+\sqrt{\Lambda^2+m^2}}{m}\right)\right]. \tag{45}$$

Integrating without any epsilon - tau substitutions, with periodic BC

$$\begin{aligned}
\frac{1}{2}\sum_n \nu &= \sum_n \left[\frac{\tau}{2}\sqrt{C_n^2+m^2} - \frac{m^2\tau\epsilon^2}{\sqrt{C_n^2+m^2}}\right] \\
&\to E_0\tau - \frac{L}{2\pi}\int_{-\infty}^{\infty} dC \left[\frac{m^2\tau\epsilon^2}{\sqrt{C^2+m^2}}\right] \\
&= E_0\tau - \frac{L}{2\pi}\lim_{\Lambda\to\infty}\int_{-\Lambda}^{\Lambda} dC\left[\frac{m^2\tau\epsilon^2}{\sqrt{C^2+m^2}}\right] \\
&= E_0\tau - \frac{Lm^2\tau\epsilon^2}{\pi}\ln\left(\frac{\Lambda+\sqrt{\Lambda^2+m^2}}{m}\right)
\end{aligned} \tag{46}$$

VI. RENORMALISATION

In order to get the energy

$$E = -\frac{d}{d\tau}\left[S_{cl} + S_{ct} - \frac{1}{2}\sum_i \nu_i\right], \tag{47}$$

we must introduce counter terms and cancel the divergences from the sum of stability angles. The quadratically divergent piece is exactly the vacuum energy of the theory so will vanish. The more problematic is the logarithmically divergent piece.

Due to the field strength renormalisation which normally comes into the two-point function, we can use the standard trick and replace $\phi = Z^{1/2}\phi_r$ in the Lagrangian and write the bear mass m_0 and g_0 instead of the physical mass and coupling constant. Then we can introduce the standard:

$$\begin{aligned}
\delta_Z &= Z - 1 \\
\delta_m &= Zm_0^2 - m^2 \\
\delta_g &= Z^2 m_0^2 g_0 - m^2 g.
\end{aligned} \tag{48}$$

Inserting this into Lagrangian:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}m_0^2\phi^2 + \frac{1}{4}m_0^2 g_0 \phi^4 \tag{49}$$

We do not need to worry about ϕ^6 term since when we insert the counter terms into the WKB energy equation, terms will have to be evaluated at the classical field solution. And since we are only solving up to order less than $O(\epsilon^4)$, we can leave the ϕ^6 term out.

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}m_0^2\phi^2 + \frac{1}{4}m_0^2 g_0 \phi^4 = \\
&= \frac{1}{2}Z\left(\partial_\mu \phi_r\right)^2 - \frac{1}{2}Zm_0^2\phi_r^2 + \frac{1}{4}Z^2 m_0^2 g_0 \phi_r^4 = \\
&= \frac{1}{2}\left(\partial_\mu \phi_r\right)^2 - \frac{1}{2}m^2\phi_r^2 + \frac{1}{4}m^2 g \phi_r^4 + \\
&\quad + \frac{1}{2}\delta_Z\left(\partial_\mu \phi_r\right)^2 - \frac{1}{2}\delta_m\phi_r^2 + \frac{1}{4}\delta_g \phi_r^4
\end{aligned} \tag{50}$$

We can proceed exactly as in Peskin chapter 10.2 where he find counter terms for the standard ϕ^4 theory. We only need to make suitable identification of parameters between our Lagrangian and the standard phi to the 4 Lagrangian.

The identification is:

$$\begin{aligned}
\phi^4 &= oscillon \\
\lambda &= -6m^2 g \\
\delta_\lambda &= -6\delta_g
\end{aligned} \tag{51}$$

Then by analyzing 2 -¿ 2 scattering we can find the counter terms.

The renormalisation conditions are the usual ones as in Peskin pp. 325.

Now

$$i\mathcal{M} = 6im^2 g + \left(6im^2 g\right)^2 \left[iV(s) + iV(t) + iV(u)\right] + 6i\delta_g, \tag{52}$$

where

$$iV(p^2) = \frac{i}{2}\int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2 - m^2}\frac{i}{(k+p)^2 - m^2}. \tag{53}$$

The renormalisation condition demands that the amplitude equals $6im^2 g$ at $s = 4m^2$ and $t = u = 0$.

Therefore

$$\delta_g = 6m^4 g^2 \left[V(4m^2) + 2V(0)\right] \tag{54}$$

Computing this integral gives after writing $l = k + xp$, Wick rotating and writing $\Delta = m^2 - x(1-x)p^2$:

$$\begin{aligned} V(p^2) &= \frac{i}{2}\int_0^1 dx \int \frac{d^2 l}{(2\pi)^2}\frac{1}{[l^2 + x(1-x)p^2 - m^2]^2} \\ &= -\frac{1}{2}\int_0^1 dx \int \frac{d^2 l_E}{(2\pi)^2}\frac{1}{\left[l_E^2 + \Delta\right]^2} \\ &= -\frac{1}{4\pi}\int_0^1 dx \int_0^\infty \frac{l_E dl_E}{\left[l_E^2 + \Delta\right]^2}. \end{aligned} \tag{55}$$

We can replace the integration limit ∞ with Λ which we later take to infinity. Now:

$$\begin{aligned} V(p^2) &= -\frac{1}{4\pi}\lim_{\Lambda\to\infty}\int_0^1 dx \frac{\Lambda^2}{2\Delta\left(\Delta + \Lambda^2\right)} \\ &= -\frac{1}{4\pi}\lim_{\Lambda\to\infty}\int_0^1 dx \frac{\Lambda^2}{2\left(m^2 - x(1-x)p^2\right)\left[\left(m^2 - x(1-x)p^2\right) + \Lambda^2\right]}. \end{aligned} \tag{56}$$

So we have:

$$\begin{aligned} V(4m^2) &= -\frac{1}{4\pi}\lim_{\Lambda\to\infty}\int_0^1 dx \frac{\Lambda^2}{2\left(m^2 - 4m^2 x(1-x)\right)\left[\left(m^2 - 4m^2 x(1-x)\right) + \Lambda^2\right]} \\ &= -\frac{1}{4\pi}\lim_{\Lambda\to\infty}\int_0^1 dx \frac{\Lambda^2}{2\left(m - 2mx\right)^2\left[\left(m - 2mx\right)^2 + \Lambda^2\right]} \\ &= -\frac{1}{8\pi}\lim_{\Lambda\to\infty}\left[\frac{1}{2m\left(m - 2mx\right)} + \frac{\arctan\left(\frac{m-2mx}{\Lambda}\right)}{2m\Lambda}\right]_0^1 \\ &= -\frac{1}{8\pi}\lim_{\Lambda\to\infty}\left[-\frac{1}{2m^2} - \frac{1}{2m^2} + \frac{\arctan\left(-\frac{m}{\Lambda}\right)}{2m\Lambda} - \frac{\arctan\left(\frac{m}{\Lambda}\right)}{2m\Lambda}\right] \\ &= \frac{1}{8\pi m^2} \end{aligned} \tag{57}$$

and

$$\begin{aligned} V(0) &= -\frac{1}{4\pi}\lim_{\Lambda\to\infty}\int_0^1 dx \frac{\Lambda^2}{2m^2\left[m^2 + \Lambda^2\right]} \\ &= -\frac{1}{8\pi m^2}. \end{aligned} \tag{58}$$

Hence

$$\begin{aligned}\delta_g &= 6m^4g^2\left[V(4m^2)+2V(0)\right]\\ &= 6m^4g^2\left[\frac{1}{8\pi m^2}-\frac{1}{4\pi m^2}\right]\\ &= -\frac{3g^2m^2}{4\pi}\end{aligned} \tag{59}$$

Now we need to get the mass renormalization factor. Proceeding as in Peskin, we need to consider the self energy loop diagram. One can show that,

$$0 = 3im^2g\int\frac{d^2k}{(2\pi)^2}\frac{i}{k^2-m^2} - i\delta_m. \tag{60}$$

Wick rotating we get:

$$\begin{aligned}\delta_m &= 3im^2g\int\frac{d^2k}{(2\pi)^2}\frac{1}{k^2-m^2}\\ &= 3m^2g\int\frac{d^2l}{(2\pi)^2}\frac{1}{l^2+m^2}\\ &= \frac{3m^2g}{2\pi}\lim_{\Lambda\to\infty}\int_0^\Lambda\frac{ldl}{l^2+m^2}\\ &= \frac{3m^2g}{4\pi}\lim_{\Lambda\to\infty}\ln\left(\frac{\Lambda^2+m^2}{m^2}\right)\\ &\to \frac{3m^2g}{2\pi}\lim_{\Lambda\to\infty}\ln\left(\frac{\Lambda}{m}\right)\end{aligned} \tag{61}$$

It is actually better to take what is given in Rajaraman which gives mass renormalisation:

$$\begin{aligned}\delta_m &= \frac{3m^2g}{2\pi}\int_0^\Lambda\frac{dk}{\sqrt{k^2+m^2}}\\ &= \frac{3m^2g}{2\pi}\ln\left(\frac{\Lambda+\sqrt{\Lambda^2+m^2}}{m}\right)\end{aligned} \tag{62}$$

We now have both counter terms for the renormalized theory. They are:

$$\begin{aligned}\delta_g &= -\frac{3m^2g^2}{4\pi}\\ \delta_m &= \frac{3m^2g}{2\pi}\lim_{\Lambda\to\infty}\ln\left(\frac{\Lambda+\sqrt{\Lambda^2+m^2}}{m}\right)\end{aligned} \tag{63}$$

VII. CANCELLATION OF DIVERGENCES

The counter term contribution to action is (note that we will take which ever sign coming from square roots (+ or -) will be convenient for canceling divergences):

$$\begin{aligned} S_{ct} &= -\frac{1}{2}\delta_m \int_0^\tau dt \int_{-\infty}^{\infty} dx \phi_{cl}^2 + \frac{1}{4}\delta_g \int_0^\tau dt \int_{-\infty}^{\infty} dx \phi_{cl}^4 \\ &= -\frac{4\tau\epsilon}{3gm}\delta_m + \frac{8\tau\epsilon^3}{9g^2m}\delta_g \\ &= -\frac{2m\tau\epsilon}{\pi}\ln\left(\frac{\Lambda+\sqrt{\Lambda^2+m^2}}{m}\right) - \frac{2m\tau\epsilon^3}{3\pi} \end{aligned} \tag{64}$$

We can neglect the ϵ^3 term coming from the g renormalization as no result to such high order in epsilon can be trusted. We can only work up to epsilon squared.

Now

$$\begin{aligned} S_{ct} - \frac{1}{2}\sum \nu &= -\frac{4\tau\epsilon}{3gm}\delta_m + m^2\tau\epsilon^2 \sum_n \frac{1}{\sqrt{C_n^2+m^2}} - \frac{\tau}{2}\sum_n \sqrt{C_n^2+m^2} + \tau E_0 \\ &= -\frac{2m\tau\epsilon}{\pi}\int_0^\infty \frac{dk}{\sqrt{k^2+m^2}} + 2m^2\tau\epsilon^2 \int_0^\infty \frac{dn}{\sqrt{C_n^2+m^2}} \\ &= \end{aligned} \tag{65}$$

$$L = \frac{2}{m\epsilon} \tag{66}$$

The same result arrises from dimensional regularization. I haven't written that up yet.

VIII. DIMENSIONAL REGULARISATION AND INFINITY CANCELLATION

Take $d = 2 - 2\alpha$

$$\begin{aligned} S_{ct} &= -\frac{1}{2}\delta_m \int_0^\tau dt \int_{-\infty}^{\infty} dx \phi_{cl}^2 + \frac{1}{4}\delta_g \int_0^\tau dt \int_{-\infty}^{\infty} dx \phi_{cl}^4 \\ &= -\frac{4\tau\epsilon}{3gm}\delta_m + \frac{8\tau\epsilon^3}{9g^2m}\delta_g \\ &= -4im\tau\epsilon \int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2-m^2} + \mathcal{O}(\epsilon^3) \\ &= -4im\tau\epsilon \left[\frac{-i}{4\pi}\frac{\Gamma(\alpha)}{\Gamma(1)}\left(\frac{1}{m^2}\right)^\alpha\right] \\ &= -\frac{m\tau\epsilon}{\pi}\left[\frac{1}{\alpha} - \gamma + \mathcal{O}(\alpha)\right] \end{aligned} \tag{67}$$

Now the sum over stability angles with the dimensions $d/2$ from the d we used in the mass renormalisation. To use the same parameter α we need to take $d = 1 - \alpha$.

$$\begin{aligned}\frac{1}{2}\sum_n \nu &= \sum_n \left[\frac{\tau}{2}\sqrt{C_n^2+m^2} - \frac{m^2\tau\epsilon^2}{\sqrt{C_n^2+m^2}}\right] \\ &\rightarrow E_0\tau - \frac{L}{2\pi}\int_{-\infty}^{\infty} dC \left[\frac{m^2\tau\epsilon^2}{\sqrt{C^2+m^2}}\right]\end{aligned} \tag{68}$$

Since C is in Euclidean space we need to first rotate it to "some sort of 1 dimensional" Minkowski space. Take $C \rightarrow -iC$

$$\begin{aligned}&\rightarrow E_0\tau + \frac{L}{2\pi}\int_{-\infty}^{\infty} dC \left[\frac{m^2\tau\epsilon^2}{\sqrt{C^2-m^2}}\right] \\ &= E_0\tau + Lm^2\tau\epsilon^2 \left[\frac{(-1)^{1/2}i}{(4\pi)^{1/2}}\frac{\Gamma(\alpha/2)}{\Gamma(1/2)}\left(\frac{1}{m^2}\right)^{\alpha/2}\right] \\ &= E_0\tau - \frac{Lm^2\tau\epsilon^2}{2\pi}\left[\frac{2}{\alpha} - \gamma + \mathcal{O}(\alpha)\right] \\ &= E_0\tau - \frac{Lm^2\tau\epsilon^2}{\pi}\left[\frac{1}{\alpha} - \gamma + \mathcal{O}(\alpha)\right]\end{aligned} \tag{69}$$

Then including the vacuum energy

$$\begin{aligned}S_{ct} - \frac{1}{2}\sum \nu &= E_0\tau - \frac{m\tau\epsilon}{\pi}\left[\frac{1}{\alpha} - \gamma\right] - E_0\tau + \frac{Lm^2\tau\epsilon^2}{\pi}\left[\frac{1}{\alpha} - \gamma\right] \\ &= \frac{Lm^2\tau\epsilon^2}{\pi\alpha} - \frac{m\tau\epsilon}{\pi\alpha}\end{aligned} \tag{70}$$

This gives

$$L = \frac{1}{m\epsilon} \tag{71}$$

(?????) should really be 2 due to definition of d = 2 - e.

Possible explanation could be that L comes in from the stability equation. And the stability equation knows nothing about the localisation of the oscillon. It treats the stability as an oscillating background.

Write the explanation containing the non-locality of stability equation.

IX. PUTTING TOGETHER THE PIECES

No ϵ^3 results can be trusted since we saw that there is no energy conservation at that level, as our classical solution of the field is not valid at orders that high. Another reason is that the stability angles which would next have ϵ^4 term would became of third order in epsilon as we integrated over C.

Therefore we can write classical action as:

$$\begin{aligned} S_{cl} &= \frac{4\epsilon}{3gm\tau}\left(4\pi^2 - m^2\tau^2\right) + \frac{4m\tau\epsilon^3}{9g} - \frac{256m\tau\epsilon^5}{243g} = \\ &= \frac{16\pi^2\epsilon}{3gm\tau} - \frac{4m\tau\epsilon}{3g} \\ &= \frac{4\left(4\pi^2 - m^2\tau^2\right)}{3gm\tau}\sqrt{1 - \frac{4\pi^2}{m^2\tau^2}} \end{aligned} \tag{72}$$

We saw that counter terms cancelled the sum over stability angles exactly. Even if they give us an additional finite value, the result will only differ slightly hence it is worth first seeing the mass spectrum in the exact cancellation case.

The expression which gives us WKB corrected quantum energy is therefore:

$$\begin{aligned} E &= -\frac{d}{d\tau}\left[S_{cl} + S_{ct} - \frac{1}{2}\sum_i \nu_i\right] \\ &= -\frac{dS_{cl}}{d\tau} = -\frac{d}{d\tau}\left[\frac{4\left(4\pi^2 - m^2\tau^2\right)}{3gm\tau}\sqrt{1 - \frac{4\pi^2}{m^2\tau^2}}\right] \\ &= \frac{4\left(8\pi^2 + m^2\tau^2\right)}{3gm\tau^2}\sqrt{1 - \frac{4\pi^2}{m^2\tau^2}} \end{aligned} \tag{73}$$

Now we shall use this expression for energy in terms of purely the period, to impose the quantization:

$$\begin{aligned} 2\pi N = W &= E\tau[E] + S_{cl} \\ &= \frac{4\left(8\pi^2 + m^2\tau^2\right)}{3gm\tau}\sqrt{1 - \frac{4\pi^2}{m^2\tau^2}} + \frac{4\left(4\pi^2 - m^2\tau^2\right)}{3gm\tau}\sqrt{1 - \frac{4\pi^2}{m^2\tau^2}} \\ &= \frac{16\pi^2}{gm\tau}\sqrt{1 - \frac{4\pi^2}{m^2\tau^2}}. \end{aligned} \tag{74}$$

We now have a system of 2 equation relating E to τ, and τ to N.

$$\begin{aligned} E &= \frac{4\left(8\pi^2 + m^2\tau^2\right)}{3gm\tau^2}\sqrt{1 - \frac{4\pi^2}{m^2\tau^2}} \\ N &= \frac{8\pi}{gm\tau}\sqrt{1 - \frac{4\pi^2}{m^2\tau^2}} \end{aligned} \tag{75}$$

By eliminating τ we get a quartic equation for the WKB quantised energy (correct up to order including ϵ^2)

$$81g^2E^4 - 36m^2\left(4 + 3g^2N^2\right)E^2 + m^4N^2\left(12 + g^2N^2\right)^2 = 0 \tag{76}$$

with solutions:

$$E_N = \pm\frac{m}{3g}\sqrt{8 + 6g^2N^2 \pm (4 - g^2N^2)^{3/2}} \tag{77}$$

Since energy has to be positive we can immediately eliminate 2 solutions. The remaining two are:

$$E_N = \frac{m}{3g}\sqrt{8 + 6g^2N^2 \pm (4 - g^2N^2)^{3/2}} \tag{78}$$

Since this is the rest energy it is the mass of the oscillon. The solution we can trust more since it has lower energy and would be more stable is:

$$M_N = \frac{m}{3g}\sqrt{8 + 6g^2N^2 - (4 - g^2N^2)^{3/2}} \tag{79}$$

where $N = 0,\ 1,\ 2,\ ...\ \frac{2}{g}$, since the energy has to be real. In the Sine-Gordon breather case the rest energy also went to 0 for $N = 0$. If we are to give any interpretation in terms of a number of particles then this is to be expected.

$$M_1 = \frac{m}{3g}\sqrt{8 + 6g^2 - (4 - g^2)^{3/2}} \tag{80}$$

then we can write in the weak coupling case of small g,

$$M_1 = m - \frac{mg^2}{96} + \mathcal{O}(g^4) \tag{81}$$

For a Nth state in terms of the 1st state we can write in the weak coupling limit:

$$\begin{aligned} M_N &= M_1\sqrt{\frac{8 + 6g^2N^2 - (4 - g^2N^2)^{3/2}}{8 + 6g^2 - (4 - g^2)^{3/2}}} \\ &= M_1 N - \frac{g^2}{96}M_1\left(N^3 - N\right) - \mathcal{O}\left(g^4\right) \end{aligned} \tag{82}$$

We can compare this seemingly unrelated mass spectrum to the Sine-Gordon breather model which has a Lagrangian:

$$\begin{aligned} \mathcal{L}_{\mathcal{SG}} &= \frac{1}{2}\left(\partial_\mu\phi\right)^2 + \frac{m^4}{\lambda}\left(\cos\left(\frac{\sqrt{\lambda}}{m}\phi\right) - 1\right) \\ &= \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{m^2}{2}\phi^2 + \frac{\lambda}{24}\phi^4 \\ &= \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{m^2}{2}\phi^2 + \frac{m^2g'}{4}\phi^4 \end{aligned} \tag{83}$$

where we can write $g' = \frac{\lambda}{6m^2}$, to make Lagrangian more similar to our oscillon Lagrangian.

The mass spectrum of a Sine-Gordon breather is (all written with primes):

$$M_{SG} = \frac{16m}{\gamma} \sin\left(\frac{N'\gamma}{16}\right) \tag{84}$$

where

$$\begin{aligned} \gamma &= \frac{\lambda/m^2}{1-\lambda/(8\pi m^2)} \\ &= \frac{6g'}{1-\frac{3g'}{4\pi}} \end{aligned} \tag{85}$$

In weak coupling the mass of 1st state for Sine-Gordon is:

$$M_1' = m - \frac{3mg'^2}{128} \tag{86}$$

then

$$\begin{aligned} M'_{N'} &= M_1'N' - \frac{1}{6}\left(\frac{\lambda}{16m^2}\right)^2 M_1'\left(N'^3 - N'\right) \\ &= M_1'N' - \frac{3g'^2}{128} M_1'\left(N'^3 - N'\right) \end{aligned} \tag{87}$$

We can therefore see that our oscillon mass spectrum is identical to Sine-Gordon, to order g^2, with identification of the coupling constants:

$$\begin{aligned} \text{Oscillon} &= \mathcal{S} - \mathcal{G} \\ \frac{g^2}{96} &= \frac{3g'^2}{128} \\ \rightarrow & \\ g = \pm\frac{3g'_{SG}}{2} &= \pm\frac{\lambda}{4m^2} \end{aligned} \tag{88}$$

We can therefore interpret our oscillon the same way as the Sine-Gordon breather in the weak coupling limit. This means that we can interpret it as a bound state of N particles with a slightly lower binding energy.

X. SINE-GORDON BREATHER

We have established that if we write the Sine-Gordon equation with g' coupling analogous to our phi to the 6 model there is a correspondence in the weak coupling limit to the coupling λ in the S-G model.

Now the breather field is also the same in the small ϵ approximation if we use the oscillon identity

$$m\tau = \frac{2\pi}{\sqrt{1-\epsilon^2}} \tag{89}$$

with abbreviation

$$\tilde{\tau} = \frac{m\tau}{2\pi} \tag{90}$$

Then

$$\begin{aligned}
\phi &= \frac{4m}{\sqrt{\lambda}} \arctan\left[\sqrt{\tilde{\tau}^2-1}\frac{\sin\left(\frac{mt}{\tilde{\tau}}\right)}{\cosh\left(mx\sqrt{\tilde{\tau}^2-1}/\tilde{\tau}\right)}\right] \\
&= \frac{4m}{\sqrt{\lambda}} \arctan\left[\frac{\epsilon}{\sqrt{1-\epsilon^2}} \sin\left(\frac{2\pi t}{\tau}\right) \operatorname{sech}(mx\epsilon)\right] \\
&\simeq \frac{4m\epsilon}{\sqrt{\lambda}} \sin\left(\frac{2\pi t}{\tau}\right) \operatorname{sech}(mx\epsilon) \\
&= \epsilon\sqrt{\frac{8}{3g'}} \sin\left(\frac{2\pi t}{\tau}\right) \operatorname{sech}(mx\epsilon)
\end{aligned} \tag{91}$$

Now for the stability angles

$$\nu = \tau\sqrt{m^2+q_n^2} \tag{92}$$

$$Lq_n + 4\arctan\left(\frac{m\sqrt{\tilde{\tau}^2-1}}{q\tilde{\tau}}\right) = 2\pi n \tag{93}$$

We can expand up to ϵ^2, solve solve for q, insert it into the equation for ν, expand again and we get using $LC_n = 2\pi n$

$$\begin{aligned}
\nu &= \tau\sqrt{m^2+\frac{4\pi^2 n^2}{L^2}} - \frac{4m\tau\epsilon}{L\sqrt{m^2+\frac{4\pi^2n^2}{L^2}}} \\
&= \tau\sqrt{m^2+C_n^2} - \frac{4m\tau\epsilon}{L\sqrt{m^2+C_n^2}}
\end{aligned} \tag{94}$$

This is again consistent with our oscillon stability angle upon identification

$$L = \frac{2}{m\epsilon}. \tag{95}$$

We therefore have more or less all ingredients exactly dual in both of the models.

XI. QUANTIZATION OF GENERAL OSCILLONS

We can say that an oscillon should go to vacuum expectation value

$$\phi(x) \to \phi_0 = \langle 0 \mid \phi \mid 0 \rangle, \tag{96}$$

when its period goes to

$$\tau \to \frac{2\pi}{m}, \tag{97}$$

where m is the mass in the Lagrangian.

XII. QUANTISATION OF GENERAL OSCILLONS

A. General Theory

We can apply the semi-classical quantization to an oscillon $\phi(x)$ by splitting the fields into rectangles of small width, similar to performing a Riemann integral by splitting the curve into rectangles and summing over them. Then each rectangle acts as a spatially homogeneous field and the results derived through the Mathieu equation can be used to obtain local quantization. Then all these regions must be summed over and a hopefully meaningful result can be obtained.

To obtain the stability angles we need to solve a partial differential equation of form

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - \sum_{n=0}^{N} a_n \phi^n(x,t) \right] \xi(x,t) = 0, \tag{98}$$

where $a_0 = m$ is the mass and ϕ the oscillon solution to the equations of motion.

Let us define an open-set rectangular function $\Pi(x)$ which takes the value 1 in the open-set neighbourhood $(x-\delta, x+\delta)$ for some $0 < \delta \ll 1$ and is 0 elsewhere. Then we can use this function to discretize an otherwise smooth (at least twice differentiable) oscillon solution by writing

$$\phi(x,t) = \sum_{y=-\infty}^{\infty} \bar{\phi}(y,t)\Pi(y-x), \text{ where } y \in \{-\infty, ..., -2\delta, 0, 2\delta, 4\delta, ..., \infty\} \tag{99}$$

and the spatially averaged value of the field is

$$\bar{\phi}(y,t) = \frac{1}{2\delta} \int_{y-\delta}^{y+\delta} \phi(z,t) dz \tag{100}$$

This means that we can write

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - \sum_{n,y} a_n \bar{\phi}^n(y,t)\Pi(y-x) \right] \xi(x,t) = 0 \tag{101}$$

The rectangular functions are othonormal in the sense that $\Pi(x)\Pi(y) = \Pi(x) = \Pi(y) = 1$ if $x = y \in (-\delta, \delta)$ and 0 otherwise. Of course $x - y = 2n\delta$, where $n \in \mathbb{Z}$. The function can also be written as a combination of Heaviside Thetas:

$$\Pi(x) = \Theta(x - \delta) - \Theta(x + \delta) \tag{102}$$

Therefore we can multiply the equality by $\Pi(y_0 - x)$ to separate the differential equation

$$\begin{aligned} \left[\Pi(y_0 - x)\frac{\partial^2}{\partial t^2} + \sum_{n,y} a_n \bar{\phi}^n(y,t)\Pi(y-x)\Pi(y_0 - x)\right]\xi(x,t) &= \Pi(y_0 - x)\frac{\partial^2}{\partial x^2}\xi(x,t) \\ \left[\frac{\partial^2}{\partial t^2} + \sum_{n} a_n \bar{\phi}^n(y_0,t)\right]\xi(y_0,t) &= \Pi(y_0 - x)\frac{\partial^2}{\partial x^2}\xi(x,t) \end{aligned} \tag{103}$$

Now since we defined $\Pi(y_0 - x)$ on an open interval $(y_0 - x - \delta, y_0 - x + \delta)$ we can see that the neighbourhood of the second derivative of $\xi(x,t)$ will equal the second derivative of ξ in the neighbourhood. However if we had defined Π with a closed interval, there would be infinite boundary terms involved coming from the fact that the derivative at the boundary of an interval is ill-defined since differentiation requires the existence of more than one point. Therefore

$$\left[\frac{\partial^2}{\partial t^2} + \sum_{n} a_n \bar{\phi}^n(y_0,t)\right]\xi_{y_0}(y,t) = \frac{\partial^2}{\partial y^2}\xi_{y_0}(y,t), \text{ for } y \in (y_0 - \delta, y_0 + \delta) \tag{104}$$

The differential equation is therefore separable in the neighbourhood of any point y_0 on the x axis. Introducing a separation constant $C^2_{y_n}$ for each of the differential equations (at different points y) and writing $\xi_{y_0}(y,t) = \chi_{y_0}(y)\psi_{y_0}(t)$ for the neighbourhood , we get

$$\begin{aligned} \left[\frac{d^2}{dt^2} + \sum_{n} a_n \bar{\phi}^n(y_0,t)\right]\psi_{y_0}(t) &= -C^2_{y_0}\psi_{y_0}(t) \\ \frac{d^2}{dy^2}\chi_{y_0}(y) &= -C^2_{y_0}\chi_{y_0}(y) \end{aligned} \tag{105}$$

Boundary conditions need to be set to get the values of constants $C^2_{y_q} \equiv C^2_q$ for $q \in \{-\infty, ..., -2\delta, 0, 2\delta, ..., \infty\}$. Introducing a new, more compact notation, y_q means the x-axis neighbourhood variable about a point $x = q$. Subscripts q tells us at which discrete point the equations are based. Therefore the set of differential equation for the whole oscillon quantization reads:

$$\begin{aligned} \left[\frac{d^2}{dt^2} + \sum_{n} a_n \bar{\phi}^n(q,t)\right]\psi_q(t) &= -C^2_q\psi_q(t) \\ \frac{d^2}{dy_q^2}\chi_q(y_q) &= -C^2_q\chi_q(y_q) \\ &\text{for } q \in \{-\infty, ..., -2\delta, 0, 2\delta, ..., \infty\} \end{aligned} \tag{106}$$

Solutions to spatial equations are

$$\chi_q(y_q) = A_q e^{iC_q y_q} + B_q e^{-iC_q y_q}, \quad \forall q \tag{107}$$

We know that $\xi(x, t+\tau) = e^{i\nu}\xi(x,t)$ and that $\psi_q(t+\tau) = e^{2\pi i\mu_q}\psi_q(t)$. Also $\xi(x,t) = \sum_q \chi_q(y_q)\psi_q(t)\Pi(q-x)$, so

$$\begin{aligned} \xi(x, t+\tau) &= \sum_q \chi_q(y_q)\psi_q(t+\tau)\Pi(q-x) \\ e^{i\nu}\sum_q \chi_q(y_q)\psi_q(t)\Pi(q-x) &= \sum_q e^{2\pi i\mu_q}\chi_q(y_q)\psi_q(t)\Pi(q-x) \end{aligned} \tag{108}$$

This implies that each region must have the same stability angles. This makes sense since all the regions are governed by the same Mathieu equation for the time evolution, with only a different amplitude of oscillation of the background potential. The Mathieu characteristic exponent is therefore q independent