ON THE TOPOLOGY OF PROBLEMS AND THEIR SOLUTIONS

THEOPHILUS AGAMA

Abstract. In this paper, we study the topology of problems and their solution spaces developed introduced in our first paper [1]. We introduce and study the notion of separability and quotient problem and solution spaces. This notions will form a basic underpinning for further studies on this topic.

1. Introduction

In [1] we studied the theory of problems and their solution spaces. We recall the following definitions

Definition 1.1. Let X denotes a solution (resp. answer) to problem Y (resp. question). Then we call the collection of all problems to be solved to provide solution X to problem Y the problem space induced by providing solution X to problem Y. We denote this space with $\mathcal{P}_Y(X)$. If K is any subspace of the space $\mathcal{P}_Y(X)$, then we denote this relation with $K \subseteq \mathcal{P}_Y(X)$. If the space K is a subspace of the space $\mathcal{P}_Y(X)$ with $K \neq \mathcal{P}_Y(X)$, then we write $K \subset \mathcal{P}_Y(X)$. We say problem V is a sub-problem of problem Y if providing a solution to problem Y furnishes a solution to problem V. If V is a sub-problem of the problem Y, then we write $V \leq Y$. If V is a sub-problem of the problem Y and $V \neq Y$, then we write $V < Y$ and we call V a proper sub-problem of Y.

Definition 1.2. Let $\mathcal{P}_Y(X)$ be the problem space induced by providing the solution X to problem Y. Then we call the number of problems in the space (size) the **complexity** of the space and denote by $\mathbb{C}[\mathcal{P}_Y(X)]$ the complexity of the space. We make the assignment $Z \in \mathcal{P}_Y(X)$ if problem Z is also a problem in this space.

Definition 1.3. Let X denotes a solution (resp. answer) to problem Y (resp. question). Then we call the collection of all solutions to problems obtained as a result of providing the solution X to problem Y the solution space induced by providing solution X to problem Y. We denote this space with $\mathcal{S}_Y(X)$. If K is any subspace of the space $\mathcal{S}_Y(X)$, then we denote this relation with $K \subset \mathcal{S}_Y(X)$. We make the assignment $T \in \mathcal{S}_Y(X)$ if solution T is also a solution in this space.

In [1], we made the following conjectures, whose proof or disprove will undoubtedly change the landscape of the theory and could illuminate certain subtle futures about this topology:

Conjecture 1.4. *Let V be a problem. If V has a minimal and a maximal sub-problem, then V must be a regular problem.*

Date: March 7, 2023.

Conjecture 1.5. *Let V be a problem with solution U and Y a problem with solution X. If V be regular and the spaces $\mathcal{P}_V(U)$ and $\mathcal{P}_Y(X)$ are highly connected, then Y must also be regular.*

In the current studies, the truth of the following result is obvious.

Proposition 1.6. *Let V and Y be any two problems in some problem space. If there exists an $L \in \mathcal{S}_Y(X)$ that solves V and Y, then either $V = Y$ or $V \equiv Y$.*

2. Alternative solutions and their corresponding solution spaces

Definition 2.1. Let Y be a problem. Then we say X and U are alternative solutions to Y if and only if U and X both solves Y. We denote this relation with $X \perp U$ or $U \perp X$.

Proposition 2.2. *Solution spaces remain invariant under replacement with alternative solutions.*

Proof. Let $\mathcal{P}_Y(X)$ be a problem space with corresponding solution space $\mathcal{S}_Y(X)$. Suppose $L \in \mathcal{S}_Y(X)$ with $L \perp K$, then there exist a problem $F \in \mathcal{P}_Y(X)$ that is solved by L. Since $L \perp K$, it follows that K also solves F. Thus we can replace $L \in \mathcal{S}_Y(X)$ with K. $\square$

3. Separable and inseparable problem and solution spaces

In this section we introduce and study the notion of separability of problem and their corresponding solution spaces. We first launch the following language.

Definition 3.1. Let $\mathcal{P}_Y(X)$ be a problem space. Then we say $\mathcal{P}_Y(X)$ is separable if and only there exist some $\mathcal{P}_V(U) \subset \mathcal{P}_Y(X)$ and $\mathcal{P}_K(L) \subset \mathcal{P}_Y(X)$ such that

$$\mathcal{P}_V(U) \cup \mathcal{P}_K(L) = \mathcal{P}_Y(X)$$

with

$$\mathcal{P}_V(U) \cap \mathcal{P}_K(L) = \emptyset$$

and $F \not\equiv G$ for any $F \in \mathcal{P}_V(U)$ and $G \in \mathcal{P}_K(L)$. Otherwise, we say the problem space is inseparable. Similarly, we say a solution space $\mathcal{S}_Y(X)$ is separable if and only if there exist some $\mathcal{S}_V(U) \subset \mathcal{S}_Y(X)$ and $\mathcal{S}_K(L) \subset \mathcal{S}_Y(X)$ such that

$$\mathcal{S}_V(U) \cup \mathcal{S}_K(L) = \mathcal{S}_Y(X)$$

with

$$\mathcal{S}_V(U) \cap \mathcal{S}_K(L) = \emptyset$$

and $R \not\perp W$ for any $R \in \mathcal{S}_V(U)$ and $W \in \mathcal{S}_K(L)$. Otherwise, we say the solution space is inseparable.

We demonstrate that the notion of separability can be passed to and fro between problems and their corresponding solution spaces. The following result is a formalization of this important concept.

Theorem 3.2. *Let $\mathcal{P}_Y(X)$ be a problem space with the corresponding solution space $\mathcal{S}_Y(X)$. Then $\mathcal{P}_Y(X)$ is separable if and only if $\mathcal{S}_Y(X)$ is separable.*

Proof. Suppose $\mathcal{P}_Y(X)$ is separable, then there exist $\mathcal{P}_V(U) \subset \mathcal{P}_Y(X)$ and $\mathcal{P}_K(L) \subset \mathcal{P}_Y(X)$ such that
$$\mathcal{P}_V(U) \cup \mathcal{P}_K(L) = \mathcal{P}_Y(X)$$
with
$$\mathcal{P}_V(U) \cap \mathcal{P}_K(L) = \emptyset$$
and $F \not\equiv G$ for any $F \in \mathcal{P}_V(U)$ and $G \in \mathcal{P}_K(L)$. For any $F \in \mathcal{P}_V(U)$ there exists some $R \in \mathcal{S}_V(U)$ that solves F and some $W \in \mathcal{S}_K(L)$ that solves G. Since $\mathcal{P}_V(U) \cap \mathcal{P}_K(L) = \emptyset$ and problems in both spaces are not equivalent, it follows that $R \not\perp W$ and $R \notin \mathcal{S}_K(L)$ and $W \notin \mathcal{S}_V(U)$. Since R and W are arbitrary , it follows that $\mathcal{S}_Y(X)$ must also be separable. Suppose without loss of generality that R solves some problem in the space $\mathcal{P}_K(L)$. In particular, there exists some $T \in \mathcal{P}_K(L)$ that is solved by R. Since R also solves F and there exists some $W \in \mathcal{S}_K(L)$ that solves T, it must be that $W \perp R$, a contradiction. In the case, $R \perp W$ then we obtain $R \in \mathcal{S}_K(L)$ and $W \in \mathcal{S}_V(U)$ by virtue of Proposition 2.2. Without loss of generality, we examine the case $R \perp W$ and $R \in \mathcal{S}_K(L)$ with $W \notin \mathcal{S}_V(U)$ then $W \in \mathcal{S}_V(U)$ by virtue of Proposition 2.2. This is also a contradiction.
Conversely, suppose the solution space $\mathcal{S}_Y(X)$ is separable. Then there exist some $\mathcal{S}_V(U) \subset \mathcal{S}_Y(X)$ and $\mathcal{S}_K(L) \subset \mathcal{S}_Y(X)$ such that
$$\mathcal{S}_V(U) \cup \mathcal{S}_K(L) = \mathcal{S}_Y(X)$$
with
$$\mathcal{S}_V(U) \cap \mathcal{S}_K(L) = \emptyset$$
and $R \not\perp W$ for any $R \in \mathcal{S}_V(U)$ and $W \in \mathcal{S}_K(L)$. Clearly R solves some $G \in \mathcal{P}_V(U)$ and W solves some $T \in \mathcal{P}_K(L)$. We claim that $T \not\equiv G$ with
$$\mathcal{P}_V(U) \cup \mathcal{P}_K(L) = \mathcal{P}_Y(X)$$
with
$$\mathcal{P}_V(U) \cap \mathcal{P}_K(L) = \emptyset.$$
Suppose $T \equiv G$ for some $T \in \mathcal{P}_K(L)$ and $G \in \mathcal{P}_V(U)$, then $R \perp W$, a contradiction. Since
$$\mathcal{S}_V(U) \cup \mathcal{S}_K(L) = \mathcal{S}_Y(X)$$
with
$$\mathcal{S}_V(U) \cap \mathcal{S}_K(L) = \emptyset$$
it follows that
$$\mathcal{P}_V(U) \cup \mathcal{P}_K(L) = \mathcal{P}_Y(X).$$
Suppose to the contrary that
$$\mathcal{P}_V(U) \cup \mathcal{P}_K(L) \subset \mathcal{P}_Y(X)$$
then there exist a problem $A \in \mathcal{P}_Y(X)$ that has no solution in $\mathcal{S}_V(U) \cup \mathcal{S}_K(L)$ but has solution in $\mathcal{S}_Y(X)$. This assertion contradicts the equality
$$\mathcal{S}_V(U) \cup \mathcal{S}_K(L) = \mathcal{S}_Y(X).$$
We note that $\mathcal{S}_V(U) \cap \mathcal{S}_K(L) = \emptyset$ implies $\mathcal{P}_V(U) \cap \mathcal{P}_K(L) = \emptyset$. Suppose that $\mathcal{P}_V(U) \cap \mathcal{P}_K(L) = \not\emptyset$. Then there exists a problem $J \in \mathcal{P}_V(U) \cap \mathcal{P}_K(L)$ so that there exists some $N \in \mathcal{S}_V(U) \cap \mathcal{S}_K(L)$ that solves J. This completes the proof. $\square$

4. Quotient problem and solution spaces

In this section, we introduce and study the notion of the quotient problem and their corresponding solution spaces. We launch the following terminologies.

Definition 4.1. Let $\mathcal{P}_Y(X), \mathcal{P}_V(U)$ be problem spaces with
$$\mathcal{P}_V(U) \subset \mathcal{P}_Y(X).$$
Then we say the quotient space induced by $\mathcal{P}_V(U)$ in $\mathcal{P}_Y(X)$ regulated by a fixed $T \in \mathcal{P}_Y(X)$, denoted by $\mathcal{P}_Y(X)/_T\mathcal{P}_V(U)$, is the collection of problems
$$\mathcal{P}_Y(X)/_T\mathcal{P}_V(U) := \{T\} \cup \mathcal{P}_V(U).$$
If $\mathcal{P}_Y(X)/_T\mathcal{P}_V(U) := \{T\} \cup \mathcal{P}_V(U) = \mathcal{P}_Y(X)$ for some $T \in \mathcal{P}_Y(X)$ then we say $\mathcal{P}_V(U)$ is a principal subspace of the space $\mathcal{P}_Y(X)$. On the other hand, if $\mathcal{P}_Y(X)/_T\mathcal{P}_V(U) := \{T\} \cup \mathcal{P}_V(U) = \mathcal{P}_V(U)$ for all $T \in \mathcal{P}_Y(X)$ $(T \neq Y)$ then we say $\mathcal{P}_V(U)$ is an ideal sub-space of the problem space $\mathcal{P}_Y(X)$.

In the sequel we use the notion of regularity and maximality to find a subspace that is ideal and at the same time principal.

Proposition 4.2. *Let $\mathcal{P}_Y(X), \mathcal{P}_V(U)$ be problem spaces with $\mathcal{P}_V(U) \subset \mathcal{P}_Y(X)$. If Y is a regular problem and V is the maximal sub-problem of Y, then the sub-space $\mathcal{P}_V(U)$ is ideal and principal.*

Proof. Suppose $\mathcal{P}_V(U) \subset \mathcal{P}_Y(X)$ and assume that Y is a regular problem and V is the maximal sub-problem of Y. It follows for the sequence of all the sub-problems $\{J_i\}_{i\geq 1}$ of Y except V, we can write
$$\cdots J_n \leq \cdots \leq V \leq Y.$$
Since every problem in the space $\mathcal{P}_V(U)$ is a sub-problem of Y, it follows that for each $T \in \mathcal{P}_Y(X)$ except Y, we must have
$$\{T\} \cup \mathcal{P}_V(U) = \mathcal{P}_V(U)$$
and the space is ideal. Similarly, if we choose $T = Y$, then we have $\{T\} \cup \mathcal{P}_V(U) = \mathcal{P}_Y(X)$ and the space is a principal space. $\square$

5. Overlapping and non-overlapping problem and solution spaces

In this section we study the notion of overlapping and non-overlapping problem and solution spaces. We launch formally the following languages.

Definition 5.1. Let $\mathcal{P}_Y(X), \mathcal{P}_V(U)$ be problem spaces. Then we say they are overlapping if and only if
$$\mathcal{P}_Y(X) \cap \mathcal{P}_V(U) \neq \emptyset.$$
Otherwise, we say they are non-overlapping. The same characterization also holds for their corresponding solution spaces.

Proposition 5.2. *Let $\mathcal{P}_Y(X), \mathcal{P}_V(U)$ be problem spaces, with their corresponding solution spaces $\mathcal{S}_Y(X), \mathcal{S}_V(U)$ such that $F \not\equiv G$ for any $F \in \mathcal{P}_Y(X)$ and $G \in \mathcal{P}_V(U)$. Then the problem spaces are non-overlapping if and only if their corresponding solution spaces are non-overlapping.*

Proof. First suppose $\mathcal{P}_Y(X) \cap \mathcal{P}_V(U) \neq \emptyset$ then there exist some $T \in \mathcal{P}_Y(X) \cap \mathcal{P}_V(U)$. Since Y is a problem with solution X and V is a problem with solution U, it follows that T must also be a solved problem. That is, there exist some $K \in \mathcal{S}_Y(X)$ that solves T. Again, $T \in \mathcal{P}_V(U)$ so that there exist some $G \in \mathcal{S}_V(U)$ that solves T. It follows that G and K must be the same solution or $G \perp K$; that is, G and K are alternative solutions to T. Since solutions spaces remain invariant under replacement with alternative solutions, it follows in particular that we can replace $G \in \mathcal{S}_V(U)$ with K and the space $\mathcal{S}_V(U)$ still remains unchanged. Conversely, suppose $\mathcal{S}_Y(X) \cap \mathcal{S}_V(U) \neq \emptyset$. It follows that for each $F \in \mathcal{S}_Y(X) \cap \mathcal{S}_V(U)$ must be a solution to some problem $T \in \mathcal{P}_Y(X) \cap \mathcal{P}_V(U)$. □

6. Symmetric problem spaces

In this section we study the notion of symmetry existing among problem spaces. We launch the following languages.

Definition 6.1. Let $\mathcal{P}_Y(X), \mathcal{P}_V(U)$ be problem spaces. We say the problem spaces are symmetric if for each problem $T \in \mathcal{P}_Y(X)$ there exist a problem $L \in \mathcal{P}_V(U)$ such that $K \equiv L$. That is, problem K and problem L are equivalent. We denote the equivalence between the space $\mathcal{P}_Y(X)$ and $\mathcal{P}_V(U)$ as

$$\mathcal{P}_Y(X) \asymp \mathcal{P}_V(U).$$

We use the notion of symmetry to justify the assertion that the problems spaces endowed with equivalent problems have indistinguishable solution spaces. In fact, it has consequences that allows us to artificially build solution spaces that can be tweak without changing the structure.

Proposition 6.2. *Let $\mathcal{P}_Y(X)$ be a problem space with a corresponding solution space $\mathcal{S}_Y(X)$. If $\mathcal{P}_Y(X) \asymp \mathcal{P}_V(U)$, then*

$$\mathcal{S}_Y(X) = \mathcal{S}_V(U).$$

Proof. Suppose $\mathcal{P}_Y(X) \asymp \mathcal{P}_V(U)$, then for each problem $T \in \mathcal{P}_Y(X)$ there exists a problem $K \in \mathcal{P}_V(U)$ such that $K \equiv T$. Since $\mathcal{S}_Y(X)$ is the corresponding solution space for $\mathcal{P}_Y(X)$, there exists some $F \in \mathcal{S}_Y(X)$ that solves T. Since problem T and problem K are equivalent problems, it follows that F also solves $K \in \mathcal{P}_V(U)$. The claim follows by iterating the argument in this manner to build the solution space $\mathcal{S}_V(U)$. □

Proposition 6.3. *Let $\mathcal{S}_Y(X)$ and $\mathcal{S}_V(U)$ be solution spaces. If for each $K \in \mathcal{S}_Y(X)$ there exist some $L \in \mathcal{S}_V(U)$, then*

$$\mathcal{P}_Y(X) \asymp \mathcal{P}_V(U).$$

Proof. Let K and L be arbitrary with $K \in \mathcal{S}_Y(X)$ and $L \in \mathcal{S}_V(U)$. Then there exists a problem $T \in \mathcal{P}_Y(X)$ solved by K and a problem $F \in \mathcal{P}_V(U)$ solved by L. Since $\equiv$ is an equivalence relation and $K \perp L$, it follows that $T \equiv F$, since L also solves T and K also solves F. The claim follows by repeating the argument with solutions in the space. □

References

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Department of Mathematics, African Institute for mathematical sciences, Ghana.
E-mail address: Theophilus@aims.edu.gh/emperordagama@yahoo.com