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THE RIEMANN HYPOTHESIS IS TRUE: THE END OF THE MYSTERY -V6-

ABDELMAJID BEN HADJ SALEM

Abstract

In 1859, Georg Friedrich Bernhard Riemann had announced the following conjecture, called Riemann Hypothesis : *The nontrivial roots (zeros) $s = \sigma + it$ of the zeta function, defined by:*

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \quad \Re(s) > 1$$

have real part $\sigma = \dfrac{1}{2}$. In this note, I give the proof that $\sigma = \dfrac{1}{2}$ using an equivalent statement of the Riemann Hypothesis concerning the Dirichlet η function.

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1. Introduction

In 1859, G.F.B. Riemann had announced the following conjecture [1] known Riemann Hypothesis:

Conjecture 1.1. *Let $\zeta(s)$ be the complex function of the complex variable $s = \sigma + it$ defined by the analytic continuation of the function:*

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \textit{ for } \Re(s) = \sigma > 1$$

over the whole complex plane, with the exception of $s = 1$. Then the nontrivial zeros of $\zeta(s) = 0$ are written as :

$$s = \frac{1}{2} + it$$

In this paper, our idea is to start from an equivalent statement of the Riemann Hypothesis, namely the one concerning the Dirichlet η function. The latter is related to Riemann's ζ function where we do not need to manipulate any expression of $\zeta(s)$ in the critical band $0 < \Re(s) < 1$. In our calculations, we will use the definition of the limit of real sequences. We arrive to give the proof that $\sigma = \dfrac{1}{2}$.

1.1. The function zeta(s) We denote $s = \sigma + it$ the complex variable of $\mathbb{C}$. For $\Re(s) = \sigma > 1$, let ζ_1 be the function defined by :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

We know that with the previous definition, the function ζ_1 is an analytical function of s. Denote by $\zeta(s)$ the function obtained by the analytic continuation of $\zeta_1(s)$ to the whole complex plane, minus the point $s = 1$, then we recall the following theorem [2]:

THEOREM 1.2. *The function $\zeta(s)$ satisfies the following :*

1. $\zeta(s)$ has no zero for $\Re(s) > 1$;

2. the only pole of $\zeta(s)$ is at $s = 1$; it has residue 1 and is simple;

3. $\zeta(s)$ has trivial zeros at $s = -2, -4, \ldots$;

4. the nontrivial zeros lie inside the region $0 \leq \Re(s) \leq 1$ (called the critical strip) and are symmetric about both the vertical line $\Re(s) = \dfrac{1}{2}$ and the real axis $\Im(s) = 0$.

The vertical line $\Re(s) = \dfrac{1}{2}$ is called the critical line.

In addition to the properties cited by the theorem 1.2 above, the function $\zeta(s)$ satisfies the functional relation [2] called also the reflection functional equation for $s \in \mathbb{C}\backslash\{0, 1\}$:

$$\zeta(1 - s) = 2^{1-s}\pi^{-s} cos\frac{s\pi}{2}\Gamma(s)\zeta(s) \tag{1.1}$$

where $\Gamma(s)$ is the *gamma function* defined only for $\Re(s) > 0$, given by the formula :

$$\Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}dt, \quad \Re(s) > 0$$

So, instead of using the functional given by (1.1), we will use the one presented by G.H. Hardy [3] namely Dirichlet eta function [2]:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)$$

The function eta is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$ [2].

We have also the theorem (see page 16, [3]):

THEOREM 1.3. *For all $t \in \mathbb{R}$, $\zeta(1 + it) \neq 0$.*

So, we take the critical strip as the region defined as $0 < \Re(s) < 1$.

1.2. A Equivalent statement to the Riemann Hypothesis Among the equivalent statements to the Riemann Hypothesis is that of the Dirichlet eta function which is stated as follows [2]:

EQUIVALENCE 1.4. *The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function :*

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s), \quad \sigma > 1 \tag{1.2}$$

that fall in the critical strip $0 < \Re(s) < 1$ *lie on the critical line* $\Re(s) = \dfrac{1}{2}$.

The series (1.2) is convergent, and represents $(1-2^{1-s})\zeta(s)$ for $\Re(s) = \sigma > 0$ ([3], pages 20-21). We can rewrite:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s), \quad \Re(s) = \sigma > 0 \tag{1.3}$$

$\eta(s)$ is a complex number, it can be written as :

$$\eta(s) = \rho.e^{i\alpha} \Longrightarrow \rho^2 = \eta(s).\overline{\eta(s)} \tag{1.4}$$

and $\eta(s) = 0 \Longleftrightarrow \rho = 0$.

2. Preliminaries of the proof that the zeros of the function eta(s) are on the critical line $\Re(s) = 1/2$

PROOF. We denote $s = \sigma + it$ with $0 < \sigma < 1$. We consider one zero of $\eta(s)$ that falls in critical strip and we write it as $s = \sigma + it$, then we obtain $0 < \sigma < 1$ and $\eta(s) = 0 \Longleftrightarrow (1-2^{1-s})\zeta(s) = 0$. We verify easily the two propositions:

$$\boxed{s, \textit{ is one zero of } \eta(s) \textit{ that falls in the critical strip, is also one zero of } \zeta(s)} \tag{2.1}$$

Conversely, if s is a zero of $\zeta(s)$ in the critical strip, let $\zeta(s) = 0 \Longrightarrow \eta(s) = (1-2^{1-s})\zeta(s) = 0$, then s is also one zero of $\eta(s)$ in the critical strip. We can write:

$$\boxed{s, \textit{ is one zero of } \zeta(s) \textit{ that falls in the critical strip, is also one zero of } \eta(s)} \tag{2.2}$$

Let us write the function η:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-sLogn} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-(\sigma+it)Logn} =$$

$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn}.e^{-itLogn}$$

$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn}(cos(tLogn) - isin(tLogn))$$

The function η is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$, but not absolutely convergent. Let s be one zero of the function eta, then :

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = 0$$

or:

$$\forall \epsilon' > 0 \quad \exists n_0, \forall N > n_0, \mid \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^s} \mid < \epsilon'$$

We definite the sequence of functions $((\eta_n)_{n\in\mathbb{N}^*}(s))$ as:

$$\eta_n(s) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^s} = \sum_{k=1}^{n} (-1)^{k-1} \frac{cos(tLogk)}{k^\sigma} - i \sum_{k=1}^{n} (-1)^{k-1} \frac{sin(tLogk)}{k^\sigma}$$

with $s = \sigma + it$ and $t \neq 0$.

Let s be one zero of η that lies in the critical strip, then $\eta(s) = 0$, with $0 < \sigma < 1$. It follows that we can write $lim_{n\longrightarrow+\infty}\eta_n(s) = 0 = \eta(s)$. We obtain:

$$lim_{n\longrightarrow+\infty} \sum_{k=1}^{n} (-1)^{k-1} \frac{cos(tLogk)}{k^\sigma} = 0$$

$$lim_{n\longrightarrow+\infty} \sum_{k=1}^{n} (-1)^{k-1} \frac{sin(tLogk)}{k^\sigma} = 0$$

Using the definition of the limit of a sequence, we can write:

$$\forall \epsilon_1 > 0 \, \exists n_r, \forall N > n_r, \mid \Re(\eta(s)_N) \mid < \epsilon_1 \Longrightarrow \Re(\eta(s)_N)^2 < {\epsilon_1}^2 \tag{2.3}$$

$$\forall \epsilon_2 > 0 \, \exists n_i, \forall N > n_i, \mid \Im(\eta(s)_N) \mid < \epsilon_2 \Longrightarrow \Im(\eta(s)_N)^2 < {\epsilon_2}^2 \tag{2.4}$$

Then:

$$0 < \sum_{k=1}^{N} \frac{cos^2(tLogk)}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} \frac{(-1)^{k+k'} cos(tLogk).cos(tLogk')}{k^\sigma k'^\sigma} < \epsilon_1^2$$

$$0 < \sum_{k=1}^{N} \frac{sin^2(tLogk)}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} \frac{(-1)^{k+k'} sin(tLogk).sin(tLogk')}{k^\sigma k'^\sigma} < \epsilon_2^2$$

Taking $\epsilon = \epsilon_1 = \epsilon_2$ and $N > max(n_r, n_i)$, we get by making the sum member to member of the last two inequalities:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma} < 2\epsilon^2 \tag{2.5}$$

We can write the above equation as :

$$0 < \rho_N^2 < 2\epsilon^2 \tag{2.6}$$

or $\rho(s) = 0$.

3. Case $\Re(s) = 1/2$

We suppose that $\sigma = \dfrac{1}{2}$. Let's start by recalling Hardy's theorem (1914) ([2], page 24):

THEOREM 3.1. *There are infinitely many zeros of $\zeta(s)$ on the critical line.*

From the propositions (2.1-2.2), it follows the proposition :

PROPOSITION 3.2. *There are infinitely many zeros of $\eta(s)$ on the critical line.*

Let $s_j = \frac{1}{2}+it_j$ one of the zeros of the function $\eta(s)$ on the critical line, so $\eta(s_j) = 0$. The equation (2.5) is written for s_j:

$$0 < \sum_{k=1}^{N} \frac{1}{k} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

If $N \longrightarrow +\infty$, the series $\sum_{k=1}^{N} \dfrac{1}{k}$ is divergent and becomes infinite. then:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Hence, we obtain the following result:

$$\boxed{lim_{N\longrightarrow+\infty} \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} = -\infty} \tag{3.1}$$

if not, we will have a contradiction with the fact that :

$$lim_{N\longrightarrow+\infty} \sum_{k=1}^{N} (-1)^{k-1} \frac{1}{k^{s_j}} = 0 \Longleftrightarrow \eta(s) \text{ is convergent for } s_j = \frac{1}{2} + it_j$$

4. Case $0 < \Re(s) < 1/2$

4.1. Case where there are zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \dfrac{1}{2}$.

Suppose that there exists $s = \sigma + it$ one zero of $\eta(s)$ or $\eta(s) = 0 \Longrightarrow \rho^2(s) = 0$ with $0 < \sigma < \frac{1}{2} \Longrightarrow s$ lies inside the critical band. We write the equation (2.5):

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma}k'^{\sigma}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N}\frac{1}{k^{2\sigma}} < 2\epsilon^2 - 2\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^{\sigma}k'^{\sigma}}$$

But $2\sigma < 1$, it follows that $lim_{N\longrightarrow+\infty}\sum_{k=1}^{N}\frac{1}{k^{2\sigma}} \longrightarrow +\infty$ and then, we obtain :

$$\boxed{\sum_{k,k'=1;k<k'}^{+\infty}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^{\sigma}k'^{\sigma}} = -\infty} \tag{4.1}$$

5. Case $1/2 < \Re(s) < 1$

Let $s = \sigma+it$ be the zero of $\eta(s)$ in $0 < \Re(s) < \frac{1}{2}$, object of the previous paragraph. From the proposition (2.1), $\zeta(s) = 0$. According to point 4 of theorem 1.2, the complex number $s' = 1-\sigma+it = \sigma'+it'$ with $\sigma' = 1-\sigma$, $t' = t$ and $\frac{1}{2} < \sigma' < 1$ verifies $\zeta(s') = 0$, so s' is also a zero of the function $\zeta(s)$ in the band $\frac{1}{2} < \Re(s) < 1$, it follows from the proposition (2.2) that $\eta(s') = 0 \Longrightarrow \rho(s') = 0$. By applying (2.5), we get:

$$0 < \sum_{k=1}^{N}\frac{1}{k^{2\sigma'}} + 2\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(t'Log(k/k'))}{k^{\sigma'}k'^{\sigma'}} < 2\epsilon^2 \tag{5.1}$$

As $0 < \sigma < \frac{1}{2} \Longrightarrow 2 > 2\sigma' = 2(1-\sigma) > 1$, then the series $\sum_{k=1}^{N}\frac{1}{k^{2\sigma'}}$ is convergent to a positive constant not null $C(\sigma')$. As $1/k^2 < 1/k^{2\sigma'}$ for all $k > 0$, then :

$$0 < \zeta(2) = \frac{\pi^2}{6} = \sum_{k=1}^{+\infty}\frac{1}{k^2} < \sum_{k=1}^{+\infty}\frac{1}{k^{2\sigma'}} = C(\sigma') = \zeta_1(2\sigma') = \zeta(2\sigma')$$

From the equation (5.1), it follows that :

$$\sum_{k,k'=1;k<k'}^{+\infty}(-1)^{k+k'}\frac{cos(t'Log(k/k'))}{k^{\sigma'}k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty \tag{5.2}$$

5.0.1. Case $t = 0$ We suppose that $t = 0 \Longrightarrow t' = 0$. The equation (5.2) becomes:

$$\sum_{k,k'=1;k<k'}^{+\infty}(-1)^{k+k'}\frac{1}{k^{\sigma'}k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty \tag{5.3}$$

Then $s' = \sigma' > 1/2$ is a zero of $\eta(s)$, we obtain :

$$\eta(s') = \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^{s'}} = 0 \tag{5.4}$$

Let us define the sequence S_m as:

$$S_m(s') = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n^{s'}} = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n^{\sigma'}} = S_m(\sigma') \tag{5.5}$$

From the definition of S_m, we obtain :

$$lim_{m \longrightarrow +\infty} S_m(s') = \eta(s') = \eta(\sigma') \tag{5.6}$$

We have also:

$$S_1(\sigma') = 1 > 0 \tag{5.7}$$

$$S_2(\sigma') = 1 - \frac{1}{2^{\sigma'}} > 0 \quad because\ 2^{\sigma'} > 1 \tag{5.8}$$

$$S_3(\sigma') = S_2(\sigma') + \frac{1}{3^{\sigma'}} > 0 \tag{5.9}$$

We proceed by recurrence, we suppose that $S_m(\sigma') > 0$.

1. $m = 2q \Longrightarrow S_{m+1}(\sigma') = \sum_{n=1}^{m+1} \frac{(-1)^{n-1}}{n^{s'}} = S_m(\sigma') + \frac{(-1)^{m+1-1}}{(m+1)^{\sigma'}}$, it gives:

$$S_{m+1}(\sigma') = S_m(\sigma') + \frac{(-1)^{2q}}{(m+1)^{\sigma'}} = S_m(\sigma') + \frac{1}{(m+1)^{\sigma'}} > 0 \Rightarrow S_{m+1}(\sigma') > 0$$

2. $m = 2q + 1$, we can write $S_{m+1}(\sigma')$ as:

$$S_{m+1}(\sigma') = S_{m-1}(\sigma') + \frac{(-1)^{m-1}}{m^{\sigma'}} + \frac{(-1)^{m+1-1}}{(m+1)^{\sigma'}}$$

We have $S_{m-1}(\sigma') > 0$, let $T = \frac{(-1)^{m-1}}{m^{\sigma'}} + \frac{(-1)^{m}}{(m+1)^{\sigma'}}$, we obtain:

$$T = \frac{(-1)^{2q}}{(2q+1)^{\sigma'}} + \frac{(-1)^{2q+1}}{(2q+2)^{\sigma'}} = \frac{1}{(2q+1)^{\sigma'}} - \frac{1}{(2q+2)^{\sigma'}} > 0 \tag{5.10}$$

and $S_{m+1}(\sigma') > 0$.

Then all the terms $S_m(\sigma')$ of the sequence S_m are great then 0, it follows that $lim_{m \longrightarrow +\infty} S_m(s') = \eta(s') = \eta(\sigma') > 0$ and $\eta(\sigma') < +\infty$ because $\Re(s') = \sigma' > 0$ and $\eta(s')$ is convergent. We deduce the contradiction with the hypothesis s' is a zero of $\eta(s)$ and:

$$\boxed{\text{The equation (5.3) is false for the case } t' = t = 0.} \tag{5.11}$$

5.0.2. Case $t' = t \neq 0$ We suppose that $t' \neq 0$. Let $s' = \sigma' + it' = 1 - \sigma + it$ a zero of $\eta(s)$, we have:

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t' Log(k/k'))}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty \tag{5.12}$$

the left member of the equation (5.12) above is finite and depends of σ' and t', but the right member is a function only of σ' equal to $-\zeta(2\sigma')/2$.

We recall the following theorem (see page 140, [3]):

THEOREM 5.1.

$$lim_{T\longrightarrow+\infty}\frac{1}{T}\int_1^T|\zeta(\sigma"+i\tau)|^2\,d\tau=\zeta(2\sigma")\quad(\sigma">\frac{1}{2}) \tag{5.13}$$

We fix $\sigma"=\sigma'$, from the theorem above, $\zeta(2\sigma')$ is independent of any $\tau\geq1>0$, then $\zeta(2\sigma')$ does non depend of t' so that $s'=\sigma'+it'$ is a root of η, it follows the contradiction with equation (5.12). Then the equation (5.12) is false.

$$\boxed{\text{It follows that the equation (5.12) is false for the case } t'\neq0.} \tag{5.14}$$

It follows that the equation (5.2) is false and $\eta(s')$ does not vanish for $\sigma'\in]1/2,1[$.

From (5.11-5.14), we conclude that the function $\eta(s)$ has no zeros for all $s'=\sigma'+it'$ with $\sigma'\in]1/2,1[$, it follows that the case of the paragraph (4) above concerning the case $0<\Re(s)<\dfrac{1}{2}$ is false too. Then, the function $\eta(s)$ has all its zeros on the critical line $\sigma=\dfrac{1}{2}$. From the equivalent statement (1.4), it follows that **the Riemann hypothesis is verified**. □

From the calculations above, we can verify easily the following known proposition:

PROPOSITION 5.2. *For all $s=\sigma$ real with $0<\sigma<1$, $\eta(s)>0$ and $\zeta(s)<0$.*

6. Conclusion

In summary: for our proofs, we made use of Dirichlet $\eta(s)$ function:

$$\eta(s)=\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s}=(1-2^{1-s})\zeta(s),\qquad s=\sigma+it$$

on the critical band $0<\Re(s)<1$, in obtaining:

- $\eta(s)$ vanishes for $0<\sigma=\Re(s)=\dfrac{1}{2}$;
- $\eta(s)$ does not vanish for $0<\sigma=\Re(s)<\dfrac{1}{2}$ and $\dfrac{1}{2}<\sigma=\Re(s)<1$.

Consequently, all the zeros of $\eta(s)$ inside the critical band $0<\Re(s)<1$ are on the critical line $\Re(s)=\dfrac{1}{2}$. Applying the equivalent proposition to the Riemann Hypothesis (1.4), we conclude that **the Riemann hypothesis is verified** and all the nontrivial zeros of the function $\zeta(s)$ lie on the critical line $\Re(s) = \dfrac{1}{2}$. The proof of the Riemann Hypothesis is thus completed.

We therefore announce the important theorem as follows:

THEOREM 6.1. *The Riemann Hypothesis is true:*
All nontrivial zeros of the function $\zeta(s)$ with $s = \sigma + it$ lie on the vertical line $\Re(s) = \dfrac{1}{2}$.

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Abdelmajid Ben Hadj Salem, Résidence Bousten 8, Av. Mosquée Raoudha, Bloc B, 1181 Soukra Raoudha, Tunisia.
e-mail: abenhadjsalem@gmail.com