Contradiction and reconstruction of axiom of *ZF* system

----Tranclosed logic princiole and its inference (5)

Jincheng Zhang

Oriental Culture Research Office, Guangde County, Anhui Province China.

Email:656790205@qq.com

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Abstract: In set theory *ZF* system, there are specious strange conclusions when set of infinite sets are interpreted into numbers. Axiom of infinity in *ZF* system confuses the difference between $\delta+1=\delta$ and $\delta+1\neq\delta$, Error of Cantor's set theory is the error of axiom of infinity. We use (δ + 1 = δ or δ + 1 ≠ δ) to establish two opposite sets of axioms *SZF* + and *SZF*-, which are contradictory systems, similar to Euclidean geometry and non-Euclidean geometry axioms.

In set theory *SZF* system, after correcting the axiom of infinity, we can prove that "for any definable predicate P (n), if it is true to finite n, then for any infinite number *a*, P(*a*) is also true." Power set axiom, axiom of separation, axiom of substitution, axiom of choice, axiom of foundation these axioms have become provable propositions.

In fact, the four axioms and several definitions of "axiom of empty set, set axioms, union axioms, and infinite axioms" are sufficient for the set theory system.

In Cantor's set theory, there are ordinal numbers and uncountable ordinal numbers. There are no uncountable ordinal numbers, so proof of uncountable ordinal number is false. The transfinite induction in Cantor's set theory is also false; in fact, transfinite induction is very simple, we can prove that: for any definable predicate P (n), if it is true to finite n, then for any infinite number *a*, P(*a*) is also true.

In classical set theory system, all difficulties are from infinite difficulties. After we have solved the axiom of infinity, infinity is as simple as finite.

1. Contradiction and reconstruction of axiom of infinity

1.1 Contradiction of axiom of infinity

We know that a set can be defined as a number and a finite set can be defined as a finite number ; an infinite set can be defined as an infinite number.

For finite natural numbers, we can define with set: (1) $0=\varnothing$, $0+1=\{0\}$,(2) $n+1=n\bigcup\{n\}$,

For any finite number, there is $n=\{0\cdot1\cdot2\cdots\cdot n-1\}$;

The infinite axiom in the classical set theory *ZF* system:

$$\exists\omega[(0\in\omega)\wedge(n\in\omega\rightarrow n+1\in\omega)],\ \ \omega=\{0\cdot1\cdot2\cdots\cdot n\cdots\}.$$

Cantor's ordinal number ω, is the generalization of finite natural number, that is a set and a number, but it is contradictory.

For finite set: $n=\{0\cdot 1\cdot 2\cdots\cdots n-1\}$,

Make limits on the both sides at the same time:

$\lim_{n\to\infty} n=\{0, 1, 2, \cdots, \lim_{n\to\infty}(n-2), \lim_{n\to\infty}(n-1)\}$,

Let $\delta=\lim_{n\to\infty} n$, only $\delta=\{0\cdot 1\cdot 2\cdots\cdots \delta-2\cdot\delta-1\}$ is obtained.

There are only two possibilities for successor of infinite number δ: $\delta+1\neq\delta$ or $\delta+1=\delta$.

(I) If $\delta+1\neq\delta$, then $\delta-2\neq\delta-1\neq\delta\neq\delta+1, \delta=\{0\cdot 1\cdot 2\cdots\cdots\delta-2\cdot\delta-1\}$;

(II) If $\delta+1=\delta$, then $\delta-2=\delta-1=\delta=\delta+1, \delta=\{0\cdot 1\cdot 2\cdots\cdots\delta\}$.

There is no doubt about Cantor's definition on ordinal numbers and finite sets are defined as finite numbers; about an infinite set is defined as an infinite number (that is, infinity) ω, we need to ask : What is the successor (ω^+) of this infinite set ? $\omega+1=\omega$? Or $\omega+1\neq\omega$?

(1) If ω meets (I), that is $\omega=\delta$,

then $\omega=\{0\cdot 1\cdot 2\cdots\cdots\omega-2\cdot\omega-1\}$ --(a1)

And general axiom of infinity holds that: $\omega=\{0\cdot 1\cdot 2\cdots\cdot n\cdots\}$ ----------------------（b1);

In form (a1), ω has successors $\omega+1, \omega+2, \cdots$, obviously there are predecessors $\omega-1, \omega-2, \cdots$.

In form (b1), if ω is regarded as limit set of n, $\omega=\lim_{n\to\infty} n=\{0, 1, 2, \cdots, n, \cdots\}$, ω only has successors $\omega+1, \omega+2, \cdots$ and so on but do not have predecessors $\omega-1, \omega-2, \cdots$;

(a1) and (b1) are inconsistent in form.

(2) If ω meets (II), $\omega=\omega+1$, $\omega=\{0\cdot 1\cdot 2\cdots\cdots\omega\}$

ω is the ∞ in the standard analysis, in the standard analysis $\infty=\infty+1$,

$\infty-n=\infty-2=\infty-1=\infty=\infty+1=\infty+2=\infty+n$,can be obtained immediately .

That is: $\omega=\omega+1$ --(a2);

According to set operation, $\omega=\{0\cdot 1\cdot 2\cdots\cdot n\cdots\}$, $\omega+1=\omega\cup\{\omega\}=\{0\cdot 1\cdot 2\cdots\cdots\omega\}$, The elements of these two sets are different.

That is: $\omega+1\neq\omega$ --(b2);

(a2) and (b2) are contradictory and obviously, ω is not ∞.

Cantor’s definition on infinite ordinal number ω does not belong to any of the above cases and we think that it is divorced from mathematics and a game of symbols of no practical significance.

(3) If calculate according to limit of standard analysis, $n=\{0,1,2,\cdots,n-1\}$ is the limit form,

$$\lim_{n\to\infty} n=\{0,1,2,\cdots,\lim_{n\to\infty}(n-2),\lim_{n\to\infty}(n-1)\},$$

Since,

$$\lim_{n\to\infty} n=\lim_{n\to\infty}(n-1)=\infty$$

$\infty=\{0,1,2,\cdots,\infty\}$ can be obtained immediately

Namely, $\infty\in\infty$.

This is a cycle set, while classical *ZF* system cannot explain significance of it but regards it as a singular set to be ruled out.

(4) ω is an infinity, according to intuitive sequence of infinite, $2^{\omega}, 3^{\omega}$ shall be between ω^2 and ω^{ω},

$$0\cdot1\cdot2\cdots\cdot\omega\cdots2\omega\cdots\omega^2\cdots\omega^3\cdots2^{\omega}\cdots3^{\omega}\cdots\omega^{\omega}\cdots\omega^{\omega^{\omega}}\cdots$$

Size of infinities is as follows:

$$\omega^2<\omega^3<\cdots<2^{\omega}<3^{\omega}<\cdots<\omega^{\omega}<\cdots \text{-------------------(a4)}$$

In the calculus definition of ordinal numbers

$$\beta+\alpha=\sup\{\beta+\gamma\mid\gamma<\alpha\},$$

$$\beta\cdot\alpha=\sup\{\beta\cdot\gamma\mid\gamma<\alpha\},$$

$$\beta^{\alpha}=\sup\{\beta^{\gamma}\mid\gamma<\alpha\}.$$

It can be concluded that:

$$\omega=2^{\omega}=3^{\omega}=n^{\omega},$$

$$\omega=2^{\omega}=3^{\omega}=n^{\omega}<2\omega<\omega^2<\omega^3<\omega^{\omega}<\omega^{\omega^{\omega}} \text{-------------------(b4)}$$

These are some strange conclusions and (a4) contradicts with (b4)

(5) In Cantor’s set theory, it is easily to conclude that:

$2^{\omega}=\omega$ (Power operation theorem of Cantor's ordinal number $2^{\omega}=\sup\left\{2^{n} \mid n<\omega\right\}=\omega$)

In Cantor's set theory, it is easy to conclude that: if two ordinal numbers are equal, their cardinal numbers must be equal.

Thus: $2^{\omega}=\omega \Rightarrow |2^{\omega}|=|\omega|$---(a5);

In Cantor's set theory, it is easy to conclude that: cardinal numbers of power set of natural numbers are larger than cardinal numbers of set of natural numbers.

$|2^{\omega}|>|\omega|$ Cantor's power set of natural numbers is an uncountable axiom -----------(b5);

(a5) contradicts with (b5);

But in some papers, to avoid these contradictions, the power operation of ordinal numbers is simply omitted or not discussed.

(6) Cantor defines all countable ordinals as a set

$$\omega_1=\{0 \cdot 1 \cdot 2 \cdots \cdots \omega \cdots \cdot 2\omega \cdots \omega^2 \cdots \omega^3 \cdots \cdot 2^{\omega} \cdots \cdot 3^{\omega} \cdots \cdot \omega^{\omega} \cdots \cdot \omega^{\omega^{\omega}} \cdots \cdot\}$$

If ω_1 is a countable ordinal number, then $\omega_1 \in \omega_1$, which is contradictory with axiom of regularity, therefore, it can be inferred that: ω_1 is a countable ordinal number.

$$\omega_0=\omega, \quad \omega_1=\{x: on(x) \wedge |x| \leq |\omega_0|\}, \quad \omega_{\lambda+1}=\{x: on(x) \wedge |x| \leq |\omega_{\lambda}|\}$$

And then a serious uncountable ordinal numbers are defined

$$\omega_1<\omega_2<\omega_3<\omega_4<\cdots$$

$x \in x$ does not necessarily lead to contradiction, above inferences on uncountability are false.

The above contradiction is essentially caused by the incorrect definition of the infinite, leading to the entire ordinal error. We must thoroughly revise the 'infinite axiom'.

We have introduced two infinite axioms, the standard infinite axiom and the non-standard infinite axiom.(See *Infinity and infinite induction---Tranclosed logic princiole and its inference(3)*)

1.2 Reconstruction of infinity axiom

Definition of the classical infinity: assume that N is set of all finite numbers, there is a δ larger than all finite numbers E, δ is infinite and it is called as infinity noted as : $\delta=\infty$,that is

$$N=\{0 \cdot 1 \cdot 2 \cdots \cdots n \cdots \cdot\}$$

$$\exists \delta\,[\,\forall E \in N\,(\delta>E)]$$

Or define with limit:

$$\lim_{n \to \delta} x_n=\delta \text{, note as: } \delta=\infty .$$

According to above classical infinite limit,

$$\lim_{n\to\infty} n = \lim_{n\to\infty} n+1, \text{ note as } \lim_{n\to\infty} n = \delta, \text{ that is: } \exists\delta\,(\delta+1=\delta),$$

Above classical infinite limit definition can also be summarized as “classical axiom of infinity”, that is:

Classical axiom of infinity: $N=\{0\cdot 1\cdot 2\cdots\cdots n\cdots\}$,

$$\exists\delta\,[\,\forall E\in N\,(\delta>E)]\wedge[\delta+1=\delta].$$

Classical infinite axiom:

$$N=\{0\cdot 1\cdot 2\cdots\cdots n\cdots\},\ \exists\delta\,[\,\forall E\in N\,(\delta>E)]\wedge[\delta+1=\delta].$$

Compared to the classical infinity axiom, we propose anticlassical axiom,

Anti-classical infinite axiom :

$$N=\{0\cdot 1\cdot 2\cdots\cdots n\cdots\},\exists\delta\,[\,\forall E\in N\,(\delta>E)]\wedge[\delta+1\neq\delta];$$

"Classical infinite axiom" and "anticlassical infinite axiom" are contradictory and cannot be established simultaneously in the same system.

In order to distinguish the different operation results of classic infinite axiom and anti-classical axiom, we introduce the following definitions:

Definition1.2.1 Standard limit and non-standard limit

We rule:

(1) If satisfy classical infinite axiom $\exists\delta\,[\,\forall E\in N\,(\delta>E)]\wedge[\delta+1=\delta]$, remember δ as $\infty=\lim_{n\to\infty} n$, then called $\lim_{n\to\infty} n$ is standard limit ;

(2) If satisfy non-classical infinite axiom $\exists\delta\,[\,\forall E\in N\,(\delta>E)]\wedge[\delta+1\neq\delta]$, remember δ as $\varpi=\mu\lim_{n\to\infty} n$, then called $\mu\lim_{n\to\infty} n$ is non-standard limit .

Finite set $n=\{0,1,2,\cdots,(n-1)\}$, when $n\to\infty$,

(1) $\lim_{n\to\infty} n=\{0,1,2,\cdots,\lim_{n\to\infty}(n-2),\lim_{n\to\infty}(n-1)\}$ is an infinite set,

$$\infty=\{0,1,2,\cdots,\infty\};$$

(2) $\mu\lim_{n\to\infty} n=\{0,1,2,\cdots,\mu\lim_{n\to\infty}(n-2),\mu\lim_{n\to\infty}(n-1)\}$ is an infinite set,

$$\varpi = \{0, 1, 2, \cdots, \varpi - 1\}.$$

Based on the above analysis, we propose the following two axioms.

Definition 1.2.2 Classical infinite axiom and anti-classical infinite axiom

1. Classical infinite axiom （Standard infinite axiom *ZF6-*）

（1）$\varnothing$ is a set，（2）If n is a set ，$n^+ = n \bigcup \{n\}$ is a set，（3）$\lim_{n\to\infty} n$ also is a set;

$$(\infty = \lim_{n\to\infty} n, \infty = \infty + 1)$$

2. Anti-classical infinite axiom (non-standard infinite axiom *ZF6+*)

（1）$\varnothing$ is a set，（2）If n is a set ，$n^+ = n \bigcup \{n\}$ is a set，（3）$\mu\lim_{n\to\infty} n$ also is a set;

$$(\varpi = \mu\lim_{n\to\infty} n, \quad \varpi \neq \varpi + 1)$$

Example 1.2.1 Difference between standard infinity and non-standard infinity

(1) In standard infinity, infinity ∞ is a number, infinitesimal is 0, $\frac{1}{\infty} = 0$, there is no bit for any number at infinity, a_∞ does not exist.

$$x = 0.a_1a_2a_3\cdots a_n\cdots$$

(2) In non-standard infinity, Multi-level non-standard infinity $\cdots, \varpi - 1, \varpi, \varpi + 1, \cdots$, infinitesimal $\cdots, \frac{1}{\varpi - 1}, \frac{1}{\varpi}, \frac{1}{\varpi + 1}, \cdots$ both are numbers; there are many infinite bits at infinity for hyperreal numbers

$$x = 0.a_1a_2a_3\cdots a_n\cdots a_{\varpi-1}a_\varpi$$

Classical infinite axiom is also called standard infinite axiom, anti-classical infinite axiom is also called non-standard infinite axiom, they are recorded in the set theory system as ***ZF6-*** and ***ZF6+***。

2. Non-Cantor set theory system *SZF*

In Cantor set theory, cycle set refers to sets in form of $x \in x$, also called singular set. A cycle set is a normal phenomenon that occurs when a set is mapped into a number and does not need to be excluded

Example 2.1 Assume that all sets containing a real number $\{x\}$ corresponds to a real number y, that is : $y = f(x)$, $f(x) = \{x\}$,

$$\text{Set}\{x\}: \quad \{2\},\ \{\frac{3}{2}\},\ \{7\},\ \{\sqrt{2}\},\ \ldots\ldots\ldots$$

Corresponds to a real number : $\{2\}=\frac{1}{3}$, $\{\frac{3}{2}\}=\frac{1}{2}$, $\{7\}=-\frac{4}{3}$, $\{\sqrt{2}\}=1-\frac{1}{3}\sqrt{2}$, ………

If: function $f(x)=1-\frac{1}{3}x, \{2\}=\frac{1}{3}, \{7\}=-\frac{4}{3}, \{\sqrt{2}\}=1-\frac{1}{3}\sqrt{2}, \cdots$ and so on.

Ask: is there is set $x\in x$? Convert to above mapping relation, that is : is there $x=f(x)$?

$$x=1-\frac{1}{3}x, x=\frac{3}{4}.$$

That is : $\frac{3}{4}=\{\frac{3}{4}\}$, namely, there exists set $\frac{3}{4}\in\frac{3}{4}$.

Similarly, we can get a cycle set by mapping the set of real numbers containing two elements, set of real numbers containing multiple elements, and even the set of real numbers containing infinite elements into a real number

Set of natural numbers $\{0,1,2,\cdots,n-1\}$ corresponds to a number $f(n)$, $f(n)=n$, that is:

$$\{0\}=1,\ \{0,1\}=2,\ \{0,1,2\}=3,\ \{0,1,2,3\}=4,\ \ldots\ldots\ldots$$

$$f(n)=\{0,1,2,\cdots,n-1\}, n=\{0,1,2,\cdots,n-1\},$$

$$\lim_{n\to\infty} n=\lim_{n\to\infty}(n-1)=\infty.$$

$$\infty=\{0,1,2,\cdots,\infty\},$$

That is $\infty\in\infty$, can be obtained immediately.

$\infty\in\infty$ which a cycle set cannot be explained by classical ZF system and regarded as a singular set needing to be excluded. A cycle set does not cause contradiction and it makes no sense to exclude it.

Thus, we delete axiom of regularity in following *SZF* system.

We use non-standard infinite axiom （***SZF6+***） and the standard infinite axiom (***SZF6-***) to substitute the infinite axiom *ZF*6 in *ZF* and construct two set theory systems ***SZF+***and ***SZF-*** respectively.

1.Primitive symbol

（1）Set: $V=\{\varnothing,x,y,z,\cdots u,v,\cdots A,B,C,\cdots\}$;

（2）Belong to ： $\in$;

（3）Logical symbol: $\neg,\to,\leftrightarrow,\wedge,\vee,\exists,\forall,\vdash$;

（4）Predicate: $A(x),B(x),\cdots A(x,y),B(x,y),\cdots$;

（5）Transfinite calculus $\lim_{n\to\infty} n, \mu\lim_{n\to\infty} n$;

（6）Parentheses：$\{\}$；（，）.

2.Formation rule of set

（1）Empty set $\varnothing$ is a set;

（2）If x, y are sets, then $\{x, y\}$ is a set ;

（3）If x, y are sets, then $x \bigcup y$ is a set .

3.Definition

（1）Union set：$A \bigcup B def = (\forall x)(x \in A \vee x \in B)$；

（2）Successor set：$x^{+} def = x \bigcup \{x\}$；

（3）Subset：$x \subseteq y \; def = (\forall v)(v \in x \rightarrow v \in y)$；

（4）Ordered pair set：$(x, y) \; def = \{\{x\}, \{x, y\}\}$；

（5）Set of direct product：$u \times v \; def = \{(x, y) \mid (x \in u) \wedge (y \in v)\}$；

（6）Power set：$\wp(y) def = (\forall x)(x \in \wp(y) \leftrightarrow x \subseteq y)$.

4.Axiom of set.

***ZF*1:** axiom of extensionality, if elements in two sets are the same, then these two sets are equal, that is:

$$x \in V \vdash (\forall A)(\forall B)(\forall x)(x \in A \leftrightarrow x \in B) \rightarrow (A = B);$$

***ZF*2:** axiom of empty set, we can construct a set $\varnothing$ without any element, that is:

$$x \in V \vdash (\exists \varnothing)(\forall x) \neg (x \in \varnothing);$$

***ZF*3:** axiom of pair set, if x, y are sets, we can construct a set only including set x, y , that is:

$$x \in V \vdash (\forall y)(\forall z)(\exists u)(\forall x)(x \in u \leftrightarrow (x = y \vee x = z));$$

***ZF*4** Axiom of union：For any given set x, there is a set y which has as members all of the members of all of the members of x. that is:

$$x \in V \vdash \forall x \exists y \forall z (z \in y \leftrightarrow \exists u (z \in u \wedge u \in x))$$

（note as $y = def \bigcup x$）

***ZF*5:** axiom of power set, if y is a set, we construct a set A including all subsets of y, that is:

$x \in V \vdash (\forall y)(\exists A)(\forall x)(x \in A \leftrightarrow x \subseteq y)$, note as $A = \wp(y)$;

(Axiom of power set is provable)

***ZF*7:** Axiom of separation**.** $A(x)$ is a property on set V. On V, a subset satisfying property $A(x)$ can be constructed, namely:

$$x \in V \vdash (\exists I)(\forall x)(x \in I \leftrightarrow A(x));$$

(under transconsistent logic, we can substitute “axiom of separation” with following “comprehension axiom”.)

***ZF*8:** Axiom of substitution. If z is a set, make use of function $A(x,y)$ to construct set t, that is:

$$x \in V \vdash (\forall x)(\exists y)A(x,y) \to (\forall z)(\exists t)(\forall u)(u \in t \leftrightarrow (\exists v)(v \in z \wedge A(v,u)));$$

5.Classical infinite axiom and anti-classical infinite axiom

***ZF6-*. Classical infinite axiom （Standard infinite axiom）**

（1）$\varnothing$ is a set，（2）If n is a set ， $n^+ = n \bigcup \{n\}$ is a set，（3）$\lim_{n\to\infty} n$ also is a set;

$$(\infty = \lim_{n\to\infty} n, \infty = \infty + 1)$$

***ZF6+*. Anti-classical infinite axiom (non-standard infinite axiom)**

（1）$\varnothing$ is a set，（2）If n is a set ， $n^+ = n \bigcup \{n\}$ is a set，（3）$\mu\lim_{n\to\infty} n$ also is a set;

$$(\varpi = \mu\lim_{n\to\infty} n, \quad \varpi \neq \varpi + 1)$$

***ZF6-*.** standard infinite axiom, construct an infinite set including original set and all successor set as elements, that is:

$$x \in V \vdash (\exists\infty)(\infty = \lim_{n\to\infty} n) \wedge ((0 \in \infty \wedge n \in \infty) \to n^+ \in \infty);$$

$$\text{Note as } \infty = \lim_{n\to\infty} n, \ \infty = \{ 0 \cdot 1 \cdot 2 \cdot 3 \cdots\cdots \infty \cdots \}.$$

***ZF6+*.** Non-standard axiom of infinity, we can construct an infinite set including original set and all successor sets, that is:

$$x \in V \vdash (\exists\varpi)(\varpi = \mu\lim_{n\to\infty} n) \wedge ((0 \in \varpi \wedge n \in \varpi) \to n^+ \in \varpi).$$

$$\text{Note } \varpi = \mu\lim_{n\to\infty} n, \ \varpi = \{ 0 \cdot 1 \cdot 2 \cdot 3 \cdots\cdots \varpi \}.$$

Definition 2.1 Axiom system of set theory SZF

(1) The axiom system composed of axioms ZF1, ZF2, ZF3, ZF4, ZF5, ZF7, ZF8 in addition with standard infinite axiom ZF6 is called the standard set axiom system, expressed as S***ZF-*;**

(2) The axiom system composed of axioms ZF1, ZF2, ZF3, ZF4, ZF5, ZF7, ZF8, in addition with

non-standard infinite axiom ZF6 + is called non-standard set axiom system, expressed as SZF +.

$$\text{Set theory system } SZF \begin{cases} \text{Standard set theory system } SZF-, (\infty + 1 = \infty); \\ \text{Non-standard set theory system } SZF + (\varpi + 1 \neq \varpi). \end{cases}$$

Power set axiom, axiom of separation, axiom of substitution, axiom of choice, axiom of foundation; In ZF systems, these axioms do not work in finite sets, but only in infinite sets.

In set theory SZF system, we correct axiom of infinity; later we can prove that "for any definable predicate P (n), if it is true to finite n, then for any infinite number *a*, P (*a*) is also true.". Power set axiom, axiom of separation, axiom of substitution, axiom of choice, axiom of foundation all have become provable propositions.

In fact, axiom of empty set, axiom of set, axiom of union and axiom of infinity these four axioms and several definitions are sufficient for set theory system. To preserve the historical appearance of set theory, we reserve these axioms. We will prove these propositions with infinite induction and will not discuss here.

With standard infinite axiom and non-standard infinite axiom, we have constructed two set-theoretic systems, *SZF-*, *SZF+*, respectively. These two systems, similar to Euclidean geometry and non-Euclidean geometry axiom systems are contradictory systems.

We use non-standard infinite axiom (***ZF6+***) and the standard infinite axiom (***ZF-***) to construct two set theory systems ***SZF+*** and ***SZF-*** respectively. These two systems which are similar to Euclidean geometry and non-Euclidean geometry axioms are mutually contradictory systems. Next we discuss ordinal numbers, cardinal numbers, infinite induction and continuum hypothesis in two systems respectively.

Note:In the following discussion on axiom system of standard set *SZF* -, "●"is used for marking.

3. Ordinal numbers And Infinite induction of Non-standard Set Theory (*SZF+*)

3.1 Ordinal number

In axiom system of non-standard set theory *SZF*+, natural numbers can be generalized to transfinite natural numbers, namely ordinal numbers.

Definition3.1.1 Ordinal numbers

(1) $\varnothing$ is an ordinal number, denoted as 0;

(2) If n is an ordinal number, then $n+1 def = n \bigcup \{n\}$ is an ordinal number;

(3) When $n \to \infty$, $\varpi = \mu \lim_{n\to\infty} n$ is an ordinal number;

(4) If α is an ordinal number, then $\alpha+1 def = \alpha\bigcup\{\alpha\}$ is an ordinal number.

Numbers formed by (1) and (2) are finite numbers, also called standard natural numbers; numbers formed by (3) and (4) are infinite numbers, also called non-standard natural numbers; ordinal numbers which are generalized by standard natural numbers are also named transfinite natural numbers.

Limit of $n=\{0\cdot 1\cdot 2\cdots\cdots n-1\}$, $\varpi=\mu\lim\limits_{n\to\infty} n$, get:

$$\varpi=\{0\cdot 1\cdot 2\cdots\cdots\varpi-1\},$$

It contains the subtraction of ordinal numbers. In the axiom system of non-standard set theory *SZF+*, the infinite ordinal number has not only the successor but also the predecessor, which is different from Cantor's infinite ordinal number.

$n-1,\ n-2,\cdots,n-k,\cdots,\ n^2-1,\ n^2-2,\cdots,n^2-k,\cdots$.....etc. are all functions on natural numbers.

When $n\to\infty$, $\varpi-1,\ \varpi-2,\cdots,\varpi-k,\cdots\ \varpi^2-1,\ \varpi^2-2,\cdots,\omega^2-k,\cdots$ are all ordinal numbers.

Different types of ordinal numbers lineages:

$0,1,2,3,\cdots,n,\cdots\cdots$

$\varpi,\varpi+1,\cdots 2\varpi,2\varpi+1,\cdots,3\varpi,3\varpi+1,\cdots\cdots$

$\varpi^2,\varpi^2+1,\cdots 2\varpi^2,2\varpi^2+1,\cdots,3\varpi^2,3\varpi^2+1,\cdots\cdots$

$\varpi^3,\varpi^3+1,\cdots 2\varpi^3,2\varpi^3+1,\cdots,3\varpi^3,3\varpi^3+1,\cdots\cdots$

$2^\varpi,2^\varpi+1,\cdots 2\times 2^\varpi,2\times 2^\varpi+1,\cdots,3\times 2^\varpi,3\times 2^\varpi+1,\cdots\cdots$

$3^\varpi,3^\varpi+1,\cdots 2\times 3^\varpi,2\times 3^\varpi+1,\cdots,3\times 3^\varpi,3\times 3^\varpi+1,\cdots\cdots$

$\varpi^\varpi,\varpi^\varpi+1,\cdots 2\varpi^\varpi,2\varpi^\varpi+1,\cdots,3\varpi^\varpi,3\varpi^\varpi+1,\cdots\cdots$

$\varpi^{\varpi^\varpi},\varpi^{\varpi^\varpi}+1,\cdots 2\varpi^{\varpi^\varpi},2\varpi^{\varpi^\varpi}+1,\cdots,3\varpi^{\varpi^\varpi},3\varpi^{\varpi^\varpi}+1,\cdots\cdots$

…………………………………

$$\mu\lim_{n\to\infty}(kn+m)=k\varpi+m,\ \mu\lim_{n\to\infty}(kn^p+m)=k\varpi^p+m,$$

$$\mu\lim_{n\to\infty}(kp^n+m)=kp^\varpi+m,\ \mu\lim_{n\to\infty}(kn^n+m)=k\varpi^\varpi+m,$$

…and so on are all ordinal numbers ($k,m,p\in N$).

Later we denote set of ordinal number with N^*:

$$N^* = \{0 \cdot 1 \cdot 2 \cdots \cdot \varpi \cdot \varpi + 1 \cdots \cdot 2\varpi \cdot 3\varpi \cdots \cdot \varpi^2 \cdot \varpi^3 \cdots \cdot$$

$$\varpi^n \cdots \cdot n^{\varpi} \cdots \cdot \varpi^{\varpi} \cdots \varpi^{\varpi^{\varpi}} \cdots \varpi^{\varpi^{\cdot^{\cdot^{\varpi}}}} \cdots\}$$

We know that size of standard natural numbers can be compared since it has certain order, so when it is generalized to ordinal numbers, does the ordinal number remain in its original order? The answer is yes.

It is easy to know that when the ordinal number is regarded as a set, satisfying following relations

$$0 \in 1 \in 2 \in 3 \in \cdots \in \varpi - 1 \in \varpi \in \varpi + 1 \in \cdots$$

$$\in 2\varpi - 1 \in 2\varpi \in 2\varpi + 1 \in \cdots \cdot \in 3\varpi \in \cdots \cdot \in \varpi^2 \in \cdots \cdot \in \varpi^3 \cdots \cdot$$

Simultaneously satisfying:

$$0 \subset 1 \subset 2 \subset 3 \subset \cdots \subset \varpi - 1 \subset \varpi \subset \varpi + 1 \subset \cdots$$

$$\subset 2\varpi - 1 \subset 2\varpi \subset 2\varpi + 1 \subset \cdots \subset 3\varpi \subset \cdots \subset \varpi^2 \subset \cdots \cdot \subset \varpi^3 \subset \cdots \cdot$$

This is the same as the Cantor’s ordinal number and can be used to compare size of ordinal numbers.

Definition3.1.2 Size comparison of ordinal numbers

Let $f(\varpi), g(\varpi)$ be ordinal numbers containing ϖ,

(1)If there exists standard natural number n_0, $\forall n(n > n_0 \to f(n) > g(n))$, then $f(\varpi) > g(\varpi)$;

(2) If there exists standard natural number n_0, $\forall n(n > n_0 \to f(n) < g(n))$, then $f(\varpi) < g(\varpi)$;

(3) If for any standard natural number n, $f(n) = g(n)$, then $f(\varpi) = g(\varpi)$.

Obviously, above definition is also true to any ordinal number.

Example 3.1.1 Size comparison of ordinal numbers

Since when $n > 0$, $n + 5 > n + 2$, $n^3 + 7 > n^3 + 1$, so $\varpi + 5 > \varpi + 2$, $\varpi^3 + 7 > \varpi^3 + 1$;

Since when $n > 2$, $n^2 > n + 2$, $n^3 > n^2 + 3$, so $\varpi^2 > \varpi + 2$, $\varpi^3 > \varpi^2 + 3$;

Since when $n > 1$, $n^7 > n^5$, $n^8 > n^5$, $n^9 > n^7$, so $\varpi^7 > \varpi^5$, $\varpi^8 > \varpi^5$, $\varpi^9 > \varpi^7$;

Since when $n > 1$, $2^n > n$, $2^{n^2+1} > n^2$, so $2^{\varpi} > \varpi$, $2^{\varpi^2+1} > \varpi^2$;

Since when $n > 1$, $10^n > 9^n > 7^n > \cdots > 2^n$, so $10^{\varpi} > 9^{\varpi} > 7^{\varpi} > \cdots > 2^{\varpi}$;

Since when $n > 5$, $(n+1)^n > n^n > 5^n > 2^n$, so $(\varpi + 1)^{\varpi} > \varpi^{\varpi} > 5^{\varpi} > 2^{\varpi}$.

It is easy to conclude:

Theorem3.1.1 Size and order of transfinite natural numbers

$$0<1<2<\cdots n<\cdots<\varpi<\varpi+1<\cdots<\varpi+n<\cdots<2\varpi<3\varpi<\cdots<n\varpi<\cdots$$

$$<\varpi^2<\varpi^3<\cdots<\varpi^n<\cdots<2^\varpi<3^\varpi<\cdots<n^\varpi<\cdots<\varpi^\varpi<\cdots<\varpi^{\varpi^\varpi}<\cdots<\varpi^{\varpi^{\cdot^{\cdot^{\varpi}}}}<\cdots$$

Prove: (omit)

This arrangement of transfinite natural numbers is consistent with the algebraic operation of standard natural numbers, and is consistent with intuition.

3.2 Infinite induction

Whether mathematical induction of transfinite natural numbers can be generalized to the set of transfinite natural numbers, whether the generalized induction of transfinite natural numbers is the Cantor’s infinite induction; the questions will be answered in the following.

Let $N^*=\{0\cdot 1\cdot 2\cdots\cdot\varpi\cdot\varpi+1\cdots 2\varpi\cdot 3\varpi\cdots\cdot\varpi^2\cdot\varpi^3\cdots\cdot$

$\varpi^n\cdots\cdot n^\varpi\cdots\cdot\varpi^\varpi\cdots\varpi^{\varpi^\varpi}\cdots\varpi^{\varpi^{\cdot^{\cdot^{\varpi}}}}\cdots\}$ is the set of transfinite natural numbers.

If for finite natural numbers, formula $A(n)$ is true, then for infinite natural numbers, is formula $A(\varpi)$ also true?

More generally: induction for standard natural numbers can be generalized to non-standard natural numbers, that is :

Theorem3.2.1 Transfinite natural number induction

Assume that n is the variable of standard natural number $n\in N$, α is the variable of transfinite natural number $\alpha\in N^*$ and A is a property defined by equality on the set of standard natural numbers,

$$\alpha\in N^* \ \vdash A(0)\wedge[A(n)\to A(n^+)]\to\forall\alpha\in N^* A(\alpha)\,.$$

In set of transfinite natural numbers N^*, induction is still true, so it is also called transfinite natural number induction.

Example3.2.1 It is easy to prove with finite natural number induction:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1} \text{ is true.}$$

Namely: $A(n)$--------Let $A(n)$ be above formula,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{\alpha(\alpha+1)}=\frac{\alpha}{\alpha+1}$$ is true,

Namely: $A(\alpha)$,

Example **3.2.2** Propositions which are true in standard natural numbers are true in non-standard natural numbers, for instance, following propositions of standard natural numbers are true,

$$1^3+2^3+3^3+\cdots+n^3=[\frac{1}{2}n(n+1)]^2;$$

$$\frac{1}{2^2-1}+\frac{1}{3^2-1}+\frac{1}{4^2-1}+\cdots+\frac{1}{n^2-1}=\frac{3}{4}-\frac{1}{2n}-\frac{1}{2(n+1)}。$$

Above formula is also true to transfinite natural number (α is a transfinite natural number),

$$1^3+2^3+3^3+\cdots+\alpha^3=[\frac{1}{2}\alpha(\alpha+1)]^2,$$

$$\frac{1}{2^2-1}+\frac{1}{3^2-1}+\frac{1}{4^2-1}+\cdots+\frac{1}{\alpha^2-1}=\frac{3}{4}-\frac{1}{2\alpha}-\frac{1}{2(\alpha+1)}.$$

"Transfinite natural number induction" also can be regarded as non-standard form of "finite natural number induction.

Finite natural number induction:

$$n\in N \ \vdash \ A(0)\wedge[A(n)\to A(n^+)]\to\forall n\in NA(n),$$

Transfinite natural number induction:

$$\alpha\in N^* \ \vdash \ A(0)\wedge[A(n)\to A(n^+)]\to\forall\alpha\in N^*A(\alpha);$$

3.3 Infinite Sets under infinite induction

Example 3.3.1 Definite of finite set

(1)Natural number $F(n)=\{0,1,2,3,\cdots,n-1\}$. If $F(n)=n$, then $n=\{0,1,2,3,\cdots,n-1\}$;

$$F(1)=\{0\},$$

$$F(2)=\{0,1\},$$

$$F(3)=\{0,1,2\},$$

....

$$F(n)=\{0,1,2,3,\cdots,n-1\},$$

(2)Set of additional layer $L(n)=\underbrace{\{\cdots\{\{\{0\}\}\}\cdots\}}_{n}$;

$$\varnothing = 0 ,$$

$$L(1)=\overset{1}{\{0\}} ,$$

$$L(2)=\overset{2}{\{\{0\}\}} ,$$

$$L(3)=\underbrace{\{\{\{0\}\}\}}_{3} ,$$

....

$$L(n)=\underbrace{\{\cdots\{\{\{0\}\}\}\cdots\}}_{n} .$$

(3)Power set of natural numbers $\wp(n)=\wp(\{0,1,2,3,\cdots,n-1\})$;

$$\varnothing = 0$$

$$\wp(1)=\wp(\{0\})=\{0,\{0\}\} ,$$

$$\wp(2)=\wp(\{0,1\})=\{0,\{0\},\{1\},\{0,\{1\}\}\} ,$$

$$\wp(3)=\wp(\{0,1,2\})=\{0,\{0\},\{1\},\{2\},\{0,1\},\{0,2\},\{1,2\},\{0,1,2\}\} ,$$

…………….

$$\wp(n)=\{0,\{0\},\{1\},\{2\},\cdots,\{n-1\},\{0,1\},\{0,2\},\{1,2\},\cdots,\{0,1,2,3,\cdots,n-1\}\} .$$

(4)If $a_1,a_2,a_3,\cdots,a_n$ are sets, union set $U(n)=\bigcup_{i=1}^{n} a_i$, direct product set $D(n)=\prod_{i=1}^{n} a_i$ are sets.

According to infinite induction, if $\forall n(F(n)=H(n))$ is true, in non-standard analysis,

$$a_1=\{n_{11},n_{12}\} ,$$

$$a_2=\{n_{21},n_{22}\} ,$$

…………..

$$a_n=\{n_{n1},n_{n2}\} ,$$

$$U(n)=\bigcup_{i=1}^{n} a_i=\{n_{11},n_{12}n_{21},n_{22},\cdots,n_{n1},n_{n2}\} .$$

$$D(n)=\prod_{i=1}^{2} a_i=a_1\times a_2=\{(n_{11},n_{21}),(n_{12},n_{21}),(n_{11},n_{22}),(n_{12},n_{22})\} .$$

$$D(n)=\prod_{i=1}^{3} a_i=a_1\times a_2\times a_3$$

$$=\left\{\begin{matrix}(n_{11},n_{21},n_{31}),(n_{12},n_{21},n_{31}),(n_{11},n_{22},n_{31}),(n_{12},n_{22},n_{31}),\\(n_{11},n_{21},n_{32}),(n_{12},n_{21},n_{32}),(n_{11},n_{22},n_{32}),(n_{12},n_{22},n_{32})\end{matrix}\right\}.$$

$\mu\lim_{n\to\infty}F(n)=\mu\lim_{n\to\infty}H(n)$, $F(\varpi)=H(\varpi)$ is also true.

Theorem 3.3.1 Infinite sets in non-standard analysis

(1) Set of infinite natural numbers

For any n, $F(n)=\{0,1,2,3,\cdots,n-1\}$ is a set, so

$$\mu\lim_{n\to\infty}F(n)=\mu\lim_{n\to\infty}\{0,1,2,3,\cdots,n-1\}$$ is set, namely,

$$F(\varpi)=\{0,1,2,3,\cdots,\varpi-1\}$$

If $F(n)=n$, then

$$\varpi=\{0,1,2,3,\cdots,\varpi-1\}.$$

(2) Set of infinite additional layer

For any n, $L(n)=\underbrace{\{\cdots\{\{\{0\}\}\}\cdots\}}_{n}$ is a set, so

$$\mu\lim_{n\to\infty}L(n)=\mu\lim_{n\to\infty}\underbrace{\{\cdots\{\{\{0\}\}\}\cdots\}}_{n}$$ is a set, namely,

$$L(\varpi)=\underbrace{\{\cdots\{\{\{0\}\}\}\cdots\}}_{\varpi}.$$

(3) Infinite power set $\wp(n)=\wp(\{0,1,2,3,\cdots,n-1\})$;

For any n, $\wp(n)=\wp(\{0,1,2,3,\cdots,n-1\})$ is a set, so

$$\mu\lim_{n\to\infty}\wp(n)=\mu\lim_{n\to\infty}\wp(\{0,1,2,3,\cdots,n-1\})$$ is a set, namely,

$$\wp(\varpi)=\wp(\{0,1,2,3,\cdots,\varpi-1\})$$

(4) Infinite union set, infinite direct product set;

If $a_1,a_2,a_3,\cdots,a_n$ are sets, for any n, $U(n)=\bigcup_{i=1}^{n}a_i$, $D(n)=\prod_{i=1}^{n}a_i$ are sets, therefore

Infinite union set and infinite direct product set

$$\mu\lim_{n\to\infty}U(n)=\mu\lim_{n\to\infty}\bigcup_{i=1}^{n}a_i,\quad \mu\lim_{n\to\infty}D(n)=\mu\lim_{n\to\infty}\prod_{i=1}^{n}a_i$$ are sets, namely,

$$U(\varpi)=\bigcup_{i=1}^{\varpi}a_i,\ D(\varpi)=\prod_{i=1}^{\varpi}a_i.$$

●4. Ordinal numbers And Infinite induction of Non-standard Set Theory（*SZF-*）

In *axiom system of* standard set theory *SZF-, since*

$$\lim_{n\to\infty} n=\infty\,,\quad \infty+n=\infty\,,$$

Calculate limit of $n=\{0\cdot1\cdot2\cdots\cdot n-1\}$, get

$$\infty=\{0\cdot1\cdot2\cdots\cdot\infty\}$$

Theorem4.1.1 In *SZF*-axiom system of the standard set theory, all infinite numbers are equal, and there is only one infinite ∞

$$\cdots=\infty-2=\infty-1=\infty=\infty+1=\infty+n=2\infty=n\infty=\cdots$$

$$=\infty^2=\infty^3=\infty^n=2^\infty=3^\infty=\infty^\infty=\infty^{\infty^\infty}=\cdots$$

Later we denote set of ordinal numbers with N, while N is the set of natural numbers:

$$N=\{0\cdot1\cdot2\cdots\infty\ \}$$

In the axiom system (*SZF*-) of standard set theory, “transfinite natural number induction” changes to following form,

Theorem4.2.1 Transfinite natural number induction

Assume that n is the variable of standard natural number, namely $n\in N$, A is a property defined on equality on the set of finite natural numbers,

$$\vdash A(0)\wedge[A(n)\to A(n^+)]\to A(\infty)\,.$$

In set of standard natural numbers, N, if $A(n)$ is true, then $A(\infty)$ is also true.

Example 4.2.1 It is easy to prove with finite natural number induction:

$$\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}\,;$$

$$\frac{1}{2^2-1}+\frac{1}{3^2-1}+\frac{1}{4^2-1}+\cdots+\frac{1}{n^2-1}=\frac{3}{4}-\frac{1}{2n}-\frac{1}{2(n+1)}\,.$$

For ∞, infinite series is also true,

$$\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\cdots+\frac{1}{i(i+1)}+\cdots=\lim_{n\to\infty}\frac{n}{n+1}=1,$$

$$\frac{1}{2^2-1}+\frac{1}{3^2-1}+\frac{1}{4^2-1}+\cdots+\frac{1}{i^2-1}+\cdots=\lim_{n\to\infty}[\frac{3}{4}-\frac{1}{2n}-\frac{1}{2(n+1)}]=\frac{3}{4}\text{。}$$

According to infinite induction, if $\forall n(F(n)=H(n))$ is true, in standard analysis,

$\lim_{n\to\infty}F(n)=\lim_{n\to\infty}H(n)$, $F(\infty)=H(\infty)$ is also true.

Theorem 4.3.1 Infinite sets in standard analysis

(1)Set of infinite natural numbers

For n, $F(n)=\{0,1,2,3,\cdots,n-1\}$ is a set, so

$$\lim_{n\to\infty}F(n)=\lim_{n\to\infty}\{0,1,2,3,\cdots,n-1\}$$ is a set, namely,

$$F(\infty)=\{0,1,2,3,\cdots,\infty\}$$

(2) Set of infinite additional layer

For n, $L(n)=\underbrace{\{\cdots\{\{\{0\}\}\}\cdots\}}_{n}$ is a set, so

$$\lim_{n\to\infty}L(n)=\lim_{n\to\infty}\underbrace{\{\cdots\{\{\{0\}\}\}\cdots\}}_{n}$$ is a set, namely,

$$L(\infty)=\underbrace{\{\cdots\{\{\{0\}\}\}\cdots\}}_{\infty}.$$

(3) Infinite power set $\wp(n)=\wp(\{0,1,2,3,\cdots,n-1\})$;

For any n, $\wp(n)=\wp(\{0,1,2,3,\cdots,n-1\})$ is a set, so

$$\lim_{n\to\infty}\wp(n)=\lim_{n\to\infty}\wp(\{0,1,2,3,\cdots,n-1\})$$ is a set, so

$$\wp(\infty)=\wp(\{0,1,2,3,\cdots,\infty\})$$

If $F_1(n)=n$, then

$$\infty=\{0,1,2,3,\cdots,\infty\}.$$

(4)Infinite union set, infinite direct product set;

If $a_1,a_2,a_3,\cdots,a_n$ are sets, for n, $U(n)=\bigcup_{i=1}^{n}a_i$, $D(n)=\prod_{i=1}^{n}a_i$ are sets, therefore

Infinite union set and infinite direct product sets

$$\lim_{n\to\infty}U(n)=\lim_{n\to\infty}\bigcup_{i=1}^{n}a_i,\ \lim_{n\to\infty}D(n)=\lim_{n\to\infty}\prod_{i=1}^{n}a_i$$ are sets, namely

$$U(\infty)=\bigcup_{i=1}^{\infty}a_i\ ,D(\infty)=\prod_{i=1}^{\infty}a_i\ .$$

In a word, Axiom of infinity in *ZF* system is false. Axiom of infinity must be established with ($\delta + 1 = \delta$ or $\delta + 1 \neq \delta$). The infinite induction in Cantor's set theory is also false; in fact, infinite induction is very simple, For any definable predicate P (n), if it is true to finite n, then for any infinite number *a*, P(*a*) is also true. After we have solved the axiom of infinity, infinity is as simple as finite.

Appendix References

[1]Hamilton, A.G: Logic for Mathematicians, Cambridge of University, 1978: 29-40, 82-92.

[2][U.S]S.C Kleene , translated by Shaokui Mo, Introduction to Meta-Mathematics, Beijing: Science Press, 1984:4-12.

[3][U.S]Boolos, Computability and Mathematical Logic [M] Beijing: Electronic Industry Press, 2002: 152-160

[4] Fangting Wang, Mathematical Logics [M], Beijing: Science Press 2001: 159-163

[5] Huacan He, Universal Infinite Theory [M] Beijing: Science Press 2011

[6]Jincheng Zhang, Logic System and Contradiction [J] System Intelligent Journal, 2012(3): 208-209

[7]Jincheng Zhang, Fixed Terms and Undecidable Propositions of Logics and Mathematic Calculus (I)[J] System Intelligent Journal, 2014(4)

[8]Jincheng Zhang, Fixed Terms and Undecidable Propositions of Logics and Mathematic Calculus (I)[J] System Intelligent Journal, 2014(5)

[9] Jincheng Zhang. *Paradox*、*Logic And Non-Cantor Set Theory* [M]. Harbin: Harbin Institute of Technology Press，2018:1.

[10]Jinwen Zhang, Introduction to Axiom Set Theory[M] Beijing: Science Press, February 1999

[11] Mumin Dai & Haiyan Chen, Introduction to Axiom Set Theory[M], Beijing Science Press, 2011: 15-17

[12] Mathematics encyclopedic dictionary, Japan Mathematics Collection [M] Beijing: Science Press 1984:7