Quasi photons

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I. MODEL

$$
\begin{aligned}
S_0 &= \int dx \left(-\frac{1}{4} F_{\mu\nu} f(-\Box) F^{\mu\nu} - \frac{\xi}{2} \partial A f(-\Box) \partial A \right) \\
&= \frac{1}{2} \int dx [-\partial_\mu A_\nu f(-\Box) \partial^\mu A^\nu + \partial_\mu A_\nu f(-\Box) \partial^\nu A^\mu + \xi A_\mu f(-\Box) \partial^\mu \partial A] \\
&= \frac{1}{2} \int dx A_\mu [g^{\mu\nu} \Box - (1-\xi) \partial^\mu \partial^\nu] f(-\Box) A_\nu \\
&= \frac{1}{2} A D_0^{-1} A
\end{aligned}
\tag{1}
$$

Feynman gauge: $\xi = 1$,

$$D_0^{-1}(p) = -g^{\mu\nu} p^2 \tag{2}$$

$$\Pi^{++\mu\nu}(q) = -g^{\mu\nu} \ell^2 (q^2)^2 \tag{3}$$

$$f(p^2) = 1 + \ell^2 (p^2)^2 \tag{4}$$

$$D(k) = \frac{1}{k^2 + i\epsilon} - \frac{1}{k^2 + \frac{1}{\ell^2} - i\epsilon} = \frac{1}{k^2 + i\epsilon} \frac{1}{1 + \ell^2 (k^2 - i\epsilon)} = \frac{1}{k^2 + \ell^2 (k^2)^2 + i\epsilon} \tag{5}$$

II. FIELD OPERATOR

A. Harmonic oscillator with negative norm

Linear space $H = H_+ \oplus H_-$, $\eta H_\sigma = \sigma H_\sigma$
Matrix elements: $\langle m|A|n\rangle = (m, \eta A n)$
Adjoint: $\langle m|\bar{A}|n\rangle = \langle n|A|m\rangle^*$, $\bar{A} = \sigma_A A$, $\sigma = \pm 1$
$A\lambda\rangle = \lambda|\lambda\rangle$, $A\rho\rangle = \rho|\rho\rangle$, $(\lambda - \sigma_A \rho^*)\langle\rho|\lambda\rangle = 0$:
real spectrum for skew-adjoint operators with non-orthogonal, degenerate eigenvectors
$\bar{q}_\sigma = \sigma q_\sigma$, $\bar{p}_\sigma = \sigma p_\sigma$, $[q_\sigma, p_\sigma] = i$
$a_\sigma = (m\omega q_\sigma + i p_\sigma)/\sqrt{2m\omega}$, $q_\sigma = (a_\sigma + \sigma \bar{a}_\sigma)/\sqrt{2m\omega}$, $p_\sigma = (a_\sigma - \sigma \bar{a}_\sigma)/\sqrt{2}i$, $[a_\sigma, \bar{a}_\sigma] = \sigma$
$H_\sigma = \sigma(\frac{1}{2m} p_\sigma^2 + \frac{m\omega^2}{2} q_\sigma^2) = \sigma\omega(\bar{a}_\sigma a_\sigma + \frac{1}{2})$
$b = a_+$ or $\bar{a}_-$, $\bar{b} = \bar{a}_+$ or a_-, $[b, \bar{b}] = 1$
basis: $\bar{b}b|\lambda\rangle = \lambda|\lambda\rangle$, $\cdots, b^n|\lambda\rangle, \cdots, b|\lambda\rangle, |\lambda\rangle, \bar{b}|\lambda\rangle, \cdots, \bar{b}^n|\lambda\rangle, \cdots$
$\bar{b}b$ eigenvalues: $\cdots, \lambda - n, \cdots, \lambda - 1, \lambda, \lambda + 1, \cdots, \lambda + n, \cdots$

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$\sigma=+1$: stops at left, $\lambda\geq 0$, $\mathrm{sign}\langle\lambda+1|\lambda+1\rangle=\mathrm{sign}\langle\lambda|\lambda\rangle$
$\sigma=-1$: stops at right, $\lambda\leq -1$, $\mathrm{sign}\langle\lambda-1|\lambda-1\rangle=-\mathrm{sign}\langle\lambda|\lambda\rangle$
$H_\sigma|\lambda+\sigma n\rangle=E_\sigma(n)|\lambda+\sigma n\rangle$, $E_\sigma(n)=n+\frac{1}{2}+\sigma\lambda$
Coordinate eigenvalue: $\bar{q}_\sigma=\sigma q_\sigma$

$$\mathbb{1}=\int dq|\sigma q\rangle\langle q|=\int dp|\sigma p\rangle\langle p| \tag{6}$$

$H=H_++H_-$,

$$H_\sigma=\sigma\left(\frac{p_\sigma^2}{2m_\sigma}+\frac{m_\sigma\omega_\sigma^2}{2}q_\sigma^2\right)=\sigma\omega_\sigma\left(\bar{a}_\sigma a_\sigma+\frac{1}{2}\right) \tag{7}$$

$$\begin{aligned}\langle q_f,-q'_f|e^{-itH}|q_i,q'_i\rangle &= \int D[p]D[p']D[q]D[q']e^{i\int dt[p\dot{q}+p'\dot{q}'-H(q,q',p,p')]}\\ &= \int D[q]D[q']e^{i\int dtL(q,q',p,p')}\end{aligned} \tag{8}$$

$$H(q,q',p,p')=\frac{\langle q,-q'|H|p,p'\rangle}{\langle q,-q'|p,p'\rangle}=\frac{\langle q|H_+|p\rangle}{\langle q|p\rangle}+\frac{\langle -q'|H_-|p'\rangle}{\langle -q'|p'\rangle}=\frac{p_+^2}{2m_+}+\frac{m_+\omega_+^2}{2}q_+^2-\frac{p_-^2}{2m_-}-\frac{m_-\omega_-^2}{2}q_-^2 \tag{9}$$

$$L(q,q',p,p')=\frac{\dot{q}_+^2}{2m_+}-\frac{m_+\omega_+^2}{2}q_+^2-\frac{\dot{q}_-^2}{2m_-}+\frac{m_-\omega_-^2}{2}q_-^2 \tag{10}$$

convergence is achieved by $\omega_\sigma\to\omega_\sigma-\sigma i\epsilon$

B. Field operator

$M^2=\ell^{-2}$

$$L=\frac{1}{2}\partial_\mu\phi_+\partial^\mu\phi_+-\frac{m^2}{2}\phi_+^2-\frac{1}{2}\partial_\mu\phi_-\partial^\mu\phi_--\frac{M^2}{2}\phi_-^2 \tag{11}$$

negative mas square in the negative norm subspace.

$$\begin{aligned}[a(\boldsymbol{p}),a^\dagger(\boldsymbol{p}')] &= (2\pi)^3 2\omega_{\boldsymbol{p}}\delta(\boldsymbol{p}-\boldsymbol{p}')\\ \left[b(\boldsymbol{p}),b^\dagger(\boldsymbol{p}')\right] &= -(2\pi)^3 2\Omega_{\boldsymbol{p}}\delta(\boldsymbol{p}-\boldsymbol{p}'),\\ \left[a(\boldsymbol{p}),b^\dagger(\boldsymbol{p}')\right] &= [a(\boldsymbol{p}),b(\boldsymbol{p}')]=0\end{aligned} \tag{12}$$

$\omega_{\boldsymbol{p}}=\sqrt{m^2+\boldsymbol{p}^2}$, $\Omega_{\boldsymbol{p}}=\sqrt{\boldsymbol{p}^2-M^2}$, $a(\boldsymbol{p})|0\rangle=b^\dagger(\boldsymbol{p})|0\rangle=0$, $\phi(x)=\phi_+(x)+\phi_-(x)$

$$\begin{aligned}\phi_+(x) &= \int\frac{d^4k}{(2\pi)^4}2\pi\delta(k^2-m^2)a(k)e^{-ikx}=\int_{m,\boldsymbol{k}}[a(\boldsymbol{k})e^{-ikx}+a^\dagger(\boldsymbol{k})e^{ikx}]\\ \phi_-(x) &= \int\frac{d^4k}{(2\pi)^4}2\pi\delta(k^2+M^2)b^\dagger(k)e^{-ikx}=\int_{M,\boldsymbol{k}}[b(\boldsymbol{k})e^{-ikx}+b^\dagger(\boldsymbol{k})e^{ikx}]\end{aligned} \tag{13}$$

$$\int_{m,\boldsymbol{k}}f_k=\int\frac{d^3k}{(2\pi)^3 2\omega_{\boldsymbol{k}}}f_{\omega_{\boldsymbol{k}},\boldsymbol{k}},\qquad \int_{M,\boldsymbol{k}}k_k=\int_{\boldsymbol{k}^2>|M^2|}\frac{d^3k}{(2\pi)^3 2\Omega_{\boldsymbol{k}}}f_{\Omega_{\boldsymbol{k}},\boldsymbol{k}} \tag{14}$$

C. Path integral

$$
\begin{aligned}
e^{iW[\hat{j}]} &= \mathrm{Tr}T[e^{-i\int_{t_i}^{t_f}dt'[H(t')-j_+(t')\phi_+(t')]}]|0\rangle\langle 0|\bar{T}[e^{i\int_{t_i}^{t_f}dt'[H(t')+j_-(t')\phi_-(t')]}]] \\
&= \int D[\hat{\phi}]e^{\frac{i}{2}\hat{\phi}\cdot\hat{D}^{-1}\cdot\hat{\phi}+i\hat{j}\cdot\hat{\phi}} = e^{-\frac{i}{2}\hat{j}\cdot\hat{D}\cdot\hat{j}}
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
i\frac{\delta^2W[\hat{j}]}{\delta ij_a^+\delta ij_b^+} &= \sum_n\langle 0|\bar{T}[e^{i\int_{t_i}^{t_f}dt'H_i(t')}]|n\rangle\langle n|T[\phi_a\phi_b e^{-i\int_{t_i}^{t_f}dt'H_i(t')}]|0\rangle = \langle 0|T[\phi_a\phi_b]|0\rangle = iD^{++}_{ab} \\
i\frac{\delta^2W[\hat{j}]}{\delta ij_a^-\delta ij_b^-} &= \sum_n\langle 0|\bar{T}[\phi_a\phi_b]|n\rangle\langle n|0\rangle = \langle 0|T[\phi_b\phi_a]|0\rangle^* = iD^{--}_{ab} = -iD^{++*}_{ba} \\
i\frac{\delta^2W[\hat{j}]}{\delta ij_a^-\delta ij_b^+} &= \sum_n\langle 0|\phi_a|n\rangle\langle n|\phi_b|0\rangle = \langle 0|\phi_a\phi_b|0\rangle = iD^{-+}_{ab} \\
i\frac{\delta^2W[\hat{j}]}{\delta ij_a^+\delta ij_b^-} &= \sum_n\langle 0|\phi_b|n\rangle\langle n|\phi_a|0\rangle = \langle 0|\phi_b\phi_a|0\rangle = iD^{+-}_{ab} = -iD^{-+*}_{ab}
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
T[\phi_a\phi_b]+\bar{T}[\phi_a\phi_b] &= \phi_a\phi_b+\phi_b\phi_a \\
D-D^\dagger &= D^{+-}-D^{+-*}
\end{aligned} \tag{17}
$$

$$
i\begin{pmatrix} D & D^{+-} \\ D^{-+} & D^{--}\end{pmatrix}^{j,k}_{x,y} = \begin{pmatrix}\langle T[\phi^j_x\phi^k_y]\rangle & \langle\phi^k_y\phi^j_x\rangle \\ \langle\phi^j_x\phi^k_y\rangle & \langle T[\phi^k_y\phi^j_x]\rangle^*\end{pmatrix} = i\begin{pmatrix} D & -D^{-+*} \\ D^{-+} & -D^\dagger\end{pmatrix}^{j,k}_{x,y} = i\begin{pmatrix} D^n+i\Im D & -D^f+i\Im D \\ D^f+i\Im D & -D^n+i\Im D\end{pmatrix} \tag{18}
$$

$$
\begin{aligned}
iD_{x,x'} &= \Theta(t-t')\langle 0|\phi_x\phi_{x'}|0\rangle+\Theta(t'-t)\langle 0|\phi_{x'}\phi_x|0\rangle \\
2\Re D_{x,x'} &= -\Theta(t-t')i\langle 0|[\phi_x,\phi_{x'}]|0\rangle-\Theta(t'-t)i\langle 0|[\phi_{x'},\phi_x]|0\rangle = -\epsilon(t-t')i\langle 0|[\phi_x,\phi_{x'}]|0\rangle = 2D^n_{x,x'} \\
2\Im D_{x,x'} &= -\Theta(t-t')\langle 0|\{\phi_x,\phi_{x'}\}|0\rangle-\Theta(t'-t)\langle 0|\{\phi_{x'},\phi_x\}|0\rangle = -\langle 0|\{\phi_x,\phi_{x'}\}|0\rangle \\
iD^{-+}_{x,x'} &= \langle 0|\phi_x\phi_{x'}|0\rangle \\
2\Re D^{-+}_{x,x'} &= -i\langle 0|\phi_x\phi_{x'}|0\rangle+i\langle 0|\phi_{x'}\phi_x|0\rangle = -i\langle 0|[\phi_x,\phi_{x'}]|0\rangle = 2D^f_{x,x'} \\
2\Im D^{-+}_{x,x'} &= -\langle 0|\phi_x\phi_{x'}|0\rangle-\langle 0|\phi_{x'}\phi_x|0\rangle = -\langle 0|\{\phi_x,\phi_{x'}\}|0\rangle \\
iD^{\overset{r}{a}}_{x,x'} &= i(D^n_{x,x'}\pm D^f_{x,x'}) = \frac{1}{2}\epsilon(t-t')\langle 0|[\phi_x,\phi_{x'}]|0\rangle\pm\frac{1}{2}\langle 0|[\phi_x,\phi_{x'}]|0\rangle = \pm\Theta(\pm(t-t'))\langle 0|[\phi_x,\phi_{x'}]|0\rangle \\
D^{\overset{r}{a}}(x) &= D(x)-D^{\pm\mp}(x) = \Theta(t)D^{-+}(x)+\Theta(-t)D^{-+}(-x)-D^{-+}(\mp x) \\
&= \Theta(t)D^{-+}(x)+\Theta(-t)D^{-+}(-x)-\Theta(t)D^{-+}(\mp x)-\Theta(-t)D^{-+}(\mp x) \\
&= \Theta(t)[D^{-+}(x)-D^{-+}(\mp x)]+\Theta(-t)[D^{-+}(-x)-D^{-+}(\mp x)] \\
&= \pm\Theta(\pm t)[D^{-+}(x)-D^{-+}(-x)]
\end{aligned} \tag{19}
$$

D. Explicit expressions

$$\begin{aligned}
iD^{-+}(x,x') &= \frac{\langle 0|\phi(x)\phi(x')|0\rangle}{\langle 0|0\rangle} \\
&= \int_{m,\boldsymbol{k},\boldsymbol{k'}} \frac{\langle 0|[a(\boldsymbol{k})e^{-ikx}+a^\dagger(\boldsymbol{k})e^{ikx}][a(\boldsymbol{k'})e^{-ik'x'}+a^\dagger(\boldsymbol{k'})e^{ik'x'}]|0\rangle}{\langle 0|0\rangle} \\
&\quad + \int_{M,\boldsymbol{k},\boldsymbol{k'}} \frac{\langle 0|[b(\boldsymbol{k})e^{-ikx}+b^\dagger(\boldsymbol{k})e^{ikx}][b(\boldsymbol{k'})e^{-ik'x'}+b^\dagger(\boldsymbol{k'})e^{ik'x'}]|0\rangle}{\langle 0|0\rangle} \\
&= \int_{m,\boldsymbol{k},\boldsymbol{k'}} \frac{\langle 0|a(\boldsymbol{k})a^\dagger(\boldsymbol{k'})e^{i(-kx+k'x')}|0\rangle}{\langle 0|0\rangle} + \int_{M,\boldsymbol{k},\boldsymbol{k'}} \frac{\langle 0|b^\dagger(\boldsymbol{k})b(\boldsymbol{k'})e^{i(kx-k'x')}|0\rangle}{\langle 0|0\rangle} \\
&= \int_{m,\boldsymbol{k}} e^{-i\omega_{\boldsymbol{k}}(t-t')+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x'})} + \int_{M,\boldsymbol{k}} e^{i\Omega_{\boldsymbol{k}}(t-t')-i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x'})} \\
iD^{+-}(x,x') &= \frac{\langle 0|\phi(x')\phi(x)|0\rangle}{\langle 0|0\rangle} = iD^{-+}(x',x)
\end{aligned} \tag{20}$$

$$\begin{aligned}
\Theta(t-t')D^{-+}(x,x') &= \int_\omega \frac{e^{-i\omega(t-t')}}{\omega+i\epsilon}\int_{m,\boldsymbol{k}} e^{-i\omega_{\boldsymbol{k}}(t-t')+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x'})} + \int_\omega \frac{e^{i\omega(t-t')}}{-\omega+i\epsilon}\int_{M,\boldsymbol{k}} e^{i\Omega_{\boldsymbol{k}}(t-t')-i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x'})} \\
&= \int_{m,\boldsymbol{k},\omega} \frac{e^{-i(\omega_{\boldsymbol{k}}+\omega)(t-t')+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x'})}}{\omega+i\epsilon} - \int_{M,\boldsymbol{k},\omega} \frac{e^{i(\omega+\Omega_{\boldsymbol{k}})(t-t')-i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x'})}}{\omega-i\epsilon} \\
&= \int_{m,\boldsymbol{k},\omega} \frac{e^{-i\omega(t-t')+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x'})}}{\omega-\omega_{\boldsymbol{k}}+i\epsilon} - \int_{M,\boldsymbol{k},\omega} \frac{e^{i\omega(t-t')-i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x'})}}{\omega-\Omega_{\boldsymbol{k}}-i\epsilon}
\end{aligned} \tag{21}$$

$$\begin{aligned}
\Theta(t'-t)D^{+-}(x,x') &= \int_\omega \frac{e^{i\omega(t-t')}}{\omega+i\epsilon}\int_{m,\boldsymbol{k}} e^{i\omega_{\boldsymbol{k}}(t-t')-i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x'})} + \int_\omega \frac{e^{-i\omega(t-t')}}{-\omega+i\epsilon}\int_{M,\boldsymbol{k}} e^{-i\Omega_{\boldsymbol{k}}(t-t')+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x'})} \\
&= \int_{m,\boldsymbol{k},\omega} \frac{e^{i(\omega_{\boldsymbol{k}}+\omega)(t-t')-i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x'})}}{\omega+i\epsilon} - \int_{M,\boldsymbol{k},\omega} \frac{e^{-i(\omega+\Omega_{\boldsymbol{k}})(t-t')+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x'})}}{\omega-i\epsilon} \\
&= \int_{m,\boldsymbol{k},\omega} \frac{e^{i\omega(t-t')-i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x'})}}{\omega-\omega_{\boldsymbol{k}}+i\epsilon} - \int_{M,\boldsymbol{k},\omega} \frac{e^{-i\omega(t-t')+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x'})}}{\omega-\Omega_{\boldsymbol{k}}-i\epsilon} \\
&= -\int_{m,\boldsymbol{k},\omega} \frac{e^{-i\omega(t-t')+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x'})}}{\omega+\omega_{\boldsymbol{k}}-i\epsilon} + \int_{M,\boldsymbol{k},\omega} \frac{e^{i\omega(t-t')-i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x'})}}{\omega+\Omega_{\boldsymbol{k}}+i\epsilon}
\end{aligned} \tag{22}$$

$$\begin{aligned}
D(x,x') &= \Theta(t-t')D^{-+}(x,x') + \Theta(t'-t)D^{+-}(x,x') \\
&= \int_{m,\boldsymbol{k},\omega} e^{-i\omega(t-t')+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x'})}\left(\frac{1}{\omega-\omega_{\boldsymbol{k}}+i\epsilon} - \frac{1}{\omega+\omega_{\boldsymbol{k}}-i\epsilon}\right) \\
&\quad - \int_{M,\boldsymbol{k},\omega} e^{i\omega(t-t')-i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x'})}\left(\frac{1}{\omega-\Omega_{\boldsymbol{k}}-i\epsilon} - \frac{1}{\omega+\Omega_{\boldsymbol{k}}+i\epsilon}\right) \\
&= \int_k \left[\frac{e^{-ikx}}{k^2-m^2+i\epsilon} - \frac{\Theta(\boldsymbol{k}^2-M^2)e^{-ikx}}{k^2+M^2-i\epsilon}\right]
\end{aligned} \tag{23}$$

$$\begin{aligned}
D^{\overset{r}{a}}(x) &= \pm\Theta(\pm t)[D^{-+}(x) - D^{-+}(-x)] \\
&= \mp\Theta(\pm t)i\left[\int_{m,\boldsymbol{k}} (e^{-i\omega_{\boldsymbol{k}}t+i\boldsymbol{k}\boldsymbol{x}} - e^{i\omega_{\boldsymbol{k}}t-i\boldsymbol{k}\boldsymbol{x}}) - \int_{M,\boldsymbol{k}} (e^{i\Omega_{\boldsymbol{k}}t-i\boldsymbol{k}\boldsymbol{x}} - e^{-i\Omega_{\boldsymbol{k}}t+i\boldsymbol{k}\boldsymbol{x}})\right]
\end{aligned} \tag{24}$$

$$
\begin{aligned}
D^{\substack{r\\a}}(x) &= \pm\Theta(\pm t)\int_k e^{-ikx}\left[\frac{1}{(k^0\pm i\epsilon)^2-\boldsymbol{k}^2-m^2}-\frac{\Theta(\boldsymbol{k}^2-M^2)}{(k^0\pm i\epsilon)^2-\boldsymbol{k}^2+M^2}\right]\\
&= \pm\Theta(\pm t)\int_k e^{-ikx}\left[\frac{1}{(k^0\pm i\epsilon-\omega_{\boldsymbol{k}})(k^0\pm i\epsilon+\omega_{\boldsymbol{k}})}-\frac{\Theta(\boldsymbol{k}^2-M^2)}{(k^0\pm i\epsilon-\Omega_{\boldsymbol{k}})(k^0\pm i\epsilon+\Omega_{\boldsymbol{k}})}\right]\\
&= \pm\Theta(t)i\int_{\boldsymbol{k}} e^{i\boldsymbol{k}\boldsymbol{x}}\left[\frac{e^{i\omega_{\boldsymbol{k}}t}-e^{-i\omega_{\boldsymbol{k}}t}}{2\omega_{\boldsymbol{k}}}-\Theta(\boldsymbol{k}^2-M^2)\frac{e^{i\omega_{\boldsymbol{k}}t}-e^{-i\omega_{\boldsymbol{k}}t}}{2\Omega_{\boldsymbol{k}}}\right]
\end{aligned}
\tag{25}
$$

E. Real space expressions for $m = 0$

the contribution at large k are negligible in a cutoff theory:

$$
\begin{aligned}
D(x) &= D_{x0}\to\int_{\boldsymbol{k}} e^{i\boldsymbol{k}\boldsymbol{x}-\epsilon' k}\int_\omega\frac{e^{-i\omega t}}{\omega^2-k^2+i\epsilon}\\
&= \int_{\boldsymbol{k}} e^{i\boldsymbol{k}\boldsymbol{x}-\epsilon' k}\int_\omega\frac{e^{-i\omega t}}{(\omega-k+i\epsilon)(\omega+k-i\epsilon)}\\
&= -\frac{i}{4\pi^2}\int dkk^2dce^{ik(rc+i\epsilon')}\left(\frac{\Theta(t)e^{-i(k-i\epsilon)t}}{2k-i\epsilon}+\frac{\Theta(-t)e^{i(k-i\epsilon)t}}{2k-i\epsilon}\right)\\
&= -\frac{i}{4\pi^2}\int_0^\infty dkk^2\frac{e^{ik(r+i\epsilon')}-e^{-ik(r-i\epsilon')}}{ikr}\frac{1}{2k}e^{-i(k-i\epsilon)|t|}\\
&= -\frac{1}{8\pi^2 r}\int_0^\infty dk(e^{ik(r-|t|+i\epsilon')}-e^{-ik(r+|t|-i\epsilon')})\\
&= \frac{1}{8\pi^2 r}\left(\frac{1}{i(r-|t|+i\epsilon')}-\frac{1}{-i(r+|t|-i\epsilon')}\right)\\
&= -\frac{i}{8\pi^2 r}\left(\frac{1}{r-|t|+i\epsilon'}+\frac{1}{r+|t|-i\epsilon'}\right)\\
&= -\frac{i}{4\pi^2(r^2-(|t|-i\epsilon')^2)}\\
&= \frac{i}{4\pi^2x^2-i\epsilon'}\\
&= P\frac{i}{4\pi^2x^2}-\frac{1}{4\pi}\delta(x^2)
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
D^{\overset{r}{a}}(x) &= D^{\overset{r}{a}}_{x0} = \int_{\boldsymbol{k}} e^{i\boldsymbol{k}\boldsymbol{x}} \int_\omega \frac{e^{-i\omega t}}{(\omega \pm i\epsilon)^2 - \boldsymbol{k}^2} \\
&= \int_{\boldsymbol{k}} e^{i\boldsymbol{k}\boldsymbol{x}} \int_\omega \frac{e^{-i\omega t}}{(\omega \pm i\epsilon - |\boldsymbol{k}|)(\omega \pm i\epsilon + |\boldsymbol{k}|)} \\
&= \underbrace{-\Theta(t) i \int_{\boldsymbol{k}} e^{i\boldsymbol{k}\boldsymbol{x}} \left(\frac{e^{-i(|\boldsymbol{k}|-i\epsilon)t}}{2|\boldsymbol{k}|} - \frac{e^{i(|\boldsymbol{k}|+i\epsilon)t}}{2|\boldsymbol{k}|} \right)}_{R} + \underbrace{\Theta(-t) i \int_{\boldsymbol{k}} e^{i\boldsymbol{k}\boldsymbol{x}} \left(\frac{e^{-i(|\boldsymbol{k}|+i\epsilon)t}}{2|\boldsymbol{k}|} - \frac{e^{i(|\boldsymbol{k}|-i\epsilon)t}}{2|\boldsymbol{k}|} \right)}_{A} \\
&= -\left(\frac{\Theta(t) i}{(2\pi)^2} \int dk k^2 dc e^{ikrc} \frac{e^{-ikt} - e^{ikt}}{2k} - \frac{\Theta(-t) i}{(2\pi)^2} \int dk k^2 dc e^{ikrc} \frac{e^{-ikt} - e^{ikt}}{2k} \right) e^{-\epsilon|t|} \\
&= -\left(\frac{\Theta(t) i}{(2\pi)^2} \int dk k^2 \frac{e^{ikr} - e^{-ikr}}{ikr} \frac{e^{-ikt} - e^{ikt}}{2k} - \frac{\Theta(-t) i}{(2\pi)^2} \int dk k^2 \frac{e^{ikr} - e^{-ikr}}{ikr} \frac{e^{-ikt} - e^{ikt}}{2k} \right) e^{-\epsilon|t|} \\
&= -\left(\frac{\Theta(t)}{2(2\pi)^2 r} \int_0^\infty dk (e^{ikr} - e^{-ikr})(e^{-ikt} - e^{ikt}) - \frac{\Theta(-t)}{2(2\pi)^2 r} \int_0^\infty dk (e^{ikr} - e^{-ikr})(e^{-ikt} - e^{ikt}) \right) e^{-\epsilon|t|} \\
&= -\frac{1}{8\pi r} \int_{-\infty}^\infty \frac{dk}{2\pi} (e^{ik(r-t)} + e^{ik(-r+t)} - e^{ik(-r-t)} - e^{ik(r+t)})(\Theta(t) - \Theta(-t)) e^{-\epsilon|t|} \\
&= -\frac{\Theta(t)\delta(r-t) + \Theta(-t)\delta(r+t)}{4\pi r} e^{-\epsilon|t|} \\
&= -\frac{\Theta(\pm t)\delta(t \mp r)}{4\pi r} e^{-\epsilon|t|} \\
&= -\Theta(\pm t) \frac{\delta(t^2 - r^2)}{2\pi} e^{-\epsilon|t|}
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
D^f(x) &= \frac{1}{2}(D^r(x) - D^a(x)) = -\frac{1}{4\pi}\delta(x^2)\epsilon(x^0) \\
D^n(x) &= \frac{1}{2}(D^r(x) + D^a(x)) = -\frac{1}{4\pi}\delta(x^2)
\end{aligned}
\tag{28}
$$

F. Real space expressions for $m \neq 0$

$$
D(k^2) = -i\text{sign}(\epsilon) \int_0^\infty ds e^{i\text{sign}(\epsilon)s(k^2 - m^2 + i\epsilon)},
\tag{29}
$$

Analytic for $\Im e^{i\theta} > 0$:

$$
\int dx e^{\frac{i}{2} a e^{i\theta} x^2} = \sqrt{\frac{2\pi}{a}} e^{i(\frac{\pi}{4} - \frac{\theta}{2})}
\tag{30}
$$

For real a:

$$
\int \frac{d^4k}{(2\pi)^4} e^{(ia-\epsilon)k^2 - ikx} = e^{-\frac{ix^2}{4a}} \int \frac{d^4k}{(2\pi)^4} e^{(ia-\epsilon)(k - \frac{x}{2a})^2} = \frac{e^{i(\pi - \frac{3}{2}\pi)}}{32\pi^2 a^2} e^{-\frac{ix^2}{4a}} = -\frac{i\text{sign}(a)}{32\pi^2 a^2} e^{-\frac{ix^2}{4a}}
\tag{31}
$$

$$
\begin{aligned}
D(x) &= -i\mathrm{sign}(\epsilon)\int\frac{d^4k}{(2\pi)^4}\int_0^\infty ds e^{i\mathrm{sign}(\epsilon')s(k^2-m^2+i\epsilon)-ikx} \\
&= -\frac{1}{32\pi^2}\int\frac{da}{a^2}e^{-i\mathrm{sign}(\epsilon)a(m^2-i\epsilon)-\frac{ix^2}{4a}} \qquad \alpha=\frac{1}{4a} \\
&= \frac{1}{8\pi^2}\int_{-\infty}^{\infty}d\alpha e^{-\frac{im^2-i\epsilon}{4\alpha}-ix^2\alpha} \\
&= \frac{1}{4\pi^2}\int_0^\infty d\alpha\cos\left(\frac{m^2}{4\alpha}+x^2\alpha\right) \\
&= \frac{1}{4\pi^2}\partial_{x^2}\int_0^\infty\frac{d\alpha}{\alpha}\sin\left(\frac{m^2}{4\alpha}+x^2\alpha\right) \qquad \alpha=\frac{\sqrt{m^2-i\epsilon}}{2\sqrt{|x^2|}}e^\theta \\
&= \frac{1}{4\pi^2}\partial_{x^2}\int_{-\infty}^{\infty}d\theta\sin\left[\frac{\sqrt{m^2-i\epsilon}\sqrt{|x^2|}}{2}\left(e^{-\theta}+\mathrm{sign}(x^2)e^\theta\right)\right] \\
&= \frac{\Theta(x^2)}{4\pi^2}\partial_{x^2}\int_{-\infty}^{\infty}d\theta\sin(\sqrt{(m^2-i\epsilon)x^2}\cosh\theta) \\
&= \frac{\Theta(x^2)}{4\pi}\partial_{x^2}J_0(\sqrt{(m^2-i\epsilon)x^2}) \qquad (32)
\end{aligned}
$$

$$
J_0' = -J_1 \qquad (33)
$$

$$
\begin{aligned}
D(x) &= \frac{1}{4\pi}\partial_{x^2}\Re H_0^{(1)}(m\sqrt{x^2}) \quad (\text{for } \sqrt{-\epsilon}=i\sqrt{\epsilon}) \\
&= -\frac{\Theta(x^2)}{8\pi}\sqrt{\frac{m^2-i\epsilon}{x^2}}J_1(\sqrt{(m^2-i\epsilon)x^2}) \\
&= \frac{1}{4\pi}\partial_{x^2}\Re H_0^{(1)}(m\sqrt{x^2}) \quad (\text{for } \sqrt{-\epsilon}=i\sqrt{\epsilon}) \\
&= \frac{1}{4\pi}\delta(x^2)-\frac{m^2}{8\pi}\Re\frac{H_1^{(1)}(m\sqrt{x^2})}{m\sqrt{x^2}} \qquad (34)
\end{aligned}
$$

Positive mass square:

$$
\begin{aligned}
D^{\overset{r}{a}}(x) &= D^{\overset{r}{a}}_{x0} = \int_{\boldsymbol{k}} e^{i\boldsymbol{kx}}\int_\omega\frac{e^{-i\omega t}}{(\omega\pm i\epsilon)^2-\boldsymbol{k}^2-m^2} \\
&= \int_{\boldsymbol{k}} e^{i\boldsymbol{kx}}\int_\omega\frac{e^{-i\omega t}}{(\omega\pm i\epsilon-\omega_{\boldsymbol{k}})(\omega\pm i\epsilon+\omega_{\boldsymbol{k}})} \\
&= \underbrace{-\Theta(t)i\int_{\boldsymbol{k}} e^{i\boldsymbol{kx}}\left(\frac{e^{-i(\omega_{\boldsymbol{k}}-i\epsilon)t}}{2\omega_{\boldsymbol{k}}}-\frac{e^{i(\omega_{\boldsymbol{k}}+i\epsilon)t}}{2\omega_{\boldsymbol{k}}}\right)}_{R}+\underbrace{\Theta(-t)i\int_{\boldsymbol{k}} e^{i\boldsymbol{kx}}\left(\frac{e^{-i(\omega_{\boldsymbol{k}}+i\epsilon)t}}{2\omega_{\boldsymbol{k}}}-\frac{e^{i(\omega_{\boldsymbol{k}}-i\epsilon)t}}{2\omega_{\boldsymbol{k}}}\right)}_{A} \\
&= -\left(\frac{\Theta(t)i}{(2\pi)^2}\int dkk^2dce^{ikrc}\frac{e^{-i\omega_kt}-e^{i\omega_kt}}{2\omega_k}-\frac{\Theta(-t)i}{(2\pi)^2}\int dkk^2dce^{ikrc}\frac{e^{-i\omega_kt}-e^{i\omega_kt}}{2\omega_k}\right)e^{-\epsilon|t|} \\
&= -\left(\frac{\Theta(t)i}{(2\pi)^2}\int dkk^2\frac{e^{ikr}-e^{-ikr}}{ikr}\frac{e^{-i\omega_kt}-e^{i\omega_kt}}{2\omega_k}-\frac{\Theta(-t)i}{(2\pi)^2}\int dkk^2\frac{e^{ikr}-e^{-ikr}}{ikr}\frac{e^{-i\omega_kt}-e^{i\omega_kt}}{2\omega_k}\right)e^{-\epsilon|t|} \\
&= -\left(\frac{\Theta(t)}{2(2\pi)^2r}\int_0^\infty\frac{dkk}{\omega_k}(e^{ikr}-e^{-ikr})(e^{-i\omega_kt}-e^{i\omega_kt})-\frac{\Theta(-t)}{2(2\pi)^2r}\int_0^\infty\frac{dkk}{\omega_k}(e^{ikr}-e^{-ikr})(e^{-i\omega_kt}-e^{i\omega_kt})\right)e^{-\epsilon|t|} \\
&= \mp\frac{1}{8\pi r}\int_{-\infty}^{\infty}\frac{dkk}{2\pi\omega_k}(e^{i(kr-\omega_kt)}+e^{i(-kr+\omega_kt)}-e^{i(-kr-\omega_kt)}-e^{i(kr+\omega_kt)})\Theta(\pm t)e^{-\epsilon|t|} \qquad (35)
\end{aligned}
$$

Negative mass square: $\tilde{\omega}_{\boldsymbol{k}} = +\sqrt{|m^2 - \boldsymbol{k}^2|} > 0$

$$
\begin{aligned}
D^r(x) &= \int_{\boldsymbol{k}} e^{i\boldsymbol{kx}} \int_\omega \frac{e^{-i\omega t}}{\omega^2 - (\boldsymbol{k}^2 - m^2)} + \text{homogeneous solutions (vanishing denomninator)} \\
&= \int_{\boldsymbol{k}>m} e^{i\boldsymbol{kx}} \int_\omega \frac{e^{-i\omega t}}{(\omega + i\epsilon - \tilde{\omega}_{\boldsymbol{k}})(\omega + i\epsilon + \tilde{\omega}_{\boldsymbol{k}})} + \int_{\boldsymbol{k}<m} e^{i\boldsymbol{kx}} \int_\omega \frac{e^{-i\omega t}}{(\omega - i\tilde{\omega}_{\boldsymbol{k}})(\omega + i\tilde{\omega}_{\boldsymbol{k}})} + \cdots \\
&= -\Theta(t) i \left[\int_{\boldsymbol{k}} e^{i\boldsymbol{kx}} \left(\frac{e^{-i(\tilde{\omega}_{\boldsymbol{k}} - i\epsilon)t}}{2\tilde{\omega}_{\boldsymbol{k}}} - \frac{e^{i(\tilde{\omega}_{\boldsymbol{k}} + i\epsilon)t}}{2\tilde{\omega}_{\boldsymbol{k}}} \right) - i \int_{\boldsymbol{k}<m} e^{i\boldsymbol{kx}} \frac{e^{-\tilde{\omega}_{\boldsymbol{k}} t}}{2\tilde{\omega}_{\boldsymbol{k}}} \right] \\
&= -\frac{\Theta(t)}{(2\pi)^2} \left[i \int_m^\infty dk k^2 dc e^{ikrc} \frac{e^{-i\tilde{\omega}_k t} - e^{i\tilde{\omega}_{\boldsymbol{k}} t}}{2\tilde{\omega}_k} + \int_0^m dk k^2 dc e^{ikrc} \frac{e^{-\tilde{\omega}_k t}}{2\tilde{\omega}_k} \right] \\
&= -\frac{\Theta(t)}{(2\pi)^2} \left[i \int_m^\infty dk k^2 \frac{e^{ikr} - e^{-ikr}}{ikr} \frac{e^{-i\tilde{\omega}_k t} - e^{i\tilde{\omega}_{\boldsymbol{k}} t}}{2\tilde{\omega}_k} + \int_0^m dk k^2 \frac{e^{ikr} - e^{-ikr}}{ikr} \frac{e^{-\tilde{\omega}_k t}}{2\tilde{\omega}_k} \right] \\
&= -\frac{\Theta(t)}{2(2\pi)^2 r} \left[\int_m^\infty \frac{dk k}{\tilde{\omega}_k} (e^{ikr} - e^{-ikr})(e^{-i\tilde{\omega}_k t} - e^{i\tilde{\omega}_{\boldsymbol{k}} t}) - i \int_0^m \frac{dk k}{\tilde{\omega}_k} (e^{ikr} - e^{-ikr}) e^{-\tilde{\omega}_k t} \right]
\end{aligned} \tag{36}
$$