Sound Relativistic Hamiltonians and Quantum Propagators

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Abstract Dirac erroneously tried to impose space-time symmetry on the time-skewed Schrödinger equation, which is the time component of a Lorentz-covariant four-vector system of equations—that system's three space-component equations specify the quantum three-momentum operator in coordinate representation. Dirac's misconception resulted in a noninteracting-particle Hamiltonian that isn't the time component of a Lorentz-covariant four-momentum times c, and which causes the noninteracting particle to spontaneously undergo immense acceleration of the order of c squared divided by the particle's Compton wavelength, and to also have a fixed unphysical speed which is c times the square root of three. Dirac's Hamiltonian has a physically untenable unbounded-below set of negative energy eigenvalues, which have been airily "reinterpreted" as (very questionably) implying propagation backward in time. Dirac's misconceived Hamiltonian is in any case irrelevant since a noninteracting particle's Lorentz-covariant four-velocity times its mass m times c has a time component which is a superbly-behaved Hamiltonian with a simple space-time propagator for quantum wave functions. Via a Lorentz-invariant action integral, Lorentz long ago extended this noninteracting-particle Hamiltonian to describe the particle's interaction with an electromagnetic four-potential. Here we modify Lorentz's Lorentz-invariant action integral to accommodate the spin-1/2 particle by adding the Lorentz-invariant extrapolation of the nonrelativistic spin-1/2 particle's magnetic-moment potential energy in a magnetic field. We also point out the important fact that when particles can be produced, perturbation contributions become increasingly invalid with increasingly high virtual momentum values, which must be cut off.

1. The misconception about Lorentz covariance which produced the Dirac equation

Lorentz transformations of the coordinate-space vector $\mathbf{r}$ and time t leave the locus $|\mathbf{r}|^2 - (ct)^2 = 0$ of the expanding spherical-shell light-wave front invariant; indeed they leave the quadratic form $(ct)^2 - |\mathbf{r}|^2$ invariant regardless of what its value is, somewhat as spatial rotations leave invariant the quadratic form $|\mathbf{r}|^2$ regardless of its value. This fundamental characteristic of Lorentz transformations spawned the notion that the time entity $x^0 \stackrel{\text{def}}{=} ct$ *is entirely equivalent for purposes of physics to the components of the space vector* $\mathbf{r}$; however *the minus sign* which occurs in the Lorentz-invariant quadratic form $(x^0)^2 - |\mathbf{r}|^2$ indicates *that this can't be entirely true,* even *notwithstanding* the fact that Lorentz transformations *intermingle* x^0 *with the components of* $\mathbf{r}$. Indeed, for any two space-time events there *always* exists some inertial frame of reference in which they occur *at the same space point sequentially in time,* or else in which they occur *at different space points simultaneously,* or else an expanding spherical-shell light-wave front that originates in one of the events intersects the other. Just as mistaken as concluding that special relativity puts x^0 on exactly the same physical footing as the components of $\mathbf{r}$ is subscribing to the *half-truth* that special relativity *requires every relation of physical significance to exhibit space-time symmetry.* However *no single one* of the four Laws of electromagnetism (i.e., *no single one* of the four Maxwell equations) *is space-time symmetric by itself;* those equations *must be organized into one of two particular equation pairs* or else *converted to second order in time and space derivatives* before space-time symmetry emerges. But that surely doesn't imply that Coulomb's Law by itself has no physical significance, or that Gauss' Law is bereft of physical significance until it is paired with Faraday's Law.

Returning now to firmer ground, it is fairly straightforward to use the Lorentz-covariant four vector $(x^0, \mathbf{r})$ and its closely-related Lorentz-invariant entity $(x^0)^2 - |\mathbf{r}|^2$ to construct many other physically-interesting Lorentz-invariant and Lorentz-covariant entities. One of these is obviously the Lorentz-invariant differential quadratic form $(dx^0)^2 - |d\mathbf{r}|^2 = (c\,dt)^2 - |d\mathbf{r}|^2$. Another is its Lorentz-invariant square root divided by c (c is a constant that is obviously Lorentz invariant itself). The latter is called the Lorentz-invariant differential time $d\tau$ because,

$$\left(\sqrt{(c\,dt)^2 - |d\mathbf{r}|^2}\,\right)/c = \sqrt{(dt)^2 - |d\mathbf{r}/c|^2} = dt\sqrt{1 - |(d\mathbf{r}/dt)/c|^2} = dt\sqrt{1 - |\dot{\mathbf{r}}/c|^2} = d\tau. \tag{1.1}$$

The Lorentz-invariant differential time $d\tau$ is *also* called *proper* differential time because it is the shorter differential time interval that is observed to be recorded by a clock traveling at velocity $\dot{\mathbf{r}}$ when the observer's own clock, which is *at rest* next to him, is observed to record the longer differential time interval dt—this observed slowing of clock mechanisms which are in motion with respect to the observer is of course called special-relativistic time dilation.

Since differential proper time $d\tau$ is Lorentz invariant, the entity,

$$(dx^0/d\tau,\ d\mathbf{r}/d\tau) = (c(dt/dt),\ d\mathbf{r}/dt)/\sqrt{1 - |(\dot{\mathbf{r}}/c|^2} = (c,\ \dot{\mathbf{r}})/\sqrt{1 - |\dot{\mathbf{r}}/c|^2}, \tag{1.2a}$$

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Lorentz transforms exactly the same way as the space-time four-vector $(x^0,\ \mathbf{r})$ Lorentz transforms; we call such entities *Lorentz-covariant four-vectors.* We in particular refer to the Lorentz-covariant four-vector $(dx^0/d\tau,\ d\mathbf{r}/d\tau) = (c,\ \dot{\mathbf{r}})/\sqrt{1-|\dot{\mathbf{r}}/c|^2}$ as *the proper-velocity four-vector,* and note that its Lorentz-invariant inner product with itself, namely $(dx^0/d\tau)^2 - |d\mathbf{r}/d\tau|^2$, which is equal to $(c^2 - |\dot{\mathbf{r}}|^2)/(1-|\dot{\mathbf{r}}/c|^2)$, *has the constant value* c^2.

If we multiply a noninteracting particle's Lorentz-covariant proper-velocity four-vector $(dx^0/d\tau,\ d\mathbf{r}/d\tau)$ by its Lorentz-invariant constant rest mass m, the result is the noninteracting particle's Lorentz-covariant relativistic-momentum four-vector $(p^0,\ \mathbf{p})$,

$$(p^0,\ \mathbf{p}) \stackrel{\text{def}}{=} (m\,dx^0/d\tau,\ m\,d\mathbf{r}/d\tau) = (mc,\ m\dot{\mathbf{r}})/\sqrt{1-|\dot{\mathbf{r}}/c|^2}, \tag{1.2b}$$

whose Lorentz-invariant inner product with itself, $(p^0)^2 - |\mathbf{p}|^2$, is of course equal to the constant value $(mc)^2$. That result implies that $p^0 = \sqrt{(mc)^2 + |\mathbf{p}|^2}$, which permits the noninteracting particle's Lorentz-covariant relativistic four-momentum $(p^0,\ \mathbf{p})$ to be expressed as,

$$(p^0,\ \mathbf{p}) = (\sqrt{(mc)^2 + |\mathbf{p}|^2},\ \mathbf{p}) = (mc\sqrt{1+|\mathbf{p}/(mc)|^2},\ \mathbf{p}). \tag{1.2c}$$

The entity $E = cp^0 = mc^2\sqrt{1+|\mathbf{p}/(mc)|^2}$ has the dimension of energy, and in the nonrelativistic limit where $|\mathbf{p}/m| \ll c$, which implies that $|\mathbf{p}/(mc)|^2 \ll 1$, E has the following approximation,

$$E = mc^2 + (|\mathbf{p}|^2/(2m))(1 + O(|\mathbf{p}/(mc)|^2)), \tag{1.2d}$$

which is the relativistic energy mc^2 of the noninteracting particle's rest mass m plus the noninteracting particle's nonrelativistic kinetic energy and Hamiltonian $(|\mathbf{p}|^2/(2m))$ together with that nonrelativistic entity's relativistic corrections; *those corrections vanish in the* $c \to \infty$ *nonrelativistic limit.*

Since $E = cp^0 = mc^2\sqrt{1+|\mathbf{p}/(mc)|^2}$ *becomes a noninteracting particle's Hamiltonian plus the constant* mc^2 *in the nonrelativistic limit* $c \to \infty$, and since E is the "time component" of a Lorentz-covariant four-vector, namely of $(E,\ c\mathbf{p}) = (mc^2\sqrt{1+|\mathbf{p}/(mc)|^2},\ c\mathbf{p}) = (cp^0,\ c\mathbf{p}) = (mc(dx^0/d\tau),\ mc(d\mathbf{r}/d\tau))$, $E = mc^2\sqrt{1+|\mathbf{p}/(mc)|^2}$ *consequently is a noninteracting particle's relativistic Hamiltonian* $H(\mathbf{p})$.

Since this relativistic Hamiltonian $H(\mathbf{p}) = mc^2\sqrt{1+|\mathbf{p}/(mc)|^2}$ for a noninteracting particle is the "time component" of a Lorentz-covariant four-vector, *it is obviously entirely skewed toward time.* In fact, a properly relativistic Hamiltonian *is always the "time component" of a Lorentz-covariant four-vector whose "space components" are* $c\mathbf{p}$, so a properly relativistic Hamiltonian *is always entirely skewed toward time.*

Since the Hamiltonian $H(\mathbf{p}) = mc^2\sqrt{1+|\mathbf{p}/(mc)|^2}$ is entirely skewed toward time, and partial differentiation with respect to time t is as well entirely skewed toward time, the Schrödinger equation,

$$i\hbar(\partial\psi(\mathbf{r},t)/\partial t) = H(\mathbf{p})\psi(\mathbf{r},t), \tag{1.2e}$$

is unobjectionable from the standpoint of special relativity. It *isn't clear,* however, how the right side of this Schrödinger equation, namely $H(\mathbf{p})\psi(\mathbf{r},t)$ should be interpreted, but the fact that the Hamiltonian $H(\mathbf{p})$ is the "time component" of the Lorentz-covariant four-vector $(H(\mathbf{p}), c\mathbf{p})$ *turns out to be helpful in that regard.* The *reason* this is so is that the partial derivative with respect to time $(\partial/\partial t)$ is *also* the "time component" of *a Lorentz-covariant four-vector of partial derivatives,* which in detail comes out to be $((\partial/\partial t), -c\nabla_{\mathbf{r}})$. By using *the two Lorentz-covariant four-vector entities* $((\partial/\partial t), -c\nabla_{\mathbf{r}})$ *and* $(H(\mathbf{p}), c\mathbf{p})$, we can write down *the Lorentz-covariant four-vector version of the* Eq. (1.2e) *time-skewed Schrödinger equation,*

$$i\hbar((\partial/\partial t), -c\nabla_{\mathbf{r}})\psi(\mathbf{r},t) = (H(\mathbf{p}), c\mathbf{p})\psi(\mathbf{r},t). \tag{1.2f}$$

Separation of this Lorentz-covariant *four-vector* Schrödinger equation into its *entirely time-skewed time-component equation* and its *entirely space-skewed three-vector of space-component equations* produces,

$$i\hbar(\partial\psi(\mathbf{r},t)/\partial t) = H(\mathbf{p})\psi(\mathbf{r},t) \text{ and } -i\hbar\nabla_{\mathbf{r}}\psi(\mathbf{r},t) = \mathbf{p}\,\psi(\mathbf{r},t). \tag{1.2g}$$

The Eq. (1.2g) *entirely space-skewed three-vector equation,* $\mathbf{p}\,\psi(\mathbf{r},t) = -i\hbar\nabla_{\mathbf{r}}\psi(\mathbf{r},t)$, *directly tells us how* $\mathbf{p}\,\psi(\mathbf{r},t)$ *is interpreted.* That interpretation of the effect of the operator $\mathbf{p}$ on functions of $\mathbf{r}$ makes it straightforward *to verify the familiar fundamental quantum-mechanics operator commutation relation,*

$$[(\mathbf{r})^i,\ (\mathbf{p})^j] = i\hbar\delta^{ij}. \tag{1.2h}$$

It is fascinating that the Eq. (1.2f) Lorentz-covariant *four-vector* version of the Schrödinger equation *also implies the* Eq. (1.2h) *fundamental quantum-mechanics operator commutation relation.* This intimate intertwining of quantum mechanics with relativity *is far too seldom pointed out.*

The Eq. (1.2f) Lorentz-covariant *four-vector* version of the Schrödinger equation *is clearly space-time symmetric,* but paradoxically *its usefulness in quantum mechanics resides in its entirely time-skewed equation* $i\hbar(\partial\psi(\mathbf{r},t)/\partial t) = H\psi(\mathbf{r},t)$ *and in its entirely space-skewed equation* $\mathbf{p}\,\psi(\mathbf{r},t) = -i\hbar\nabla_{\mathbf{r}}\psi(\mathbf{r},t)$.

Dirac unfortunately had never laid eyes on the *space-time symmetric* Eq. (1.2f) Lorentz-covariant four-vector version of the Schrödinger equation, and furthermore was *so imbued* with the *half-truth* that in special relativity *every relation of physical significance must exhibit space-time symmetry* that he was altogether unable to deal appropriately with the noninteracting particle's relativistic Hamiltonian $H(\mathbf{p}) = mc^2\sqrt{1 + |\mathbf{p}/(mc)|^2}$, which, as *the time component* of the Lorentz-covariant four-vector $(H(\mathbf{p}), c\mathbf{p})$, *is entirely time-skewed,* exactly as the Schrödinger equation it enters into *is entirely time-skewed.*

Dirac set about *modifying* $H(\mathbf{p})$ *in order to force physically-inappropriate space-time symmetry on the entirely time-skewed Schrödinger equation it enters into.* Noting that $H(\mathbf{p}) = cp^0$, and that cp^0 *is linear in* p^0, Dirac declared that space-time symmetry requires that $H(\mathbf{p})$ *be superseded by an entity* $H_D(\mathbf{p})$ *which is correspondingly linear in* $\mathbf{p}$. But *in order to attempt to preserve in* $H_D(\mathbf{p})$ *the physical facts which* $H(\mathbf{p})$ *expresses,* Dirac further declared that $(H_D(\mathbf{p}))^2$ *must be equal to* $(H(\mathbf{p}))^2$. To facilitate *achieving* $(H_D(\mathbf{p}))^2 = (H(\mathbf{p}))^2$, Dirac proposed for $H_D(\mathbf{p})$ the following form, which is *inhomogeneously linear in* $\mathbf{p}$,

$$H_D(\mathbf{p}) = mc^2\beta \ + c\mathbf{p}\cdot\vec{\alpha}, \tag{1.3a}$$

where β and $\vec{\alpha}$ are dimensionless. The upshot of requiring that $(H_D(\mathbf{p}))^2 = (H(\mathbf{p}))^2$ is that $\beta^2 = (\alpha^1)^2 = (\alpha^2)^2 = (\alpha^3)^2 = 1$, and that β *anticommutes with* α^1, α^2 *and* α^3, α^1 *anticommutes with* α^2 *and* α^3, *and* α^2 *anticommutes with* α^3. Realization of this algebra, which is very similar to the Pauli algebra for the three Hermitian matrices σ^1, σ^2 and σ^3, requires Hermitian matrices for β, α^1, α^2 and α^3 (the Dirac Hamiltonian $H_D(\mathbf{p})$ *must of course be Hermitian)* whose dimension is at least 4×4; hence Dirac's 4-spinor wave functions.

The *linearity* in $\mathbf{p}$ of the Dirac Hamiltonian $H_D(\mathbf{p})$, which Dirac *insisted* on, causes $(H_D(\mathbf{p}), c\mathbf{p})$ *to fail to be Lorentz-covariant,* and *relatedly* renders $H_D(\mathbf{p})$ *utterly incapable of conforming to the physically-correct energy-momentum relation* $H_D(\mathbf{p}) \approx mc^2 + (|\mathbf{p}|^2/(2m))$ *of the nonrelativistic regime when* $|\mathbf{p}/m| \ll c$ because the nonrelativistic energy $mc^2 + (|\mathbf{p}|^2/(2m))$ *has no term whatsoever which is linear in* $\mathbf{p}$!

The *failure* of $(H_D(\mathbf{p}), c\mathbf{p})$ to be Lorentz-covariant *is devastatingly reflected by the Dirac particle's speed operator* $|\dot{\mathbf{r}}|$. To obtain its velocity operator $\dot{\mathbf{r}}$ we apply the two quantum commutation relations,

$$\dot{\mathbf{r}} = (-i/\hbar)[\mathbf{r},\ H_D(\mathbf{p})] = (-i/\hbar)[\mathbf{r},\ mc^2\beta + c\mathbf{p}\cdot\vec{\alpha}] \ \text{ and } \ [(\mathbf{r})^i,\ (\mathbf{p})^j] = i\hbar\delta^{ij}, \tag{1.3b}$$

which yield $\dot{\mathbf{r}} = c\vec{\alpha}$. Therefore the Dirac particle's speed operator $|\dot{\mathbf{r}}|$ is,

$$|\dot{\mathbf{r}}| = c|\vec{\alpha}| = c\sqrt{(\vec{\alpha}\cdot\vec{\alpha})} = c\sqrt{(\alpha^1)^2 + (\alpha^2)^2 + (\alpha^3)^2} = c\sqrt{3}, \tag{1.3c}$$

which always grossly violates special relativity's speed limit $|\dot{\mathbf{r}}| < c$ for a particle of nonzero mass m.

Furthermore, the Dirac velocity operator $\dot{\mathbf{r}} = c\vec{\alpha}$ *doesn't commute with the Dirac Hamiltonian* $H_D(\mathbf{p}) = mc^2\beta + c\mathbf{p}\cdot\vec{\alpha}$ because, inter alia, *every component of* $\vec{\alpha}$ *anticommutes with* β. Consequently, the *noninteracting* Dirac particle undergoes *acceleration* whose order of magnitude is $(mc^3/\hbar)$, which when m is the electron mass ($mc^2 = 0.511$ MeV), is around $10^{28}g$, where $g = 9.8m/s$, the acceleration of gravity at the Earth's surface. This *stupendous acceleration* of the *noninteracting* Dirac particle of $10^{28}g$ is yet another *staggering defect* of Dirac's Hamiltonian. Further study of the time behavior of the Dirac particle's coordinate vector $\mathbf{r}$ under the influence of the ostensibly *noninteracting* Dirac Hamiltonian $H_D(\mathbf{p})$ shows this immense acceleration to be an aspect of a *spontaneous* violent oscillatory *zitterbewegung* motion of the Dirac particle. The *speed* of that oscillatory motion is around the speed of light c, which is no doubt why the Dirac particle's speed operator $|\dot{\mathbf{r}}|$ has the fixed unphysical value $c\sqrt{3}$. Such *spontaneous* violent oscillatory motion on the part of the *noninteracting* Dirac particle shows that Dirac *erred* in *insisting* that special relativity *requires all physically significant relations to exhibit space-time symmetry,* in *pursuit* of which he created a noninteracting-particle Hamiltonian $H_D(\mathbf{p})$ *which is linear in* $\mathbf{p}$, heedless of the two related facts *that a Hamiltonian which is linear in* $\mathbf{p}$ *violates the nonrelativistic regime,* and that $(H_D(\mathbf{p}), c\mathbf{p})$ *fails to be Lorentz-covariant.*

Contrariwise, for the noninteracting-particle Hamiltonian $H(\mathbf{p}) = mc^2\sqrt{1 + |\mathbf{p}/(mc)|^2}$, $(H(\mathbf{p}), c\mathbf{p})$ *is Lorentz-covariant.* Here the particle's velocity operator $\dot{\mathbf{r}}$ *also* follows via the commutator of $\mathbf{r}$ with $H(\mathbf{p})$,

$$\dot{\mathbf{r}} = (-i/\hbar)\left[\mathbf{r},\, mc^2\sqrt{1+|\mathbf{p}/(mc)|^2}\right] = mc^2\nabla_{\mathbf{p}}\left(\sqrt{1+|\mathbf{p}/(mc)|^2}\right) = (\mathbf{p}/m)/\sqrt{1+|\mathbf{p}/(mc)|^2}, \tag{1.4a}$$

where we have used the fact that in momentum representation $\mathbf{r} = i\hbar\nabla_{\mathbf{p}}$; the fundamental commutation relation $[(\mathbf{r})^i, (\mathbf{p})^j] = i\hbar\delta^{ij}$ implies that in coordinate representation $\mathbf{p} = -i\hbar\nabla_{\mathbf{r}}$ and in momentum representation $\mathbf{r} = i\hbar\nabla_{\mathbf{p}}$. We see from the Eq. (1.4a) result for $\dot{\mathbf{r}}$ that,

$$|\dot{\mathbf{r}}| = c\left(|\mathbf{p}/(mc)|/\sqrt{1+|\mathbf{p}/(mc)|^2}\right) < c. \tag{1.4b}$$

That $|\dot{\mathbf{r}}| < c$ *is precisely what is expected of the speed* $|\dot{\mathbf{r}}|$ *of a properly relativistic particle which has nonzero mass.* This relativistically *sensible* result stands *in immense contrast* to the grossly relativity-violating result $|\dot{\mathbf{r}}| = c\sqrt{3}$ that follows from the Dirac Hamiltonian $H_D(\mathbf{p})$.

We next ask whether the relativistic velocity operator $\dot{\mathbf{r}} = (\mathbf{p}/m)/\sqrt{1+|\mathbf{p}/(mc)|^2}$ *commutes with the relativistic Hamiltonian operator* $H(\mathbf{p}) = mc^2\sqrt{1+|\mathbf{p}/(mc)|^2}$. Since *both* of these entities *contain no operators aside from functions of the operator* $\mathbf{p}$ (and are also free of noncommuting matrices), they do indeed commute with each other, so here *the noninteracting particle experiences no acceleration whatsoever,* which of course *is precisely what is expected of a noninteracting particle.* Again, this physically *sensible* result for $H(\mathbf{p})$ stands *in immense contrast* to the Dirac $H_D(\mathbf{p})$ result, which shows the *noninteracting* particle to undergo *spontaneous* violent *zitterbewegung* oscillations which have *stupendous acceleration.*

We next consider the eigenvalue spectrum of $H(\mathbf{p})$ and $H_D(\mathbf{p})$. Since $H(\mathbf{p}) = mc^2\sqrt{1+|\mathbf{p}/(mc)|^2}$, any arbitrary momentum eigenfunction (plane wave in coordinate-$\mathbf{r}$ representation) whose momentum eigenvalue is $\mathbf{p}_0$ is an eigenfunction of $H(\mathbf{p})$ with the eigenvalue $mc^2\sqrt{1+|\mathbf{p}_0/(mc)|^2}$, which can be any positive energy value that is greater than or equal to mc^2. Therefore the eigenvalue spectrum of $H(\mathbf{p})$ consists of all positive energy values which are greater than or equal to mc^2.

If we apply the noninteracting-particle Dirac Hamiltonian $H_D(\mathbf{p}) = mc^2\beta + c\mathbf{p}\cdot\vec{\alpha}$ to an arbitrary momentum eigenfunction whose momentum eigenvalue is $\mathbf{p}_0$, the result is $H_D(\mathbf{p}_0) = mc^2\beta + c\mathbf{p}_0\cdot\vec{\alpha}$, a 4×4 Hermitian matrix which has the dimension of energy. Because of the particular properties of the four 4×4 matrices β, α^1, α^2 and α^3, multiplying the matrix $H_D(\mathbf{p}_0)$ by itself produces the positive number $((mc^2)^2 + c^2|\mathbf{p}_0|^2)$, which has the dimension of energy squared, times the 4×4 identity matrix I,

$$(H_D(\mathbf{p}_0))^2 = ((mc^2)^2 + c^2|\mathbf{p}_0|^2)\mathrm{I}. \tag{1.5}$$

Eq. (1.5) is *the characteristic equation* of the 4×4 Hermitian matrix $H_D(\mathbf{p}_0)$. The eigenvalues of a Hermitian matrix *are the roots of its characteristic equation.* Therefore the eigenvalues of the 4×4 Hermitian matrix $H_D(\mathbf{p}_0)$ are $mc^2\sqrt{1+|\mathbf{p}_0/(mc)|^2}$ and $-mc^2\sqrt{1+|\mathbf{p}_0/(mc)|^2}$

Since $\mathbf{p}_0$ can be any three-vector of real numbers which have the dimension of momentum, an eigenvalue of $H_D(\mathbf{p})$ can be any positive number with the dimension of energy which is greater than or equal to mc^2, or any negative number with the dimension of energy which is less than or equal to $-mc^2$. Therefore the eigenvalue spectrum of $H_D(\mathbf{p})$ consists of all positive energy values which are greater than or equal to mc^2 and all negative energy values which are less than or equal to $-mc^2$

It is now clear that the Hamiltonian $H(\mathbf{p}) = mc^2\sqrt{1+|\mathbf{p}/(mc)|^2}$ describes a single noninteracting relativistic particle of mass m, but this definitely *isn't* the case for the Dirac Hamiltonian $H_D(\mathbf{p}) = mc^2\beta + c\mathbf{p}\cdot\vec{\alpha}$ because a single noninteracting relativistic particle of mass m *absolutely must have only energy eigenvalues which are greater than or equal to* mc^2.

The "solution" has been to propagate the negative-energy eigenstates of the Dirac equation backward in time and to interpret those as positive energy antiparticles propagating forward in time. This superficially may seem to cleverly "cut the Gordian knot", but it cannot cope with the fact that adding an arbitrary constant energy to almost any Hamiltonian has no physical consequences, so the *sign* of a physical state's energy *can't be definitively linked to its behavior* in the way which is implied. Can we sensibly assert that the bound states of the hydrogen atom propagate backward in time because they are customarily assigned negative energy values? Moreover, is there even so much as a *gedanken* experiment *test* of whether a physical system is propagating backward in time? The concept of a physical system propagating backward in time seems *ill-defined,* and might horrify a faithful adherent of thermodynamics. In sum, this sophomoric idea *whose very obvious underlying motivation was to "save" the Dirac Hamiltonian and the Klein-Gordon equation from richly-merited oblivion* doesn't bear scrutiny.

Moreover, this confection *diverts attention from the core issue that the Dirac Hamiltonian flouts special relativity.* If the Dirac Hamiltonian *accorded* with special relativity, *there would be no negative-energy eigenstates in the first place* to propagate backward in time; *the completely positive energy-eigenvalue spectrum of the legitimately relativistic noninteracting-particle Hamiltonian* $H(\mathbf{p}) = mc^2\sqrt{1 + |\mathbf{p}/(mc)|^2}$ makes that *obvious.* The *essence* of the violation of special relativity by the Dirac Hamiltonian $H_D(\mathbf{p})$ *is the failure of* $(H_D(\mathbf{p}), c\mathbf{p})$ *to be Lorentz-covariant,* and *a clear confirmation* of that violation of special relativity by $H_D(\mathbf{p})$ is the result that the *speed operator* $|\dot{\mathbf{r}}|$ of the Dirac particle is $c|\vec{\alpha}| = c\sqrt{3} > c$. Also *the linearity in the particle momentum* $\mathbf{p}$ of the Dirac Hamiltonian $H_D(\mathbf{p}) = mc^2\beta + c\mathbf{p}\cdot\vec{\alpha}$ renders *the crucial nonrelativistic regime,* $H_D(\mathbf{p}) \approx mc^2 + (|\mathbf{p}|^2/(2m))$ when $|\mathbf{p}/m| \ll c$, of special relativity *outright impossible.*

Therefore the relativistic physics "issue" of negative energy eigenvalues which are unbounded below *is a bogus one* that is entirely rooted *in the mistreatment of relativistic physics by the physically misguided Dirac Hamiltonian and the physically damaged Klein-Gordon equation.* Since there is absolutely *no* "problem" in *legitimate relativistic physics* with negative energy eigenvalues which are unbounded below, the supposed "solution" of that *non-problem,* namely interpreting negative-energy states as propagating backward in time, *vanishes with that non-problem like a mirage.*

One therefore wants to know what theoretical-physics principles and constructs support the existence of antiparticles and pair production. The global symmetry principle of the charge conjugation invariance of the second-quantized Hamiltonian certainly enforces *the existence of antiparticles.* To in addition enforce the existence of pair production, a second, more detailed global symmetry principle is necessary, namely the invariance of the second-quantized Hamiltonian under the interchange of particle annihilation operators and antiparticle creation operators, and likewise the invariance of that Hamiltonian under the interchange of particle creation operators and antiparticle annihilation operators. Enforcement of the existence of antiparticles and pair production thus moves to the domain of global symmetry principles at the second-quantized level, where it doubtless should always have been.

The noninteracting particle by itself is of course entirely insufficient to usefully model physics. We next study the issue of how to add relativistic interactions to the relativistic noninteracting particle.

2. Adding relativistic interactions to the relativistic noninteracting particle

Adding relativistic interactions to the relativistic noninteracting particle is almost certainly more easily done at the Lagrangian level than it is at the Hamiltonian level. For dynamics to be relativistic the action integral must be Lorentz invariant, and the Lagrangian is more simply related to the action integral than the Hamiltonian is. Given a Lagrangian $L(\dot{\mathbf{r}}, \mathbf{r}, t)$ the action integral is simply $\int L(\dot{\mathbf{r}}, \mathbf{r}, t)\, dt$. Now differential time dt isn't Lorentz-invariant, but proper differential time $d\tau = dt\sqrt{1 - |\dot{\mathbf{r}}/c|^2}$ of course *is* Lorentz-invariant. Therefore, if one should happen to have in hand a Lorentz-invariant entity $I(\dot{\mathbf{r}}, \mathbf{r}, t)$, then $\big(I(\dot{\mathbf{r}}, \mathbf{r}, t)\sqrt{1 - |\dot{\mathbf{r}}/c|^2}\,\big)$ is a candidate for a relativistic Lagrangian because it makes the action integral Lorentz-invariant. Also, if $L(\dot{\mathbf{r}}, \mathbf{r}, t)$ is a legitimately relativistic Lagrangian, then it should be the case that $\big(L(\dot{\mathbf{r}}, \mathbf{r}, t)/\sqrt{1 - |\dot{\mathbf{r}}/c|^2}\,\big)$ is a recognizable Lorentz-invariant.

Let's see if this last contention can be verified in the case of our noninteracting-particle relativistic Hamiltonian $H(\mathbf{p}) = mc^2\sqrt{1 + |\mathbf{p}/(mc)|^2}$. To pass from this Hamiltonian to the corresponding Lagrangian $L(\dot{\mathbf{r}})$, we need the velocity $\dot{\mathbf{r}} = \nabla_{\mathbf{p}} H(\mathbf{p}) = (\mathbf{p}/m)/\sqrt{1 + |\mathbf{p}/(mc)|^2}$, a relation which one can verify implies that $\mathbf{p}(\dot{\mathbf{r}}) = \big(m\dot{\mathbf{r}}/\sqrt{1 - |\dot{\mathbf{r}}/c|^2}\,\big)$. Therefore, $H(\mathbf{p}(\dot{\mathbf{r}})) = mc^2\sqrt{1 + |\mathbf{p}(\dot{\mathbf{r}})/(mc)|^2} = \big(mc^2/\sqrt{1 - |\dot{\mathbf{r}}/c|^2}\,\big)$. Finally, for passing from the Hamiltonian to the Lagrangian we have the relation $L(\dot{\mathbf{r}}) = \dot{\mathbf{r}}\cdot\mathbf{p}(\dot{\mathbf{r}}) - H(\mathbf{p}(\dot{\mathbf{r}})) = \big(m|\dot{\mathbf{r}}|^2 - mc^2)/\sqrt{1 - |\dot{\mathbf{r}}/c|^2}\,\big) = -mc^2\sqrt{1 - |\dot{\mathbf{r}}/c|^2}$. Therefore, $\big(L(\dot{\mathbf{r}})/\sqrt{1 - |\dot{\mathbf{r}}/c|^2}\,\big) = -mc^2$, which is indeed a recognizable Lorentz-invariant.

Therefore to add a relativistic interaction to the noninteracting-particle Lagrangian $-mc^2\sqrt{1 - |\dot{\mathbf{r}}/c|^2}$ we need a Lorentz-invariant entity $I(\dot{\mathbf{r}}, \mathbf{r}, t)$ which represents that interaction (to gain *physical understanding* of the Lorentz-invariant interaction entity $I(\dot{\mathbf{r}}, \mathbf{r}, t)$, one can examine it *in the particle rest frame* where $\dot{\mathbf{r}} = \mathbf{0}$). With $I(\dot{\mathbf{r}}, \mathbf{r}, t)$ in hand, the relativistic interacting-particle Lagrangian is $L(\dot{\mathbf{r}}, \mathbf{r}, t) = \big(-mc^2 + I(\dot{\mathbf{r}}, \mathbf{r}, t)\big)\sqrt{1 - |\dot{\mathbf{r}}/c|^2}$. So adding a relativistic interaction to the noninteracting particle at the Lagrangian level *entails adding a relativistic-invariant to* $(-mc^2)$ *and then multiplying the result by* $\sqrt{1 - |\dot{\mathbf{r}}/c|^2}$.

One then still has the burdensome task of passing from that relativistic interacting-particle Lagrangian $L(\dot{\mathbf{r}}, \mathbf{r}, t)$ to the corresponding Hamiltonian $H(\mathbf{r}, \mathbf{p}, t)$ To proceed to the Hamiltonian, one first obtains the canonical momentum $\mathbf{p} = \nabla_{\dot{\mathbf{r}}} L(\dot{\mathbf{r}}, \mathbf{r}, t)$, following which *one needs to solve the vector relation* $\mathbf{p} = \nabla_{\dot{\mathbf{r}}} L(\dot{\mathbf{r}}, \mathbf{r}, t)$ *for* $\dot{\mathbf{r}}(\mathbf{r}, \mathbf{p}, t)$. With $\dot{\mathbf{r}}(\mathbf{r}, \mathbf{p}, t)$ in hand, the Hamiltonian is obtained as $H(\mathbf{r}, \mathbf{p}, t) = \dot{\mathbf{r}}(\mathbf{r}, \mathbf{p}, t) \cdot \mathbf{p} - L(\dot{\mathbf{r}}(\mathbf{r}, \mathbf{p}, t), \mathbf{r}, t)$. Unfortunately, there is *no guarantee* that the vector relation $\mathbf{p} = \nabla_{\dot{\mathbf{r}}} L(\dot{\mathbf{r}}, \mathbf{r}, t)$ can be solved for $\dot{\mathbf{r}}(\mathbf{r}, \mathbf{p}, t)$ *in closed form.* If there is *no* closed-form solution of the vector relation $\mathbf{p} = \nabla_{\dot{\mathbf{r}}} L(\dot{\mathbf{r}}, \mathbf{r}, t)$ for $\dot{\mathbf{r}}(\mathbf{r}, \mathbf{p}, t)$, one needs to make an approximation, or possibly even a sequence of successive approximations.

A classic example of the above procedure was the development very long ago by Lorentz of the relativistic Lagrangian and Hamiltonian for a charged particle (that has no spin) interacting with the electromagnetic four-potential $(\phi(\mathbf{r}, t), \mathbf{A}(\mathbf{r}, t))$. A Lorentz-invariant entity that has the dimension of energy and suitably represents the particle's interaction with the electromagnetic four-potential is the particle's charge e times the Lorentz-invariant contraction of the electromagnetic four-potential with the particle's four-velocity $(c, \dot{\mathbf{r}})/\sqrt{1 - |\dot{\mathbf{r}}/c|^2}$ divided by c, which equals $\big(e\phi(\mathbf{r}, t) - (e/c)\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)\big)/\sqrt{1 - |\dot{\mathbf{r}}/c|^2}$. This Lorentz-invariant entity reduces, in the particle's rest frame where $\dot{\mathbf{r}} = \mathbf{0}$, *to the potential energy* $e\phi(\mathbf{r}, t)$, so it is to be *subtracted* from the Lorentz-invariant term $(-mc^2)$, and then that Lorentz-invariant difference *is multiplied by* $\sqrt{1 - |\dot{\mathbf{r}}/c|^2}$ *to produce the interacting particle's relativistic Lagrangian* $L(\dot{\mathbf{r}}, \mathbf{r}, t)$,

$$L(\dot{\mathbf{r}}, \mathbf{r}, t) = -mc^2\sqrt{1 - |\dot{\mathbf{r}}/c|^2} - \big(e\phi(\mathbf{r}, t) - (e/c)\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)\big). \tag{2.1a}$$

We next use this relativistic Lorentz Lagrangian to obtain the corresponding relativistic Lorentz Hamiltonian for a charged particle interacting with an electromagnetic four-potential. The canonical momentum is,

$$\mathbf{p} = \nabla_{\dot{\mathbf{r}}} L(\dot{\mathbf{r}}, \mathbf{r}, t) = (m\dot{\mathbf{r}})/\big(\sqrt{1 - |\dot{\mathbf{r}}/c|^2}\,\big) + (e/c)\mathbf{A}(\mathbf{r}, t). \tag{2.1b}$$

Fortunately the relation $\mathbf{p} = (m\dot{\mathbf{r}})/\big(\sqrt{1 - |\dot{\mathbf{r}}/c|^2}\,\big) + (e/c)\mathbf{A}(\mathbf{r}, t)$ is readily solved for $\dot{\mathbf{r}}$ in terms of $\mathbf{r}$, $\mathbf{p}$ and t,

$$\dot{\mathbf{r}}(\mathbf{r}, \mathbf{p}, t) = ((\mathbf{p} - (e/c)\mathbf{A}(\mathbf{r}, t))/m)/\big(\sqrt{1 + |(\mathbf{p} - (e/c)\mathbf{A}(\mathbf{r}, t))/(mc)|^2}\,\big) = (\mathbf{q}/m)/\big(\sqrt{1 + |\mathbf{q}/(mc)|^2}\,\big), \tag{2.1c}$$

where $\mathbf{q} \stackrel{\text{def}}{=} (\mathbf{p} - (e/c)\mathbf{A}(\mathbf{r}, t))$. We next obtain the Hamiltonian H($\mathbf{r}$, $\mathbf{p}$, t) by inserting the Eq. (2.1c) result for $\dot{\mathbf{r}}(\mathbf{r}, \mathbf{p}, t)$ into $H(\mathbf{r}, \mathbf{p}, t) = \dot{\mathbf{r}}(\mathbf{r}, \mathbf{p}, t) \cdot \mathbf{p} - L(\dot{\mathbf{r}}(\mathbf{r}, \mathbf{p}, t), \mathbf{r}, t)$ where $L(\dot{\mathbf{r}}, \mathbf{r}, t) = -mc^2\big(\sqrt{1 - |\dot{\mathbf{r}}/c|^2}\,\big) - e\phi(\mathbf{r}, t) + (e/c)(\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t))$, as given by Eq. (2.1a). After using Eq. (2.1c) to establish that $\sqrt{1 - |\dot{\mathbf{r}}/c|^2} = \big(1/\sqrt{1 + |\mathbf{q}/(mc)|^2}\,\big)$, we obtain for $H(\mathbf{r}, \mathbf{p}, t)$,

$$\begin{gathered} H(\mathbf{r}, \mathbf{p}, t) = \Big\{\big[((\mathbf{q}/m) \cdot \mathbf{p}) + mc^2 - ((\mathbf{q}/m) \cdot ((e/c)\mathbf{A}(\mathbf{r}, t)))\big]\,/\sqrt{1 + |\mathbf{q}/(mc)|^2}\Big\} + e\phi(\mathbf{r}, t) = \\ \Big\{\big[mc^2 + (|\mathbf{q}|^2/m)\big]\,/\sqrt{1 + |\mathbf{q}/(mc)|^2}\Big\} + e\phi(\mathbf{r}, t) = mc^2\sqrt{1 + |(\mathbf{q}/(mc)|^2} + e\phi(\mathbf{r}, t) = \\ mc^2\sqrt{1 + |(\mathbf{p} - (e/c)\mathbf{A}(\mathbf{r}, t))/(mc)|^2} + e\phi(\mathbf{r}, t), \end{gathered} \tag{2.1d}$$

which is Lorentz's relativistic Hamiltonian for the interaction of a particle which has charge e, mass m and no spin with the electromagnetic four-potential $(\phi(\mathbf{r}, t), \mathbf{A}(\mathbf{r}, t))$. It of course reduces to the noninteracting-particle relativistic Hamiltonian $mc^2\sqrt{1 + |(\mathbf{p}/(mc)|^2}$ when the particle's charge e is put to zero. Lorentz's unquestionably relativistic Hamiltonian *of course differs from the electromagnetic interaction expression for the no-spin Klein-Gordon particle,* which, like the Dirac particle, *suffers from totally unphysical unbounded-below negative energies* that are *by fiat* declared to (very questionably) *propagate backward in time.*

We next obtain the quantum propagator for the relativistic noninteracting particle, which is useful for developing perturbation approximations.

3. The relativistic noninteracting particle's quantum space-time propagator

The Schrödinger equation is first-order in time, so if the wave function is specified at a particular time t, the Schrödinger equation determines it at all other times t'. Since the Schrödinger equation is *homogeneously linear,* it is sufficient to solve it for *point wave functions* $\delta^{(3)}(\mathbf{r}' - \mathbf{r})$ at the particular time t, since we can *superpose* those solutions to produce the solution *for any choice of the wave function* at the particular time t. We would usually have no interest in the Schrödinger equation solutions at times that are *earlier* than the particular time t; we can cut such earlier-time information out of a Schrödinger equation solution by multiplying it by the time unit step function $\theta(t' - t)$, which is equal to 1 when $t' > t$, but is equal to 0 when $t' < t$. Schrödinger equation solutions which are *point wave functions* $\delta^{(3)}(\mathbf{r}' - \mathbf{r})$ *at the particular time*

t, and *are multiplied by* $(-i/\hbar)\theta(t'-t)$ are known as *the space-time propagators* $G((t'-t),\mathbf{r}',\mathbf{r};\widehat{H})$ of their Hamiltonian operator $\widehat{H}$, and we shall now show that they are explicitly given by,

$$G((t'-t),\mathbf{r}',\mathbf{r};\widehat{H}) = (-i/\hbar)\theta(t'-t)\langle\mathbf{r}'|\exp(-i\widehat{H}(t'-t)/\hbar)|\mathbf{r}\rangle. \tag{3.1a}$$

The Schrödinger equation with the Hamiltonian operator $\widehat{H}$ is, in coordinate representation,

$$i\hbar\partial\langle\mathbf{r}'|\psi(t')\rangle/\partial t' = \langle\mathbf{r}'|\widehat{H}|\psi(t')\rangle, \tag{3.1b}$$

and when $|\psi(t')\rangle = \exp(-i\widehat{H}(t'-t)/\hbar)|\mathbf{r}\rangle$, which is the case for the entity $\langle\mathbf{r}'|\exp(-i\widehat{H}(t'-t)/\hbar)|\mathbf{r}\rangle$ in Eq. (3.1a), then the Schrödinger equation of Eq. (3.1b) is in fact satisfied. Furthermore, when $t' = t$, then the entity $\langle\mathbf{r}'|\exp(-i\widehat{H}(t'-t)/\hbar)|\mathbf{r}\rangle = \langle\mathbf{r}'|\mathbf{r}\rangle = \delta^{(3)}(\mathbf{r}'-\mathbf{r})$, which is the required point wave function for $t' = t$. Therefore the entity $G((t'-t),\mathbf{r}',\mathbf{r};\widehat{H})$ of Eq. (3.1a) fulfills the requirements for being a propagator of the Hamiltonian operator $\widehat{H}$. The presence of the time step function $\theta(t'-t)$ in this propagator entity $G((t'-t),\mathbf{r}',\mathbf{r};\widehat{H}) = (-i/\hbar)\theta(t'-t)\langle\mathbf{r}'|\exp(-i\widehat{H}(t'-t)/\hbar)|\mathbf{r}\rangle$ prevents it from *itself* satisfying the Schrödinger equation, but since the derivative of a unit step function is a delta function, specifically, $\partial\theta(t'-t)/\partial t' = \delta(t'-t)$, the propagator $G((t'-t),\mathbf{r}',\mathbf{r};\widehat{H})$ instead satisfies the closely related equation,

$$\begin{aligned}(i\hbar\partial/\partial t' - \widehat{H})G((t'-t),\mathbf{r}',\mathbf{r};\widehat{H}) = \delta(t'-t)\langle\mathbf{r}'|\exp(-i\widehat{H}(t'-t)/\hbar)|\mathbf{r}\rangle = \\ \delta(t'-t)\langle\mathbf{r}'|\mathbf{r}\rangle = \delta(t'-t)\delta^{(3)}(\mathbf{r}'-\mathbf{r}),\end{aligned} \tag{3.1c}$$

so the propagator $G((t'-t),\mathbf{r}',\mathbf{r};\widehat{H})$ of Eq. (3.1a) is *the retarded Green's function* of the Schrödinger equation with the Hamiltonian operator $\widehat{H}$. It is the *retarded* Green's function *because it vanishes when* $t' < t$. The fact that it is *a Green's function* makes it *useful for developing perturbation approximations.*

The eigenstates of the relativistic noninteracting-particle Hamiltonian $H(\widehat{\mathbf{p}}) = mc^2\sqrt{1+|\widehat{\mathbf{p}}/(mc)|^2}$ are those of the momentum operator $\widehat{\mathbf{p}}$, so we can expand the operator $\exp(-iH(\widehat{\mathbf{p}})(t'-t)/\hbar)$ out in momentum eigenstates, and in consequence we can expand the propagator of the Hamiltonian $H(\widehat{\mathbf{p}})$ out in plane waves,

$$\begin{aligned}G((t'-t),\mathbf{r}',\mathbf{r};H(\widehat{\mathbf{p}})) = (-i/\hbar)\theta(t'-t)\langle\mathbf{r}'|\exp(-iH(\widehat{\mathbf{p}})(t'-t)/\hbar)|\mathbf{r}\rangle = \\ (-i/\hbar)\theta(t'-t)\int d^3\mathbf{p}\langle\mathbf{r}'|\mathbf{p}\rangle\exp(-iH(\mathbf{p})(t'-t)/\hbar)\langle\mathbf{p}|\mathbf{r}\rangle = \\ (-i/\hbar)\theta(t'-t)\int d^3\mathbf{p}\,(1/(2\pi\hbar))^3\exp(i(\mathbf{p}\cdot(\mathbf{r}'-\mathbf{r}))/\hbar)\exp(-iH(\mathbf{p})(t'-t)/\hbar).\end{aligned} \tag{3.2a}$$

We next reexpress the time-dependent factor $(-i/\hbar)\theta(t'-t)\exp(-iH(\mathbf{p})(t'-t)/\hbar)$ of the Eq. (3.2a) presentation of $G((t'-t),\mathbf{r}',\mathbf{r};H(\widehat{\mathbf{p}}))$ in terms of its Fourier tranformation from time to energy; we then can likewise reexpress $G((t'-t),\mathbf{r}',\mathbf{r};H(\widehat{\mathbf{p}}))$ in terms of its Fourier transformation from space-time to momentum-energy,

$$\begin{aligned}(-i/\hbar)\theta(t'-t)\exp(-iH(\mathbf{p})(t'-t)/\hbar) = \\ \int_{-\infty}^{\infty} dE\,(1/(2\pi\hbar))\exp(-iE(t'-t)/\hbar)\int_{-\infty}^{\infty} d\tau\exp(iE\tau/\hbar)(-i/\hbar)\theta(\tau)\exp(-iH(\mathbf{p})\tau/\hbar) = \\ \int_{-\infty}^{\infty} dE\,(1/(2\pi\hbar))\exp(-iE(t'-t)/\hbar)\int_{0}^{\infty} d\tau(-i/\hbar)\exp(i(E-H(\mathbf{p}))\tau/\hbar)\exp(-\epsilon\tau/\hbar) = \\ \int_{-\infty}^{\infty} dE\,(1/(2\pi\hbar))\exp(-iE(t'-t)/\hbar)(-i/\hbar)(\hbar/(-i(E-H(\mathbf{p})+\epsilon)) = \\ \int_{-\infty}^{\infty} dE\,(1/(2\pi\hbar))\exp(-iE(t'-t)/\hbar)(1/(E-H(\mathbf{p})+i\epsilon)).\end{aligned} \tag{3.2b}$$

We now insert the Eq. (3.2b) result into Eq. (3.2a) in order to reexpress the noninteracting-particle propagator $G((t'-t),\mathbf{r}',\mathbf{r};H(\widehat{\mathbf{p}}))$ in terms of its Fourier transformation from space-time to momentum-energy,

$$\begin{aligned}G((t'-t),\mathbf{r}',\mathbf{r};H(\widehat{\mathbf{p}})) = \\ \int d^3\mathbf{p}\,(1/(2\pi\hbar))^3\exp(i(\mathbf{p}\cdot(\mathbf{r}'-\mathbf{r}))/\hbar)\int_{-\infty}^{\infty} dE\,(1/(2\pi\hbar))\exp(-iE(t'-t)/\hbar)(1/(E-H(\mathbf{p})+i\epsilon)),\end{aligned} \tag{3.2c}$$

where $H(\mathbf{p}) = mc^2\sqrt{1+|\mathbf{p}/(mc)|^2}$, the noninteracting particle's relativistic energy. Thus the noninteracting particle's relativistic energy-momentum propagator is,

$$G_0(E,\mathbf{p}) = (1/(E - mc^2\sqrt{1+|\mathbf{p}/(mc)|^2} + i\epsilon)). \tag{3.3}$$

This relativistic noninteracting-particle propagator can be used in perturbation approximations in conjunction with "vertices". A basic vertex for the interaction of a particle which has no spin with the electromagnetic four-potential $(\phi(\mathbf{r},t),\mathbf{A}(\mathbf{r},t))$ is the *difference* $\left[H(\mathbf{r},\mathbf{p},t)-mc^2\sqrt{1+|\mathbf{p}/(mc)|^2}\,\right]$, where $H(\mathbf{r},\mathbf{p},t)=e\phi(\mathbf{r},t)+mc^2\sqrt{1+|(\mathbf{p}-(e/c)\mathbf{A}(\mathbf{r},t))/(mc)|^2}$, the Lorentz Hamiltonian for the interaction of a particle which has no spin with the electromagnetic four-potential $(\phi(\mathbf{r},t),\mathbf{A}(\mathbf{r},t))$. To first order in e, the particle's charge, this vertex has the value, $e\phi(\mathbf{r},t)-\left[(e(\mathbf{p}/(mc))\cdot\mathbf{A}(\mathbf{r},t))/\sqrt{1+|\mathbf{p}/(mc)|^2}\,\right]$. Therefore with the relativistic noninteracting-particle propagator $G_0(E,\mathbf{p})$ in hand, it is feasible to develop systematic perturbation approximations *in the spirit of the Feynman diagrams,* albeit *very different in detail.*

Since the quantum propagator is a space-time or energy-momentum entity, it is sometimes thought to be space-time symmetric or even Lorentz invariant. But one glance at its Eq. (3.1a) general form reveals it to be *as entirely time-skewed as the Schrödinger equation,* whose retarded Green's function it after all is.

In the next section we modify the Lorentz Lagrangian to accommodate the spin-1/2 particle by adding a Lorentz-invariant entity which in the particle's rest frame is the nonrelativistic Pauli Hamiltonian's potential energy of the spin-1/2 particle's magnetic moment in a magnetic field.

4. Modification of the Lorentz Hamiltonian to accommodate the spin-1/2 particle

We regard the potential energy of a spin-1/2 particle's magnetic moment in a magnetic field that is given by the nonrelativistic Pauli Hamiltonian, namely $-((e\hbar)/(2mc))(\vec{\sigma}\cdot\mathbf{B}(\mathbf{r},t))$, as being correct in the particle's rest frame, and therefore for the purpose of modifying the Lorentz Lagrangian to describe the interaction of the magnetic moment of a spin-1/2 particle with the electromagnetic field we seek *a relativistic invariant* which *is* this Pauli potential energy in the particle's rest frame. Since in *other* frames of reference a magnetic field transforms partially into an electric field, we need to use the *entire* antisymmetric electromagnetic field tensor $F^{\mu\nu}(\mathbf{r},t)=-F^{\nu\mu}(\mathbf{r},t)$ to construct the desired relativistic invariant. Therefore it is convenient to incorporate the three components of the Pauli spin vector $\vec{\sigma}$ into the six independent components of an antisymmetric second-rank tensor $\sigma^{\mu\nu}=-\sigma^{\nu\mu}$ in such a way that the contraction $(\sigma^{\mu\nu}F_{\mu\nu})$ is proportional to the dot product $(\vec{\sigma}\cdot\mathbf{B})$. Since $F_{ij}=-F_{ji}=-(\mathbf{B})^k$, where ijk is any cyclic permutation of 123, the standard choice for the antisymmetric spin-1/2 tensor $\sigma^{\mu\nu}=-\sigma^{\nu\mu}$ is $\sigma^{ij}=-\sigma^{ji}=(\vec{\sigma})^k$, where ijk is any cyclic permutation of 123, with the remaining ten components of $\sigma^{\mu\nu}$ being equal to zero, namely $\sigma^{00}=\sigma^{i0}=-\sigma^{0i}=\sigma^{ii}=0$, where $i=1,2,3$. This standard choice for $\sigma^{\mu\nu}$ is readily seen to yield for the contraction $(\sigma^{\mu\nu}F_{\mu\nu}(\mathbf{r},t))$,

$$(\sigma^{\mu\nu}F_{\mu\nu}(\mathbf{r},t))=-2(\vec{\sigma}\cdot\mathbf{B}(\mathbf{r},t)),\tag{4.1a}$$

which implies that,

$$((e\hbar)/(4mc))(\sigma^{\mu\nu}F_{\mu\nu}(\mathbf{r},t))=-((e\hbar)/(2mc))(\vec{\sigma}\cdot\mathbf{B}(\mathbf{r},t)).\tag{4.1b}$$

Eqs. (4.1a) and (4.1b) are relevant *only* in the particle's *rest frame* where its velocity $\dot{\mathbf{r}}=\mathbf{0}$. In any *other* particular frame of reference where the particle's velocity $\dot{\mathbf{r}}\neq\mathbf{0}$, we need to apply to the particle's rest-frame spin tensor $\sigma^{\mu\nu}$ *the Lorentz transformation from the particle's rest frame to that particular frame of reference.* Since a Lorentz transformation is a *dimensionless* 4×4 symmetric matrix, expressions for its *components* become more compact if one works with with the *dimensionless* scaled particle velocity $\mathbf{b}\stackrel{\mathrm{def}}{=}(\dot{\mathbf{r}}/c)$ instead of with $\dot{\mathbf{r}}$ directly. Another prominent *dimensionless* ingredient of the components of the symmetric Lorentz transformation matrix is the entity $\gamma\stackrel{\mathrm{def}}{=}1/\sqrt{1-|\mathbf{b}|^2}$. The components of the dimensionless *symmetric* Lorentz transformation matrix $\Lambda^\kappa_\alpha(\mathbf{b})=\Lambda^\alpha_\kappa(\mathbf{b})$ are,

$$\Lambda^0_0(\mathbf{b})=\gamma;\ \Lambda^0_i(\mathbf{b})=\Lambda^i_0(\mathbf{b})=-\gamma(\mathbf{b})^i,\ i=1,2,3;$$
$$\Lambda^i_j(\mathbf{b})=\Lambda^j_i(\mathbf{b})=\delta^{ij}+(\gamma^2/(\gamma+1))(\mathbf{b})^i(\mathbf{b})^j,\ i,j=1,2,3.\tag{4.2a}$$

We denote the particle's spin tensor in the reference frame where the particle's velocity is $\dot{\mathbf{r}}=c\mathbf{b}$ as $\sigma^{\mu\nu}(\mathbf{b})$. That spin tensor $\sigma^{\mu\nu}(\mathbf{b})$ is of course *the Lorentz transformation of* $\sigma^{\mu\nu}$ *by the symmetric Lorentz transformation matrix* $\Lambda^\kappa_\alpha(\mathbf{b})$,

$$\sigma^{\mu\nu}(\mathbf{b})=\Lambda^\mu_\alpha(\mathbf{b})\sigma^{\alpha\beta}\Lambda^\nu_\beta(\mathbf{b})=\sum_{i,j=1;\ i\neq j}^{3}\Lambda^\mu_i(\mathbf{b})\sigma^{ij}\Lambda^\nu_j(\mathbf{b})\ =\ (\vec{\sigma})^3(\Lambda^\mu_1(\mathbf{b})\Lambda^\nu_2(\mathbf{b})-\Lambda^\mu_2(\mathbf{b})\Lambda^\nu_1(\mathbf{b}))+$$
$$(\vec{\sigma})^1(\Lambda^\mu_2(\mathbf{b})\Lambda^\nu_3(\mathbf{b})-\Lambda^\mu_3(\mathbf{b})\Lambda^\nu_2(\mathbf{b}))+(\vec{\sigma})^2(\Lambda^\mu_3(\mathbf{b})\Lambda^\nu_1(\mathbf{b})-\Lambda^\mu_1(\mathbf{b})\Lambda^\nu_3(\mathbf{b})),\tag{4.2b}$$

which tells us that $\sigma^{\mu\nu}(\mathbf{b}) = -\sigma^{\nu\mu}(\mathbf{b})$, i.e., $\sigma^{\mu\nu}(\mathbf{b})$ is *antisymmetric,* just as $\sigma^{\mu\nu}$ is. Therefore $\sigma^{00}(\mathbf{b}) = \sigma^{ii}(\mathbf{b}) = 0$, where $i = 1, 2, 3$. To calculate $\sigma^{10}(\mathbf{b})$ using Eq. (4.2b) we first note from Eq. (4.2a) that $(\vec{\sigma})^1(\Lambda^1_2(\mathbf{b})\Lambda^0_3(\mathbf{b}) - \Lambda^1_3(\mathbf{b})\Lambda^0_2(\mathbf{b})) = (\vec{\sigma})^1(-(\gamma^3/(\gamma+1))((\mathbf{b})^1(\mathbf{b})^2(\mathbf{b})^3 - (\mathbf{b})^1(\mathbf{b})^3(\mathbf{b})^2) = 0$, however contrariwise, $(\vec{\sigma})^2(\Lambda^1_3(\mathbf{b})\Lambda^0_1(\mathbf{b}) - \Lambda^1_1(\mathbf{b})\Lambda^0_3(\mathbf{b})) = (\vec{\sigma})^2(-(\gamma^3/(\gamma+1))((\mathbf{b})^1(\mathbf{b})^3(\mathbf{b})^1 - (\mathbf{b})^1(\mathbf{b})^1(\mathbf{b})^3) + \gamma(\mathbf{b})^3) = \gamma(\vec{\sigma})^2(\mathbf{b})^3$ and $(\vec{\sigma})^3(\Lambda^1_1(\mathbf{b})\Lambda^0_2(\mathbf{b}) - \Lambda^1_2(\mathbf{b})\Lambda^0_1(\mathbf{b})) = (\vec{\sigma})^3(-\gamma(\mathbf{b})^2 - (\gamma^3/(\gamma+1))((\mathbf{b})^1(\mathbf{b})^1(\mathbf{b})^2 - (\mathbf{b})^1(\mathbf{b})^2(\mathbf{b})^1) = -\gamma(\vec{\sigma})^3(\mathbf{b})^2$. Therefore $\sigma^{10}(\mathbf{b}) = \gamma((\vec{\sigma})^2(\mathbf{b})^3 - (\vec{\sigma})^3(\mathbf{b})^2)$, and also $\sigma^{20}(\mathbf{b}) = \gamma((\vec{\sigma})^3(\mathbf{b})^1 - (\vec{\sigma})^1(\mathbf{b})^3)$ and $\sigma^{30}(\mathbf{b}) = \gamma((\vec{\sigma})^1(\mathbf{b})^2 - (\vec{\sigma})^2(\mathbf{b})^1)$. Therefore,

$$\sigma^{i0}(\mathbf{b}) = \gamma(\vec{\sigma}\times\mathbf{b})^i = -\sigma^{0i}(\mathbf{b}),\ i = 1, 2, 3. \tag{4.2c}$$

Similarly one obtains that $\sigma^{12}(\mathbf{b}) = (\vec{\sigma})^1(-(\gamma^2/(\gamma+1))(\mathbf{b})^1(\mathbf{b})^3) + (\vec{\sigma})^2(-(\gamma^2/(\gamma+1))(\mathbf{b})^2(\mathbf{b})^3) + (\vec{\sigma})^3(1 + (\gamma^2/(\gamma+1))((\mathbf{b})^1(\mathbf{b})^1 + (\mathbf{b})^2(\mathbf{b})^2)) = (\vec{\sigma})^3(1 + (\gamma^2/(\gamma+1))|\mathbf{b}|^2) - (\gamma^2/(\gamma+1))(\vec{\sigma}\cdot\mathbf{b})(\mathbf{b})^3 = \gamma(\vec{\sigma})^3 - (\gamma^2/(\gamma+1))(\vec{\sigma}\cdot\mathbf{b})(\mathbf{b})^3$. Likewise, $\sigma^{23}(\mathbf{b}) = \gamma(\vec{\sigma})^1 - (\gamma^2/(\gamma+1))(\vec{\sigma}\cdot\mathbf{b})(\mathbf{b})^1$ and $\sigma^{31}(\mathbf{b}) = \gamma(\vec{\sigma})^2 - (\gamma^2/(\gamma+1))(\vec{\sigma}\cdot\mathbf{b})(\mathbf{b})^2$. Therefore,

$$\sigma^{ij}(\mathbf{b}) = \gamma(\vec{\sigma})^k - (\gamma^2/(\gamma+1))(\vec{\sigma}\cdot\mathbf{b})(\mathbf{b})^k = -\sigma^{ji}(\mathbf{b}), \text{ where } ijk \text{ is a cyclic permutation of 123.} \tag{4.2d}$$

Since $F_{i0} = -(\mathbf{E})^i = -F_{0i}$, $i = 1, 2, 3$ and $F_{ij} = -(\mathbf{B})^k = -F_{ji}$, where ijk is a cyclic permutation of 123, Eqs. (4.2c) and (4.2d) imply that,

$$\begin{aligned}(\sigma^{\mu\nu}(\mathbf{b})F_{\mu\nu}(\mathbf{r},t)) = -2\gamma\big((\vec{\sigma}\cdot\mathbf{B}(\mathbf{r},t)) - ((\gamma/(\gamma+1))(\vec{\sigma}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{B}(\mathbf{r},t))) + ((\vec{\sigma}\times\mathbf{b})\cdot\mathbf{E}(\mathbf{r},t))\big) =\\ -2\gamma\big((\vec{\sigma}\cdot\mathbf{B}(\mathbf{r},t)) - ((\gamma/(\gamma+1))(\vec{\sigma}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{B}(\mathbf{r},t))) - (\mathbf{b}\cdot(\vec{\sigma}\times\mathbf{E}(\mathbf{r},t)))\big),\end{aligned} \tag{4.2e}$$

which implies that,

$$\begin{aligned}((e\hbar)/(4mc))(\sigma^{\mu\nu}(\mathbf{b})F_{\mu\nu}(\mathbf{r},t)) =\\ -((e\hbar)/(2mc))\gamma\big((\vec{\sigma}\cdot\mathbf{B}(\mathbf{r},t)) - ((\gamma/(\gamma+1))(\vec{\sigma}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{B}(\mathbf{r},t))) - (\mathbf{b}\cdot(\vec{\sigma}\times\mathbf{E}(\mathbf{r},t)))\big).\end{aligned} \tag{4.2f}$$

Eqs. (4.2e) and (4.2f) extend the particle rest-frame Eqs. (4.1a) and (4.1b) to the frame where the particle's velocity is $\dot{\mathbf{r}} = c\mathbf{b}$. The Lorentz-invariant entity $((e\hbar)/(4mc))(\sigma^{\mu\nu}(\mathbf{b})F_{\mu\nu}(\mathbf{r},t))$ of Eq. (4.2f) extrapolates the nonrelativistic Pauli potential energy of the spin-1/2 particle's magnetic moment in a magnetic field from its rest frame to the frame where its velocity $\dot{\mathbf{r}} = c\mathbf{b}$. In order to insert spin-1/2 dynamics into the Eq. (2.1a) Lorentz Lagrangian we subtract from it that Lorentz-invariant entity multiplied by $\sqrt{1 - |\dot{\mathbf{r}}/c|^2} = (1/\gamma)$,

$$\begin{aligned}L(c\mathbf{b},\mathbf{r},t) = -mc^2\sqrt{1-|\mathbf{b}|^2} - e\big(\phi(\mathbf{r},t) - (\mathbf{b}\cdot\mathbf{A}(\mathbf{r},t))\big) +\\ \big((e\hbar)/(2mc)\big)\big((\vec{\sigma}\cdot\mathbf{B}(\mathbf{r},t)) - ((\gamma/(\gamma+1))(\vec{\sigma}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{B}(\mathbf{r},t))) - (\mathbf{b}\cdot(\vec{\sigma}\times\mathbf{E}(\mathbf{r},t)))\big).\end{aligned} \tag{4.3a}$$

The Eq. (4.3a) modified Lorentz Lagrangian for the spin-1/2 particle interacting with an electromagnetic field includes the negative of $-\big((e\hbar)/(2mc)\big)\big((\vec{\sigma}\cdot\mathbf{B}(\mathbf{r},t)) - ((\gamma/(\gamma+1))(\vec{\sigma}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{B}(\mathbf{r},t)))\big)$, the non-relativistic potential energy of the spin-1/2 particle's magnetic moment in the magnetic field $\mathbf{B}(\mathbf{r},t)$ *and its order* $|\mathbf{b}|^2$ *relativistic correction.*

To obtain the Eq. (4.3a) Lagrangian's corresponding Hamilton, we need *the canonical momentum* $\mathbf{p}$,

$$\begin{aligned}\mathbf{p} = (1/c)\nabla_{\mathbf{b}}L(c\mathbf{b},\mathbf{r},t) = \big(mc\mathbf{b}/\sqrt{1-|\mathbf{b}|^2}\,\big) + (e/c)\mathbf{A}(\mathbf{r},t) - ((e\hbar)/(2mc^2))(\vec{\sigma}\times\mathbf{E}(\mathbf{r},t)) -\\ ((e\hbar)/(2mc^2))(\gamma/(\gamma+1))\big(\vec{\sigma}(\mathbf{b}\cdot\mathbf{B}(\mathbf{r},t)) + (\vec{\sigma}\cdot\mathbf{b})\mathbf{B}(\mathbf{r},t)\big).\end{aligned} \tag{4.3b}$$

The entity $-((e\hbar)/(2mc^2))(\gamma/(\gamma+1))\big(\vec{\sigma}(\mathbf{b}\cdot\mathbf{B}(\mathbf{r},t)) + (\vec{\sigma}\cdot\mathbf{b})\mathbf{B}(\mathbf{r},t)\big)$, which is a contribution to the canonical momentum $\mathbf{p}$ of Eq. (4.3b), *is of order* $|\mathbf{b}|$, and therefore can be regarded *as a relativistic correction* to the *rest* of the Eq. (4.3b) canonical momentum $\mathbf{p}$. Obtaining the Eq. (4.3a) Lagrangian's corresponding Hamiltonian requires *solving* the Eq. (4.3b) relation for the particle's velocity $\dot{\mathbf{r}} = c\mathbf{b}$ in terms of $\mathbf{r}$, $\mathbf{p}$ and t. The *presence* of the above-written *relativistic-correction contribution to* $\mathbf{p}$ makes it impossible to obtain *a closed-form solution* for $\dot{\mathbf{r}} = c\mathbf{b}$ in terms of $\mathbf{r}$, $\mathbf{p}$ and t. We therefore content ourselves with the *approximation* to the particle's Hamiltonian which entails obtaining $\dot{\mathbf{r}}(\mathbf{r},\mathbf{p},t)$ *without including* the above-written *relativistic-correction contribution to* $\mathbf{p}$,

$$\dot{\mathbf{r}}(\mathbf{r},\mathbf{p},t) \approx (\mathbf{q}/m)/\sqrt{1+|\mathbf{q}/(mc)|^2}, \text{ where } \mathbf{q} \stackrel{\text{def}}{=} \big(\mathbf{p} - (e/c)\mathbf{A}(\mathbf{r},t) + ((e\hbar)/(2mc^2))(\vec{\sigma}\times\mathbf{E}(\mathbf{r},t))\big). \tag{4.3c}$$

Inserting this approximation to $\dot{\mathbf{r}}(\mathbf{r},\mathbf{p},t)$ into $\big(\dot{\mathbf{r}}(\mathbf{r},\mathbf{p},t)\cdot\mathbf{p} - L(\dot{\mathbf{r}}(\mathbf{r},\mathbf{p},t),\mathbf{r},t)\big)$ yields the following approximation to the Hamiltonian $H(\mathbf{r},\mathbf{p},t)$ for the spin-1/2 particle interacting with an electromagnetic field,

$$H(\mathbf{r},\mathbf{p},t) \approx mc^2\sqrt{1+|\mathbf{q}/(mc)|^2} + e\phi(\mathbf{r},t) - ((e\hbar)/(2mc))(\vec{\sigma}\cdot\mathbf{B}(\mathbf{r},t)) +$$
$$\big((e\hbar)/(2mc)\big)\big(1/\big(1+\big(1/\sqrt{1+|\mathbf{q}/(mc)|^2}\,\big)\big)\big)\big((\vec{\sigma}\cdot\mathbf{q})(\mathbf{q}\cdot\mathbf{B}(\mathbf{r},t))/((mc)^2+|\mathbf{q}|^2)\big),$$
$$\text{where } \mathbf{q} \stackrel{\text{def}}{=} \big(\mathbf{p} - (e/c)\mathbf{A}(\mathbf{r},t) + ((e\hbar)/(2mc^2))(\vec{\sigma}\times\mathbf{E}(\mathbf{r},t))\big). \tag{4.3d}$$

The *effect* of the approximation we have made is *to entirely keep* what at the Lagrangian level *is a relativistic correction* to the basic nonrelativistic potential energy $-((e\hbar)/(2mc))(\vec{\sigma}\cdot\mathbf{B}(\mathbf{r},t))$ of the spin-1/2 particle's magnetic moment in a magnetic field; we have dropped *only the effect which that relativistic correction has on the particle's canonical momentum.*

The primary *utility* of the Eq. (4.3d) Hamiltonian that includes spin-1/2 particle relativistic effects *is to serve as a source of vertex factors in systematic perturbation calculations which are based on the noninteracting-particle relativistic propagator* $G_0(E,\mathbf{p}) = (1/(E - mc^2\sqrt{1+|\mathbf{p}/(mc)|^2} + i\epsilon))$.

We next briefly point out that in theories where particles are produced, perturbation contributions become increasingly invalid with increasingly high virtual momentum values, which must be cut off.

5. Virtual momentum cutoff needed in perturbation contributions if particles are produced

In single-particle quantum mechanics, contributions to perturbation approximations which entail integration over arbitrarily high virtual momentum values of that particle *are justified by completeness.*

Completeness *is vastly more involved,* however, in quantum theories *which permit particles to be produced.* In those, completeness sums *are necessarily over the momenta of any possible finite number of particles.* The standard development of perturbation approximations for quantum theories which permit particles to be produced doesn't *adequately* take into account this seismic shift in the character of completeness sums. Consequently those contributions to standard perturbation approximations which entail integration over arbitrarily high virtual momentum values *fail to adequately accommodate the natural growth in virtual particle multiplicity which accompanies such a growth in virtual energy availability;* the additional particles *are unavailable because they have been shifted to other perturbation contributions of higher order.* This physically inappropriate suppression *of virtual particle multiplicity* in any *individual* perturbation contribution which entails integration over arbitrarily high virtual momentum values causes *the parts of that perturbation contribution which are involved with progressively higher virtual momentum values to progressively lose physical validity,* so *there exists an optimal cutoff on the virtual momentum of any perturbation contribution which entails integration over arbitrarily high virtual momentum values.* To *estimate* that optimal virtual-momentum cutoff for any *given* perturbation contribution which entails integration over arbitrarily high virtual momentum values, the *higher-order* perturbation contributions *which are the next higher up in particle multiplicity* need to be *studied.*

Since standard perturbation approximations systematically suppress virtual particle multiplicity, perhaps single-particle propagators such as the one described by Eq. (3.2c) will someday be *superseded* by entities *which propagate an indefinite number of particles.* Entities called coherent quantum states, which describe an indefinite number of particles, already exist, and perhaps those, or more flexible extensions of the coherent-state construct, will someday be made the basis of propagators which are much better attuned to the possibility of an indefinitely large number of virtual particles that completeness sums permit.