An Inconsistent Hierarchy of Sets in [0, 1]

By Jim Rock

Abstract: Two contradictory arguments are developed from a hierarchy of sets in [*0, 1*]. One argument is a proof by contradiction and its conclusion is true. The other argument is an existence argument and while its conclusion is not true, it follows logically from the a valid assumption followed by three true statements that precede the conclusion.

Introduction. For all rational numbers $\boldsymbol{a}$ in the closed interval [*0, 1*] and {0} define the collection of all R_a sets equal { y is a rational number | $0 \le y < a$ } and {0}

The following four true statements characterize the collection of R_a sets and {0}.

a) The collection forms a hierarchy of sets with R_1 at the top and {0} on the bottom.

b) Each R_a set contains all the elements in sets below it in the set hierarchy.

c) Each set is a proper subset of all the R_a sets above it in the set hierarchy.

d) *Used in Arg #1 step 4*. Each individual R_a set contains at least one element that is not in any of the sets below it in the hierarchy. Otherwise, the entire hierarchy would collapse.

Argument #1: R_1 **contains a largest element.**
1) Let $\boldsymbol{c}$ and $\boldsymbol{d}$ be two elements of R_1 with $\boldsymbol{c > d}$.
2) $\boldsymbol{d}$ is an element of R_c, which is a proper subset of R_1.
3) For any two elements in R_1 the smaller element is contained in a proper subset of R_1.
4) **d)** R_1 contains a largest element not contained in any set below it in the set hierarchy.

Argument #2: R_1 **contains no largest element.**
1) Suppose there is a largest element a' in R_1.
2) $a' < (a + a')/2 < a$.
3) Let $b = (a + a')/2$.
4) Then b is in R_1 and $a' < b$.

When a largest element is assumed in Argument #2 it leads to a contradiction so there is no largest element in R_1. A valid proof by contradiction.

The difference between the two arguments is no attempt is made to specify a largest element in argument #1. It is an existence argument only.

But in argument #1, step 1 is a valid assumption and statements 2, 3, and **d)** from the **Introduction** are true statements. Step 4 follows logically from steps 1, 2, 3, and **d).**

Thus, we have two contradictory arguments that can be developed in any formal system containing sets, arithmetic, and relations between the rational numbers.

If you wish, email comments to Jim Rock at **collatz3106@gmail.com**.