An Information Based Theory of Stationary Action

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24 March 2023

The quantization of energy proposed by Planck to account for the observed spectrum of black body radiation has associated with it a quantization of entropy. This in turn implies a quantization of observable information, directly implying observational uncertainty on the order of Planck's constant. The effect of that uncertainty is analyzed. In order to adhere strictly to the use of observable quantities, a probability measure is employed based on the distinguishability of statistical samples. This leads directly to the description of probability in terms of the absolute square of a complex amplitude. The Feynman rules may then be applied naturally for indistinguishable events without contradiction to the conventional rules for distinguishable events. This enables the straightforward calculation of the probability that a particle moves from one arbitrary point to another. The Feynman formulation of quantum phenomena and the principle of stationary action results when it is assumed that the classical action represents the measure of distinguishability. Parallel analysis on a Lorentz manifold yields the geodesic principle.

Introduction

Early efforts to understand black body radiation within the confines of classical physics focused on the entropy of electromagnetic radiation. In 1884 Boltzmann studied black body radiation in a perfectly reflecting enclosure. Treating the radiation pressure as that of a continuum gas he was able to define its entropy [1]. From that followed a theoretical basis for Stefan's empirically determined dependence of total radiated power on the fourth power of temperature.

Several years earlier Boltzmann had shown a correspondence between classical entropy and the discrete quantity that was later called the "statistical multiplicity"[2] of a molecular gas, though he made no use of this in his black body work. It was not until the mid twentieth century introduction of information theory [3] that entropy could be understood as information lost due to the statistical treatment of trajectories, in lieu of a more complete microscopic model of molecular states [4].

By 1900, Planck had developed a classical model of the entropy of a black body at temperature T in which electromagnetic dipole resonators operated in equilibrium with the radiant energy in Boltzmann's reflective enclosure. Comparing the latest empirical data to his model, he found it necessary to introduce the constant h limiting radiation energy to discrete multiples of $h\nu$ [5]. This quantization of energy has the effect of limiting the entropy to increments of $h\nu/T$ when expressed in the prevailing units, those of Boltzmann's constant k.

The appearance of entropy in discrete increments is consistent with the situation in statistical mechanics. In that case discrete values replace the continuum model since entropy is now based on the discrete statistical multiplicity of macro-states. As in the case of the continuum gas model, the entropy in the black body model will be interpreted as a paucity of accurate information, in this case due to some inherent natural limit.

Consider an ideal gas consisting of a single molecule. The Planck entropy implies an inability of the classical model to describe its trajectory in phase space with uncertainty less than h. It may be that a more accurate description of the trajectory is not possible. Alternately, the trajectory may be fully deterministic but some inherent limit in observational accuracy produces the uncertainty, even with perfectly accurate measuring equipment.

In either case the description of observable quantities in the classical model, like that of a continuum gas is only approximate, with the action of observed natural phenomena differing from the classical description.

The result is an observed stochastic component of order h in the molecular trajectory. In this situation, Planck's constant h is a more convenient unit of entropy.

Let η be a system dependent parameter, then let

$$I_1 = \eta h \tag{1}$$

represent the information in a hypothetical, more accurate and possibly fully deterministic model that supersedes the inaccurate part of the information in the classical description. Let H_1 represent the entropy of the more accurate model and H_0 the entropy of the classical model. Then [6]

$$H_1 = H_0 - I_1 \tag{2}$$

If the more accurate model is both fully deterministic and completely accurate

$$H_1 = 0 \tag{3}$$

Then

$$H_0 = I_1 = \eta h \tag{4}$$

The analysis that follows explicitly acknowledges statistical uncertainties in observations of physical systems. Following classical practice, we assume no explicit limit on the accuracy of measuring instruments. Also, as in the classical model, the explicit effect of a measurement is not assumed in advance to significantly affect its own result, nor the results of future measurements.

The existence of uncertainty requires that the inherently stochastic nature of the result of observation be incorporated in the analysis. The choice of probability measure can be of profound importance [7]. Following modern practice, careful attention is paid to ensuring our analysis is based strictly on what can be observed. To that end we employ a probability measure based on stochastic outcomes that are equal in statistical distinguishability from one another.

Distinguishing one experimental outcome from another is necessarily a matter of distinguishing between their probability distributions. The distinguishability of probability distributions has been studied by Wootters [8]. It is measured by the quantity *statistical distance* on a probability space.

Consider two N sided loaded die with different loadings, where the differences in the probabilities of corresponding faces are $\delta p_1 \cdots \delta p_N$. The die are said to be distinguishable in n trials if

$$\frac{\sqrt{n}}{2}\left[\sum_{i=1}^{N}\frac{(\delta p_i)^2}{p_i}\right] > 1 \tag{5}$$

Then the statistical distance $\mathcal{S}$ between these dice on the appropriate probability space is defined by

$$\mathcal{S} = \lim_{n\to\infty}\frac{1}{\sqrt{n}}\times\begin{bmatrix}\text{the maximum number of intermediate outcomes each of}\\ \text{which is distinguishable (in } n \text{ trials) from its neighbors}\end{bmatrix} \tag{6}$$

Along with the stochastic analysis of what can be observed, the question remains: are these stochastic processes consistent with more deterministic, or even fully deterministic underlying natural processes, even if some of the parameters necessary to make use of a more deterministic model are inherently unknowable due to some natural limitation on their observability.

Analysis

Let $x = (x_0, x_1, x_2, x_3)$ represent the ordinary space and time of classical physics where x_0 represents time and x_1, x_2 and x_3 represent three-dimensional Euclidean space. Let us consider a particle that moves from start point A to end point B by an unknown trajectory through this space and time. Let $x(t)$ be an arbitrary trajectory with the same end points, where t is a time like parameter. We stipulate, in view of uncertainty, that for any value of the parameter t assigning a definite time and location on $x(t)$ there may be a nonzero probability $p(x,t)$ that the particle can be observed at any time and location x. Let $p(x,t)$ be piecewise differentiable with respect to t.

The statistical distance between points in the physical space may then be expressed as the statistical distance $\mathcal{S}$ between corresponding points in the associated probability space [9].

$$d\mathcal{S}(t) = \frac{1}{2}\left\{\int_{-\infty}^{\infty} dx_0 \iiint_{-\infty}^{\infty} dx_1 dx_2 dx_3 \frac{1}{p(x,t)}\left[\frac{dp(x,t)}{dt}\right]^2\right\}^{1/2} dt \tag{7}$$

This expression may be simplified by the substitution $\zeta(x,t) = p^{1/2}(x,t)$. Then

$$d\mathcal{S}(t) = \left\{\int_{-\infty}^{\infty} dx_0 \iiint_{-\infty}^{\infty} dx_1 dx_2 dx_3 \left[\frac{d\zeta(x,t)}{dt}\right]^2\right\}^{1/2} dt \tag{8}$$

This new expression defines an element of length $d\mathcal{S}$ in an infinite dimensional Euclidean ζ space.

Since $\zeta^2(x,t)$ is a probability, it's integral over all of space and time must equal unity for any value of the parameter t. Thus $d\mathcal{S}$ lies on the surface of an infinite dimensional unit hypersphere. This defines a probability space. Then statistical distance $\mathcal{S}$ between points on $x(t)$ is measured by the length of the arc traced on the surface of the hypersphere as the progress of parameter t traces out the trajectory in space and time between them [10].

Now let us divide the integral on the right-hand-side of (8) in two. Let $[da(t)/dt]^2$ represent the portion of the integral for which $x_0 < t$ and $[db(t)/dt]^2$ the portion for which $x_0 > t$.

$$\left[\frac{da(t)}{dt}\right]^2 = \int_{-\infty}^{t} dx_0 \iiint_{-\infty}^{\infty} dx_1 dx_2 dx_3 \left[\frac{d\zeta(x,t)}{dt}\right]^2 \tag{9}$$

$$\left[\frac{db(t)}{dt}\right]^2 = \int_{t}^{\infty} dx_0 \iiint_{-\infty}^{\infty} dx_1 dx_2 dx_3 \left[\frac{d\zeta(x,t)}{dt}\right]^2 \tag{10}$$

Then a represents events nominally in the past and b, the future. Let us form the complex quantity $a^2 + ib^2 = e^{i\theta}$ where $\theta = \tan^{-1} b/a$. The unit hypersphere is collapsed into the unit circle on the Argand plane, while the angle $\theta(t)$ measures statistical distance.

$$d\mathcal{S}(t) = d\theta(t) \tag{11}$$

Now in order to express the probability $p[x(t)]$ of the test particle being found at a point on the trajectory $[x(t)]$ as an explicit function of the statistical distance $\theta(t)$ we may define the probability amplitude

$$\varphi[x(t)] \equiv \zeta[x(t)]e^{i\theta(t)} \tag{12}$$

Then

$$p[x(t)] = |\varphi[x(t)]|^2 \tag{13}$$

<u>The Feynman Rules</u>

The identification of a probability with the absolute square of a complex quantity constitutes the Feynman *amplitude-probability rule* [11], [12].

While the Feynman rules are explicitly for the purpose of quantum mechanical calculation, the conditions assumed here in developing the probability amplitude consist of no more than a random variable with probability described by a function on space and time, piecewise differentiable with respect to time. This allows the Feynman rules to be treated as a feature of probability theory under these circumstances, when a measure based on statistical distinguishability is employed. Indeed, the peculiarities of quantum theory, depending upon what can and cannot be measured, may be regarded as depending upon distinguishability.

The conventional Laplace rule for the probability p_L of an event that may occur by any of m alternative means is

$$p_L = \sum_{i=1}^{m} p_i = \sum_{i=1}^{m} |\varphi_i|^2 \tag{14}$$

where p_i represents the probability of each of the alternatives. The Laplace rule is empirical in nature, verified by counting occurrences of the various alternatives [13]. We know from a century of experience with quantum phenomena that when the alternatives are *indistinguishable* the probability p_F is [14].

$$p_F = |\varphi_F|^2 = \left|\sum_{i=1}^{m} \varphi_i\right|^2 \tag{15}$$

This constitutes the *Feynman amplitude sum rule* [15], [16]. Alternatives are generally indistinguishable because no sufficiently accurate means of detection is present, making distinction impossible.

The conventional Laplace rule for the probability p_L that m events all occur is

$$p_L = \prod_{i=1}^{m} p_i = \prod_{i=1}^{m} |\varphi_i|^2 \tag{16}$$

where the values of p_i represent the probability of each event. Our experience with quantum phenomena informs us that when the m events are indistinguishable the probability p_F is [17].

$$p_F = |\varphi_F|^2 = \left|\prod_{i=1}^{m} \varphi_i\right|^2 \tag{17}$$

This constitutes the *Feynman amplitude product rule* [18], [19]. While the two formulae yield identical probabilities, the latter establishes the phase $\theta(t)$ of the probability amplitude for the combined event.

Remarkably, Goyal and Knuth have shown that the Feynman rules can coexist free of conflict with conventional probability [20]. Earlier Sykora noted that while probabilities are often described in terms of a single number, a measure is also necessary. Though it is frequently not explicitly stated, the need for this second real number is never-the-less implied [21]. Indeed, the complex representation of probability developed here and exhibited in quantum formulations has been shown to be a necessary result of any probability represented by a pair of real numbers [22]. In this case amplitude $|\varphi|$ and phase θ respectively describe the probability and the distinguishability of alternatives. The complex probability amplitude allows the former to be conveniently formulated in terms of the latter.

Feynman [23], [24] citing von Neumann [25] has shown how the introduction of a means of observation produces an arbitrary unknown phase shift in the probability amplitudes of previously unobservable events. Let $p_F = |\varphi_1 + \varphi_2|^2$ be the probability of an event with two indistinguishable alternative ways of occurring, signified by the two subscripts. Suppose now that a means of observing the alternatives is provided. The presence of the measuring equipment perturbs the phase of the probability amplitudes by arbitrary unknown amounts θ_1 and θ_2. Now $p_F = \left|\varphi_1\, e^{i\theta_1} + \varphi_2\, e^{i\theta_2}\right|^2$. The multiple observations required to observe the statistical frequency of these alternatives require that their phases be averaged over all angles. This results in reversion to the Laplace rule $p_L = |\varphi_1|^2 + |\varphi_2|^2$ as the sample size approaches infinity.

In view of this one may argue that (15) be treated as the more fundamental empirical rule of probability applying to any indistinguishable alternatives lying on a piecewise differentiable space and time continuum, while the conventional Laplace rule (14) becomes derivative of it. Until the possibility of an observation is present, only an amplitude exists without a corresponding probability. Where observation is possible, a frequency may be observed and a probability emerges. The real valued probability is independent of phase, and the phase of the probability amplitude is lost with the emergence of a probability.

Von Neumann's arbitrary phase shift was determined based on the properties of the Dirac-von Neumann model of quantum phenomena. Given the undeniable long-term success of that model, the von Neumann phase shift has been treated here as an empirical result. The origin of the multiplication and summation rules as well as this phase shift behavior remain questions of interest.

The Path Integral

Let $\Delta\theta = \int_{x(t)}[d\theta(t)/dt]\,dt$. This is the statistical distance traced by a particle as it traverses the path $x(t)$. Let m be an arbitrary integer and let $\delta\theta = \Delta\theta/m$ so that $x[\theta(t)]$ is divided into m equally distinguishable segments. Then the probability amplitude for the j^{th} segment is $\varphi_j = \mathcal{A}_j^{(m)} e^{i\delta\theta}$ where $\mathcal{A}_j^{(m)} = \left[p_j^{(m)}\right]^{1/2}$ while $p_j^{(m)}$, dependent on the value of m, is the probability for the j^{th} segment were a measurement possible.

The probability amplitude $\varphi_{\mathcal{P}}[x(t)]$ for a test particle following an arbitrary path $\mathcal{P}$ $=x(t)$, when the individual points cannot be observed, is the product of the probability amplitudes that the particle is found at each of the m intervals on $x[\theta(t)]$.

$$\varphi_{\mathcal{P}}[x(t)] = \lim_{m\to\infty}\prod_{j=1}^{m}\left[\mathcal{A}_j^{(m)} e^{i\delta\theta}\right] = \mathcal{A}e^{i\Delta\theta} \tag{18}$$

where $\mathcal{A}^2 = \left[\lim_{m\to\infty}\prod_{j=1}^{m}\mathcal{A}_j^{(m)}\right]^2$ is the probability that the path $x(t)$ is followed when observation of the path is possible.

We, know based on many decades of empirical experience verifying the Feynman formulation of quantum mechanics, that the proper expression for $\varphi_{\mathcal{P}}[x(t)]$ is [26]

$$\varphi_{\mathcal{P}}[x(t)] = \text{Const}\; e^{iS/\hbar} \tag{19}$$

where S is the classical action. Then the substitution

$$S/\hbar = \Delta\theta \tag{20}$$

yields the Feynman result. The classical action S corresponds to the statistical distance $\mathcal{S}$ traced by a particle as it traverses $x(t)$ while the classical Lagrangian corresponds to the rate of change of statistical distance with time.

The probability that a test particle goes from start point A to end point B by any path is then computed according to the Feynman amplitude sum rule with the path integral replacing the summation in (15).

$$p(BA) = \left|\int_A^B \varphi(BA)\,\mathcal{D}x(t)\right|^2 \tag{21}$$

Quantum Gravity

The derivation of the Feynman formalism for ordinary quantum mechanics developed here does not rely on the Euclidean nature of the physical space to which those rules are applied. A parallel derivation can be applied to a particle trajectory on a Reimann manifold provided an uncertainty is postulated in the observability of the *position* of a particle. This leads directly to the geodesic principle of general relativity in lieu of Hamilton's principle. The equivalence of these two principles in the flat space limit [27] establishes a correspondence between the two forms of uncertainty.

Let $x^* = (x_0^*, x_1^*, x_2^*, x_3^*)$ represent the four-dimensional Lorentzian manifold of general relativity, where x_0^* represents time and x_1^*, x_2^* and x_3^* represent the three spatial dimensions. Substituting x^* for x in (7) through (11) we find the equivalent definition of statistical distance. Continuing in the same vein through (18), we find the equivalent expression for the probability amplitude of an arbitrary path in terms of the statistical length of the path $\Delta\theta$.

We omit (19) and (20) which identify statistical distance with the classical action, and substitute s/s_c for $\Delta\theta$ where $s = \int_{x^*(t)} [dx^*(t)/dt]\, dt$ is the length of $x^*(t)$ and s_c is a small increment of length on the Lorentzian manifold characterizing uncertainty in position. Then the probability amplitude $\varphi^*_{\mathcal{P}}[x^*(t)]$ for a particle to follow an arbitrary path $x^*(t)$ from A to B is

$$\varphi^*_{\mathcal{P}}[x^*(t)] = \mathcal{A}^* e^{is/s_c} \tag{22}$$

As the path integral (21) yields Hamilton's principle, the new path integral

$$p^*(BA) = \left| \int_A^B \varphi^*_{\mathcal{P}}(BA)\, \mathcal{D}x^*(t) \right|^2 \tag{23}$$

yield the geodesic principle. The dynamic of general relativity is recovered directly while the principle of stationary action follows in the flat space limit [28]. This yields the relativistic equivalent to the Feynman space time formulation of non-relativistic quantum mechanics [29] providing a plausible and equally general model of quantum gravity subject to empirical validation.

It remains to be seen whether preexisting fully deterministic general relativity can provide an equally deterministic model of what cannot be observed, fully consistent with what can be observed. A closer look at the black body spectrum can yield some insight.

<u>The Black Body Spectrum Revisited</u>

When viewed in the laboratory frame, the uncertainty of position in four-space must appear as a small apparently random motion. This in turn results in a small zero-point energy. As early as 1913 employing purely classical analysis, Einstein and Stern showed that the assumption of zero-point energy $h\upsilon$ in Planck's dipole oscillators led to the Planck spectrum *without the independent assumption of energy quantization* [30].

Milonni has extended that analysis [31] noting each Planck oscillator is in equilibrium with an associated field mode of Planck's cavity. Employing the same zero-point energy, the equipartition theorem requires the oscillator and field each have energy $h\upsilon/2$.

Let us now momentarily assume that deterministic physical laws resembling the laws of classical physics continue to govern even at the microscopic level, subject to some undefined source of observational uncertainty (More on this shortly). Since the apparently random motion of the electron and the field at the same location are identical, we would expect no coupling between the zero-point motion of the dipole oscillator and the zero-point component of the field under this assumption.

Returning to Milonni, still employing purely classical analysis he demonstrates that when there is no interaction between the random components of the oscillator and the field, the black body spectral density is

$$\rho(\upsilon) = \frac{8\pi h\upsilon^3/c^3}{e^{h\upsilon/kT} - 1} + 4\pi h\upsilon^3/c^3 \tag{24}$$

in agreement with quantum electrodynamic theory. This hints that fully deterministic laws resembling the classical ones may prevail, even at the unobservable level.

Discussion

The present analysis leads naturally to the Feynman formulation under the assumption that our ability to know the state of physical phenomena is inherently imperfect. It relies on a supplemented form of conventional probability theory that employs the Feynman rules as the empirically determined rules of probability for indistinguishable states.

A similar point of view can be applied to the Dirac-von Neumann formalism. Consistent with the "ψ-epistemic" view [32], the probability amplitude represents the state of information available about the system. When the amplitude is defined this way, its collapse does not represent a change in the physical system. Instead, it indicates the state of available information about the system has changed as the result of a measurement. A probability has emerged, while simultaneously the phase of the probability amplitude has vanished.

The probability amplitude may be regarded as representing the available information about a conditional probability [33] based on the state of information prior to measurement. The amplitude after measurement represents a new conditional probability based on the new state of information generated by the measurement.

Goyal's analysis has shown [34] that the logic of the Feynman formalism is equivalent to that of Dirac-von Neumann when it is supplemented with a *no-disturbance postulate*. This posits that there exists a class of *trivial measurements* that can be made on the system which do not change the probability amplitude or its future evolution. Trivial measurements are defined by the property that they yield no new information about the system. This principle is a natural consequence of the epistemic interpretation of the probability amplitude. With no change in information the conditional probabilities of subsequent outcomes are unchanged.

The no-disturbance postulate resides uncomfortably alongside the notion that uncertainty is caused by the process of measurement. The apparent conflict is eliminated with the adoption of the ψ-epistemic viewpoint coupled with the proposed entropic origin of uncertainty.

The entropy and information considerations that motivate the present analysis allow for fully deterministic underlying processes that are beyond our ability to fully observe, hence appearing stochastic in nature. Alternatively, they also allow for these underlying processes to be inherently stochastic in nature, not just in appearance. Similarly, nothing in the ψ-epistemic viewpoint prevents the existence of fully deterministic underlying processes, nor of inherently stochastic ones [35].

Determinism

As noted earlier, the Einstein-Hoff-Milonni (EHM) black body result, though an isolated one, suggests that deterministic physical processes may continue to operate in the quantum regime even though not fully subject to observation.

The simplest model assumes that those underlying physical processes are the familiar classical ones as in Milonni's analysis. If this is so, some undefined source of zero-point energy $h\nu$ is required to account for uncertainty. This source of energy may originate outside the relativistic model, and its flat space limit, though the dynamics of the resulting motion may still be in keeping with it. This would manifest itself in the form of random perturbations of the space time metric along with the resulting dynamics. Let us call this the external-source model.

We may also describe an alternative internal-source model in which the small apparently random motions remain fully accounted for in the relativistic model. Correspondence between classical physics and general relativity occurs when energies of objects under observation are suitably small, and there are no variations in the space-time metric due to events outside the range of observation [36]. The latter of these conditions precludes from consideration a background level of broadband gravitational radiation.

If such background exists then, it can be expected to impose on the classical picture a small apparently random source of zero-point energy. Even in the full general relativistic model, background gravitational radiation that appears stochastic to an observer with only local knowledge must add an apparently random component to the predictable trajectories of ponderable masses.

The full nature of such a stochastic background of gravitational radiation is an open question [37]. Neither a spectrum nor a characteristic time for the gravitational background is known. That this is the source of uncertainty is of course speculation. If this is the case, the general relativistic model is the not-fully-predictable but fully deterministic model. The lack of predictability stems from our inability to know the background gravitational radiation in anything but stochastic terms.

The (EHM) black body model employs classical electromagnetic theory with no quantization assumed. It derives a result previously requiring quantum electrodynamic analysis. This raises the possibility that more generally at the unobservable level, electromagnetic phenomena may be treated classically with boson quantum particle behavior arising due to the presence of observational uncertainty. The entropy associated with the black body spectrum has long been understood to impart particle like particle to the radiation [38], [39]. The Feynman rules, of course, already impart substantial wave like properties to classical particle behavior in the presence of the same observational uncertainty.

The (EHM) black body model fits comfortably within the flat space limit of general relativity. An extension of this approach for broader applicability may suggest exploration of an extension of Maxwell's equations operative on the Lorentz manifold. It must of course be borne in mind that the (EHM) avenue to the black body spectrum is an isolated result. Further investigation is called for, as was the case with Planck's isolated black body result. Only then can generalizations be made with confidence.

Randomness

In the two speculative models suggested so far, the preexisting laws of classical physics including relativity are assumed to be in place while quantum phenomena are caused by an additional small stochastic component of some sort. In a third alternative, the preexisting laws are not assumed. Let us call this the bootstrap model in recognition of how these laws may come into being.

The deterministic laws of physics are generally described by differential equations. It is in the nature of these equations that macroscopic behavior follows directly from behavior at the microscopic level. Experience with quantum phenomena indicate a substantial degree of randomness at the microscopic level. Let us suppose for a moment that this randomness has no preexisting underlying order as in the previously suggested models. How then could it come to be that differential equations provide a near perfect description of nature at the coarser level? The analysis presented here is entirely classical in nature up to and including the Feynman amplitude probability rule. The product and sum rules are then postulated based on empirical quantum mechanical experience. How do these unintuitive rules come to govern the probability of physical systems?

In the face of randomness at the microscopic level, the multiplication and summation rules provide a mechanism for physical laws to emerge from randomness. The path integral assures that the observability of particles that just happen to follow paths of near stationary length in space time will be coherently amplified, while the observability of those that do not will be coherently suppressed. Despite randomness at the microscopic level, a system at least macroscopically described by deterministic differential equations may be observed. In this way the general relativistic model, indicative of what can be observed in a universe highly random at the microscopic level, may arise out of that randomness. In this model then, the Feynman multiplication and addition rules are elevated to fundamental laws of nature giving birth to the laws of mechanics. It is for now unclear whether this viewpoint can be squared with the EHM black body result.

All of these are simply speculative, illustrative examples. As always, only further investigation can provide additional clarity.

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