Elementary proof of the Collatz conjecture
(also called Syracuse conjecture)

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Abstract :

Variables used in the proof are :
x and (x + V) are positive integer variables. (x + V) is the successor of x.
V is an integer variable of adjustment, at first step $V_0 = 2$ and $x_0 = y_0 > 2$.
Representation : $x = 2^{\alpha} *(y)$ and $V = 2^{\beta}*(z)$, $\alpha*\beta = 0$, α and β are non negative integer variables, y and z are odd integer variables, y > 0.

The Collatz algorithm (3*x + 1) is applied **simultaneously** to x and (x + V), so we have for rule 2 - (3*x + 1) - of the algorithm and adjustment :

$$\mathbf{(x + V) := (2^{\alpha}*(3*y + 1)) + (2^{\beta}*(3*z) - (2^{\alpha} - 1)) = 3(2^{\alpha}*(y) + 2^{\beta}*(z)) + 1} > 0$$

That gives :

$$x := 2^{\alpha}*(3*y+1) = 2^{\alpha'}*(y'),\ V := 2^{\beta}*(3*z) - (2^{\alpha} -1) = 2^{\beta'}*(z') \quad \mathbf{(1)}$$

We deduce the rule :

$$(x := 2^{\alpha}*(3*y + 1)) \wedge (V := 2^{\beta}*(3*z) - (2^{\alpha} - 1)) \implies \mathbf{V < x}$$

By recurrence we have : $x_{i+1} + V_{i+1} := 3(x_i + V_i) + 1 > 0 \implies \mathbf{x + V > 0}$

As x + V > 0 and V < x, we deduce the rule :

$$\mathbf{(V < x) \wedge (x + V > 0) \implies (0 < x + V < 2*x)}. \quad (0< x_i+ V_i <2*x_i)$$

By hypothesis $S(x_0)$ is a sequence of Syracuse, the rule shows that sequence $S(x_0+2)$ is bounded because it is upper bounded by sequence of Syracuse $S(2*x_0) = 2*S(x_0)$ and lower bounded by 0.

The **bounded sequence $S(x_0+2)$** and the sequence of Syracuse $S(2*x_0) = 2*S(x_0)$ – **upper bound** - converge to the only trivial cycle : [4, 2, 1].
So by recurrence, every positive integer gives a sequence of Syracuse.

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Collatz conjecture (also called Syracuse conjecture)

Algorithm of Collatz :
Let x a positive integer number.
1 - if x is even then x := x/2
2 - if x is odd then x := x * 3 + 1
We repeat 1 - 2 until obtain a cycle (is only cycle ?) or x tends to infinity.
The cycle [4, 2, 1] is the Collatz conjecture.
The symbol := means : assign value on right to variable on left.

Representation of variables :

x and (x + V) are positive integer variables. (x + V) is the successor of x.
V is a variable of adjustment, at first step $V_0 = 2$ and $x_0 = y_0 > 2$.
The variables x and V are written in the form :
$x := a*(y)$ with $a := 2^\alpha$ and $V := b*(z)$ with $b := 2^\beta$.
α and β are non negative integer variables, such as α*β = 0.
y and z are odd integer variables, y > 0.
$(x + V) := a*(y) + b*(z) = 2^\alpha*(y) + 2^\beta*(z)$ and α*β = 0.

Application of the Collatz algorithm :

The Collatz algorithm (3*x + 1) is applied **simultaneously** to x and (x + V).
The coefficient a is power of 2, the algorithm is applied to the odd part y of x := a*(y) giving a sequence of Syracuse $S(x_0)$ and the odd part z of V := b*(z) is multiplied by 3 plus an adjustment.
In operation 3*x + 1, x := a*(3*y + 1) = a'*(y'), x is increased by (a - 1) to subtract from V and we have for V in x + V : V := b*(3*z) - (a-1) = b'*(z') .
a' and b' are power of 2, y' and z' are odd integer variables.
So we have the equality
a*(3*y+1) + b*(3*z) - (a-1) = a*(3*y) +1 + b*(3*z) = 3 * (a*(y) + b*(z)) + 1 > 0**, giving 3*(x + V) + 1, with x and V of before the operation 3* + 1, according to the rule 2 of the algorithm.**
The rule 2 and adjustment give :

$$x := 2^\alpha*(3*y+1),\ V := 2^\beta*(3*z) - (2^\alpha - 1) \qquad \textbf{(2)}$$

We deduce the rule:

$$(x := 2^\alpha*(3*y + 1)) \wedge (V := 2^\beta*(3*z) - (2^\alpha - 1)) \Rightarrow \mathbf{V < x}.\ (V_i < x_i)$$

In the line a'*(y') + b'*(z'), a' and b' are divided by gcd(a',b') according to the rule 1 of the algorithm.
If gcd(a',b') = 1 then division by 2 is deferred and then we have :

$$x := 2^{\alpha'}*(y'),\quad V := 2^{\beta'}*(z') \text{ and } \alpha'*\beta' = 0.$$

Evaluation of variable of adjustment V :

When x is multiplied by 3 then + 1, V is multiplied by 3.
When x is divided by 2, V is divided by 2.
When x = a(3*y+1), x is increased by (a - 1), V is decreased by (a - 1).
We deduce that V is always less than x.
We deduce the rule :

$$\mathbf{(V < x) \Longrightarrow (x + V < 2*x)}. \qquad (x_i+V_i < 2*x_i)$$

By hypothesis $S(x_0)$ is a sequence of Syracuse, the rule shows that sequence $S(x_0+2)$ is bounded because it is upper bounded by sequence of Syracuse $S(2*x_0) = 2*$**$S(x_0)$**.

By recurrence we have : $x_{i+1} + V_{i+1} := 3(x_i + V_i) + 1 > 0$ ==> **x + V > 0**
This shows (x + V) is always positive and therefore the sequence $S(x_0+2)$ is lower bounded by 0 :

(x + V) > 0

We deduce the rule :

(V < x) ^ (x + V > 0) ==> (0 < x + V < 2*x). $(0 < x_i+V_i < 2*x_i)$

Conclusion :

The **bounded sequence $S(x_0+2)$** and the sequence of Syracuse $S(2*x_0) = 2*S(x_0)$ – **upper bound** - converge to the only trivial cycle : [4, 2, 1].
So by recurrence, every positive integer gives a sequence of Syracuse.
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Generation of sequences of Syracuse $S(x_0)$ and $S(x_0 + V_0)$:

Application of Collatz algorithm simultaneously to x and (x + V) to generate sequences of Syracuse $S(x_0)$ and $S(x_0 + V_0)$.

Generation of sequences of Syracuse S(17) and S(17 + 2)=S(19) :

$x_0 = 17$ et $x_0 + V_0 = 17 + 2 = 19$

S(17) = S(17 + 2) = S(19) =

1(**17**) + 2(1) = 19 = 1(**19**)

1(**52**) + 2(3) = 58 = 2(**29**)

4(**13**) + 2(3) = 58 = 2(29)

2(13) + 1(3) = 29 = 1(29)

2(**40**) + 1(9) - 1 = 88 = 8(**11**)

16(**5**) + 8(1) = 88 = 8(11)

2(5) + 1(1) = 11 = 1(11)

2(**16**) + 1(3) - 1 = 34 = 2(**17**)

32(**1**) + 2(1) = 34 = 2(17)

16(1) + 1(1) = 17 = 1(17)

16(4) + 1(3) - 15 = 52 = 4(**13**)

64(1) + 4(-3) = 52 = 4(13)

16(1) + 1(-3) = 13 = 1(13)

16(4) + 1(-9) - 15 = 40 = 8(**5**)

64(1) + 8(-3) = 40 = 8(5)

8(1) + 1(-3) = 5 = 1(5)

8(4) + 1(-9) - 7 = 16 = 16(**1**)

32(1) + 16(-1) = 16 = 16(1)

2(1) + 1(-1) = 1 = 1(**1**)

2(4) + 1(-3) - 1 = 4 = 4(1)

8(1) + 4(-1) = 4 = 4(1)

2(1) + 1(-1) = 1 = 1(1)