The paradox and uncloseness of calculation theorem

----Tranclosed logic princiole and its inference(1)

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【Key words】 Paradox, unclosed terms ,extra-field term, diagonal method of proof, Tranclosed logic.

【Abstract】 For a long time, there is a "diagonal method of proof" dominating the mathematics field; with it, Russel finds the paradox of set theory; with it, Cantor proves that "the power set of natural numbers is uncountable" and " the set of real numbers is uncountable"; with it , Gödel proves that " natural number system ***PA*** is incomplete"; with it , Turning proves that " halting problem" is undecidable and proves that " there is non-recursive sets on sets of natural numbers" in recursion theory and so on; proofs of these significant propositions all apply the same mathematic method which is praised as " a golden diagonal".

On the basis of analyzing paradoxes, the paper finds that paradoxes are unclosed terms on closed calculus (that is extra-field term). Classical logic system cannot handle such extra-field terms, so it is transformed to the logic systems ***SL, SK*** that may handle unclosed calculus.

It can be found that "diagonal proof method" is to construct paradoxes in nature through further analysis, and it is an unclosed proof method, which can prove that real numbers constructed by Cantor's "diagonal proof method are extra-field terms which will not affect count-ability of sets of real numbers; The Gödel's undeterminable proposition is an extra-field term, which will not affect completeness of system ***PA***. The undeterminable Turing machine in the Turing halt problem is also an extra-field term.

So, the proof that real number is uncountable is wrong; the proof of Gödel's incomplete theorem and diagonal method of proof, all of them are wrong, should be completely corrected.

1. The Paradox is an unclosed term of logic calculus

2.Transformation of extra-term fields and classical logic system

1.The paradox is an unclosed term in logical calculus

Paradox is the self-reference operation of contradictory propositions, while the self-reference operation must produce unclosed terms. Paradoxes are the non-closed terms which can be defined as a new term, namely, an extra-field term.

1.1 Self-reference and paradoxes

Definition 1.1.1 Positive sets and inverse sets

Let property P be a binary partition for set $U=\{x_1, x_2, \cdots, x_i, \cdots\}$,satisfying

$+\alpha=\{x \mid P(x) \wedge x \in U\},-\alpha=\{x \mid \neg P(x) \wedge x \in U\},U=+\alpha \cup -\alpha$

(1) Set made up of member x satisfying property P is a positive set, that is proposition $P(x)$ is true and it noted as $+\alpha=\{x \mid P(x) \wedge x \in U\}$, while members in the positive set are positive terms;

(2) Set made up of member x dissatisfying property P is an inverse set, that is proposition $\neg P(x)$ is true and it is noted as $-\alpha=\{x \mid \neg P(x) \wedge x \in U\}$, while members in inverse sets are inverse terms.

Example 1.1.1 Positive sets and inverse sets on integers

Let $U=\{\cdots-2,-1,0,1,2,\cdots\}$, that is a set of all integers; assume that $P(x): x$ is an even number, then $P(x)$ is a binary partition to U.

Positive set: the set of even numbers, that is $+\alpha=\{x \mid x=2n, n \in J\}$;

Inverse set: the set of odd numbers, that is $-\alpha=\{x \mid x=1-2n, n \in J\}$;

Definition 1.1.2 Mapping on positive sets and inverse sets

Let $U=+\alpha \cup -\alpha$, f is the mapping from positive set $+\alpha$ to inverse set $-\alpha$, $y=f(x)$

$x \in +\alpha \rightarrow f(x) \in -\alpha$, note as $f: +\alpha \rightarrow -\alpha$, if $f: +\alpha \rightarrow -\alpha$ is bijective, note as $f: +\alpha \sim -\alpha$

Example 1.1.2 Bijective functions on positive sets and inverse sets

Let $U=\{\cdots-2,-1,0,1,2,\cdots\}$, that is set of all integers, assume that $P(x)$: x is an even number, then $P(x)$ is a binary partition to U.

$$+\alpha=\{x \mid x=2n, n \in J\},-\alpha=\{x \mid x=1-2n, n \in J\};$$

Bijective function of positive sets and inverse sets is $f(x)=1-x$

$$f:+\alpha \sim -\alpha , x \in +\alpha \leftrightarrow f(x) \in -\alpha ,$$

Theorem 1.1.1 Duality transformation theorem for positive sets and inverse sets

Let universal set $U=\{x_1,x_2,\cdots,x_i,\cdots\}$ is a defined set, U can be divided into positive and inverse symmetric sets $U=+\alpha\bigcup-\alpha$, $+\alpha=\{x\,|\,P(x)\}$; $-\alpha=\{x\,|\,\neg P(x)\}$; if property P is a binary partition on U; for any an x, x satisfies P, $f(x)$ satisfies property $\neg P$, bijective relation on positive sets and inverse sets $f:+\alpha\sim-\alpha$. $x\in+\alpha\leftrightarrow f(x)\in-\alpha$, then for $x\in U$, there is $P(x)\leftrightarrow\neg P(f(x))$.

Prove:

（1）$\vdash x\in+\alpha\leftrightarrow f(x)\in-\alpha$ ------------------------------- $f:+\alpha\sim-\alpha$ is bijective.

（2）$\vdash x\in+\alpha\leftrightarrow P(x)$， $f(x)\in-\alpha\leftrightarrow\neg P[f(x)]$-------------------------by definition,

（3）$\vdash [P(x)\leftrightarrow\neg P(f(x))]$---(1)(2)

（4）$\vdash$ $[x\in+\alpha\leftrightarrow f(x)\in-\alpha]\leftrightarrow[P(x)\leftrightarrow\neg P(f(x))]$----------------(1)(3).

Example 1.1.3 Duality transformation for positive sets and inverse sets

Let $U=\{\cdots-2,-1,0,1,2,\cdots\}$, that is set of all integers, assume that $P(x)$: x is an even number, then $P(x)$ is a binary partition to U.

$$+\alpha=\{x\,|\,x=2n,n\in J\},-\alpha=\{x\,|\,x=1-2n,n\in J\};$$

The bijective function for positive sets and inverse sets is $f(x)=1-x$,

$$f:+\alpha\sim-\alpha, x\in+\alpha\leftrightarrow f(x)\in-\alpha, P(x)\leftrightarrow\neg P(f(x))$$

Definition 1.1.1 Self-reference equation

Generally, function $y=f(x)$, $x\in R$, if substitute y with x, function equation $x=f(x)$ is obtained, then $x=f(x)$ is called as a self-reference equation of $y=f(x)$.

Example 1.1.4 Self-reference equations of functions

Let $U=\{\cdots-2,-1,0,1,2,\cdots\}$, that is set of all integers, assume that $P(x)$: x is an even number, then $P(x)$ is a binary partition for U.

Bijective function for the positive set and the inverse set is $f(x)=1-x$,

$$f:+\alpha\sim-\alpha, x\in+\alpha\leftrightarrow f(x)\in-\alpha,$$

$$x=f(x), x=1-x \text{ is self-inference equation.}$$

Theorem 1.1.2 Normal form of paradoxes

Let $U=\{x_1,x_2,\cdots,x_i,\cdots\}$ be a defined set, U can be divided into positive and inverse symmetric sets $U=+\alpha\cup-\alpha$, $+\alpha=\{x\mid P(x)\}$; $-\alpha=\{x\mid\neg P(x)\}$; if property P is a binary partition on U; for any an x, x satisfies P, $f(x)$ satisfies property $\neg P$, bijective relation on positive sets and inverse sets $f:+\alpha\sim-\alpha$.

When $x=f(x)=x_P$, namely when x_P is a fixed term, $P(x_P)\leftrightarrow\neg P(x_P)$ represents as paradox.

Prove: according to "duality transformation theorem for positive sets and inverse sets"

(1)$\vdash$ $[P(x)\leftrightarrow\neg P(f(x))]$------------------- $x\in+\alpha\leftrightarrow f(x)\in-\alpha$,

(2)$\vdash$ $P(x_P)\leftrightarrow\neg P(x_P)$--------------------------- （1）, $x=f(x)=x_P$.

In "Duality transformation theorem for positive sets and inverse sets"

$P(x)\leftrightarrow\neg P(f(x))$, $x\in+\alpha$, $f(x)\in-\alpha$;

it is not contradictory, however, when $x=f(x)=x_P$, namely when x_P is a fixed point, there is $P(x_P)\leftrightarrow\neg P(x_P)$,

Assume that $P(x_P)\leftrightarrow A$, then $A\leftrightarrow\neg A$, it is a paradox.

Example 1.1.5 Paradox form on sets of integers

Let $U=\{\cdots-2,-1,0,1,2,\cdots\}$, that is set of all integers, assume that $P(x)$: x is an even number, then $P(x)$ is a binary partition to U.

$$+\alpha=\{x\mid x=2n,n\in J\},-\alpha=\{x\mid x=1-2n,n\in J\};$$

Bijective function for the positive set and the inverse set is $f(x)=1-x$,

$$f:+\alpha\sim-\alpha, x\in+\alpha\leftrightarrow f(x)\in-\alpha, P(x)\leftrightarrow\neg P(f(x))$$

When $x=f(x), x=1-x, x=x_P$,

Since $x_P=f(x_P)$, similar paradox proposition $P(x_P)\leftrightarrow\neg P(x_P)$ is formed.

1.2 The paradox is a logical unclosed calculus

Definition 1.2.1 uncloseness of calculation

Establish $U=\{x_1,x_2,\cdots,x_i,\cdots\}$ is the domain of definition for one kind of unary operation or multiple operation $\odot$,

If $\forall a\in U, \forall b\in U\Rightarrow a\odot b\in U$, then U is closed for calculation $\odot$.

If $\exists a\in U, \exists b\in U\Rightarrow a\odot b\notin U$, then U is unclosed for calculation $\odot$.

Example 1.2.1 uncloseness of calculation

$N=\{0,1,2,\cdots,n,\cdots\}$

$$\forall a\in N, \forall b\in N\Rightarrow a+b\in N$$
$$\forall a\in N, \forall b\in N\Rightarrow a\times b\in N$$

So, N is closed for additive operation and multiplication calculation.

$$2\in N, 7\in N\Rightarrow 2-7\notin N,$$
$$2\in N, 7\in N\Rightarrow 2\div 7\notin N;$$

So, N is not closed for subtraction and division operation .

Example 1.2.2 uncloseness of calculation

Q is a set of rational number

$$\forall a\in Q, \forall b\in Q\Rightarrow a-b\in Q,$$

$$\forall a\in Q, \forall b\in Q\Rightarrow a\div b\in Q;$$

So, Q is closed for subtraction and division calculation.

$$2\in Q\Rightarrow\sqrt{2}\notin Q,$$

$$-3\in Q\Rightarrow\sqrt{-3}\notin Q;$$

So, Q is not closed for extration of square root .

Definition 1.2.2 Extra-field term

Let $U=\{x_1,x_2,\cdots,x_i,\cdots\}$ be a set, mapping $f:U\to U$, satisfying solution x_0 of equation $x=f(x)$. If element $x_0\notin U$, then element x_0 is called as extra-field term. Extra-field term is unclosed term for calculus essentially.

Theorem 1.2.1 Self-reference unclosed theorem on positive sets and inverse sets

Let universal set $U=\{x_1,x_2,\cdots,x_i,\cdots\}$ be a defined set and U can be divided into positive and inverse symmetric sets $U=+\alpha\bigcup-\alpha$,$+\alpha=\{x\mid P(x)\}$; $-\alpha=\{x\mid\neg P(x)\}$; if property P is a binary division on U, $f:+\alpha\sim-\alpha$ is an bijective function on positive sets and inverse sets; for any x satisfying property P, $f(x)$ satisfies property $\neg P$, then,

(A) $x\in U\vdash P(x)\leftrightarrow\neg P(f(x))$;

(B) When $x=f(x)=x_P$, $P(x_P)\leftrightarrow\neg P(x_P)$ is a paradox;

(C) If calculus on U is consistent, then term x_P is an extra-field term, that is $x_P\notin U$.

Prove:

(1) $x\in U\vdash x\in+\alpha\leftrightarrow f(x)\in-\alpha$ ------------------------- $f:+\alpha\to-\alpha$ is a bijective function,

(2)$\vdash x\in+\alpha\leftrightarrow P(x)$, $f(x)\in-\alpha\leftrightarrow\neg P[f(x)]$------------------------------by definition,

(3) $x\in U\vdash[P(x)\leftrightarrow\neg P(f(x))]$ ---(1)(2),

(4) $x_P\in U\vdash[P(x_P)\leftrightarrow\neg P(f(x_P))]$ --------------------------- x_P is substituted into (3)

(5) $x_P\in U\vdash P(x_P)\leftrightarrow\neg P(x_P)$ --- $x_P=f(x_P)$, (4),

(6) $\neg(x_P\in U)$ -------------------------(5) proof by contradiction, which is contradictory with U is consisent.

Above (3), (5) , (6) prove foregoing theorems (A), (B) , (C)

Namely $x_P\notin U$, which indicates that self-reference calculus on positive sets and inverse sets are unclosed.

Example 1.2.3 Unclosed calculus and the paradox on integer sets

Let $U=\{\cdots-2,-1,0,1,2,\cdots\}$, namely a set of all integers, let $P(x)$: x be an even number, then $P(x)$ is a binary partition for U.

$$+\alpha=\{x\mid x=2n,n\in U\};-\alpha=\{x\mid x=1-2n,n\in U\};$$

The bijective function of the positive set and the inverse set is

$$f(x)=1-x,x\in+\alpha\leftrightarrow f(x)\in-\alpha,$$

Dual transformation for the positive set and inverse set proposition,

$$P(x)\leftrightarrow\neg P(f(x)),$$

When $x=f(x),x=1-x,x=x_P$,

Because $x_P=f(x_P)$, similar paradox proposition $P(x_P)\leftrightarrow\neg P(x_P)$ is formed.

The paradox can be popularly expressed as

If: " x_P is an even number"$\Rightarrow$"$1-x_P$ is an odd number "$\Rightarrow x_P=1-x_P$, " x_P is an odd number";

If: " x_P is an odd number"$\Rightarrow$"$1-x_P$ is an even number"$\Rightarrow x_P=1-x_P$, " x_P is an even number";

That is the paradox: " x_P is an odd number" $\Leftrightarrow$ " x_P is an even number",

We have known that: $x_P=1-x_P, x_P=\frac{1}{2},\frac{1}{2}\notin U,\frac{1}{2}$ is an unclosed term on set of integers (extra-field term).

Example 1.2.4 Unclosed calculus and the paradox on the set of rational numbers

Let $U=Q^{+}$ is a set of all positive rational numbers, given a partition $P(x)$: $x^2>2$.

$$+\alpha=\{x\mid x^2>2,x\in Q^{+}\};\ -\alpha=\{x\mid x^2<2,x\in Q^{+}\}.$$

The bijective function of the positive set and the inverse set is

$$f(x)=\frac{2}{x},\ \ x\in+\alpha\leftrightarrow f(x)\in-\alpha,$$

Dual transformation for the positive set and inverse set proposition,

$$P(x) \leftrightarrow \neg P(f(x)),$$

When $x = f(x)$. $x = \frac{2}{x}$, $x = x_P$,

Because $x_P = f(x_P)$, similar paradox proposition $P(x_P) \leftrightarrow \neg P(x_P)$ is formed.

The paradox can be popularly expressed as:

If: "$x^2 > 2$"$\Rightarrow$"$x^2 = \frac{4}{x^2}$"$\Rightarrow \frac{4}{x^2} > 2, \Rightarrow$ "$x^2 < 2$";

If: "$x^2 < 2$"$\Rightarrow$"$x^2 = \frac{4}{x^2}$"$\Rightarrow \frac{4}{x^2} < 2, \Rightarrow$ "$x^2 > 2$";

That is paradox: "$x^2 < 2$"$\Leftrightarrow$"$x^2 > 2$"

We have known that: $x_P = \frac{2}{x_P}, x_P = \sqrt{2}, \sqrt{2} \notin U$, $\sqrt{2}$ is a unclosed term on the set of rational numbers (extra-field term).

Example 1.2.5 Unclosed calculus and the paradox on the set of real numbers

Let $U = (-\infty, 0) \cup (0, +\infty)$ is a set of all real numbers that are not 0, given a partition $P(x)$: $x > 0$.

$+\alpha = \{x \mid x > 0, x \in U\} = (0, +\infty)$; $-\alpha = \{x \mid x < 0, x \in U\} = (-\infty, 0)$.

Bijective function for the positive set and inverse set is $f(x) = -\frac{1}{x}, x \in +\alpha \leftrightarrow f(x) \in -\alpha$

Dual transformation for the positive set and inverse set proposition,

$$P(x) \leftrightarrow \neg P(f(x))$$

When $x = f(x), x = -\frac{1}{x}$, $x = x_P$,

Because $x_P = f(x_P)$, then similar paradox proposition $P(x_P) \leftrightarrow \neg P(x_P)$ is formed.

The paradox can be popularly expressed as:

If: "$x > 0$"$\Rightarrow$"$x = -\frac{1}{x}$"$\Rightarrow -\frac{1}{x} > 0$, $\Rightarrow$ "$x < 0$";

If: "$x < 0$"$\Rightarrow$"$x = -\frac{1}{x}$"$\Rightarrow -\frac{1}{x} < 0, \Rightarrow$ "$x > 0$";

That is paradox: "$x < 0$" $\Leftrightarrow$"$x > 0$"

We have known that: $x = -\frac{1}{x}, x = \sqrt{-1} = i, i \notin U$, i is an unclosed term on set of real numbers (extra-field term).

The more general case of theorem **1.2.1** is:

Theorem 1.2.2 Unclosed theorem of self-reference operation on the positive and inverse set

Let universal set $U=\{x_1,x_2,\cdots,x_i,\cdots\}$ be a defined set, if property P is a binary partition on U, U can be divided into a positive set and an inverse set $U=+\alpha\cup-\alpha$, $+\alpha=\{x\mid P(x)\}$;

$-\alpha=\{x\mid\neg P(x)\}$, g is arbitrary operation on U **,** if $T\in+\alpha\leftrightarrow g(T)\in-\alpha$, then

(1) When $T=g(T)$, $P(T)\leftrightarrow\neg P(T)$ is a paradox;

(2) If the above calculations on U are consistent, then term T is an extra-field, that is $T\notin U$.

Definition 1.2.3 Undefined term

Let universal set $U=\{x_1,x_2,\cdots,x_i,\cdots\}$ be a defined set, while fixed term x_P does not belong to positive set $+\alpha$ nor belong to the inverse set $-\alpha$, that is fixed term x_P does not belong to the defined set, $x_P\notin U$; we name a set for the fixed terms solely as the undefined set relative to U, that is fixed term $e=\{x_P\}$; the extra-field x_P is called as the undefined term relative to U; if x_P is an undefined term, P is a predicate on U, then $P(x_P)$ is called as an undefined proposition.

Example 1.2.6 Inference form mistake in Russell's paradox

One category of set is element of itself, that is $x\in x$, $+\alpha=\{x\mid x\in x\}$;

The other category of set is not element of itself, that is $\neg(x\in x)$, $-\alpha=\{x\mid\neg(x\in x)\}$;

Now construct the set made up of all the 2nd category of sets (that is $-\alpha$), denote with $R=\{x\mid\neg(x\in x)\}$,

That is $x\in R\leftrightarrow\neg(x\in x)$, question that set R is what's kind of set? Namely self-refer with R.

Assume that $P(x)$ denotes proposition $x \in x$, $+\alpha = \{x \mid P(x)\}$, then $\neg P(x)$ denotes proposition $\neg(x \in x)$, $-\alpha = \{x \mid \neg P(x)\}$,$U = +\alpha \cup -\alpha$, $R = \{x \mid \neg P(x)\}$,

False inference form:

$$\vdash x \in R \leftrightarrow \neg(x \in x),$$

$$\vdash R \in R \leftrightarrow \neg(R \in R),$$

$$\vdash P(R) \leftrightarrow \neg P(R),$$

Above false inference form leads to paradox.

Right inference form is:

$$x \in U \vdash \vdash x \in R \leftrightarrow \neg P(x),$$

$$R \in U \vdash R \in R \leftrightarrow \neg P(R),$$

$$R \in U \vdash P(R) \leftrightarrow \neg P(R),$$

$$\vdash R \notin U .$$

Therefore $R = \{x : x \notin x\}$, $R \notin U$, that is: Russell's paradox is an extra-field term.

Right inference form will not lead to paradox.

Example 1.2.7 Inference form mistake in axiom of comprehension

Axiom of comprehension defines a set with property $A(x)$ of a member, $x \in I \leftrightarrow A(x)$, $I = \{x \mid A(x)\}$.

False inference form:

$$\vdash x \in I \leftrightarrow A(x),$$

Right inference form:

$$x \in U \vdash x \in I \leftrightarrow A(x).$$

2 Transformation of extra-field term and classical logic system

Classical logic system is a calculus system on unclosed domain *U* either true or false, which cannot accommodate paradoxes. When paradox proposition is added to the classical logic system, its calculus inside *U* is consistent, and its calculus outside *U* is inconsistent. This is an open logic system and the paradox is an either true or false extra-field proposition. We extend

the classical predicate calculus system *K* to a system *SK* containing a logic unclosed term calculus, and an extra-field term is a third value either true or false.

2.1 Predicate calculus system *SK*

Let $U=\{x_1,x_2,x_3,\cdots,x_n,\cdots\}$ be a set of terms, P be a property on U deviding U into a positive set and an inverse set, $U=+\alpha\bigcup-\alpha$, $+\alpha=\{x\mid P(x)\}$, $-\alpha=\{x\mid\neg P(x)\}$. As to a bijective function $f:+\alpha\to-\alpha$ on U, $P(x)\leftrightarrow\neg P(f(x))$ is true in predicate calculus system *K*.

$\vdash P(x)\leftrightarrow\neg P(f(x))$-- (1)

Inference form of (1) is false obviously, which should be modified as following (2), which is true inference form.

$x\in U\vdash P(x)\leftrightarrow\neg P(f(x))$-- (2)

There should be restriction conditions $x\in U$ for all axioms in predicate calculus system *K*.

We introduce axiom $x\in U\vdash P(x)\leftrightarrow\neg P(f(x))$" on classical predicate system K and construct a new predicate logic system-**predicate system *SK*.**

1. Basic symbol

Term: $U=\{x_1,x_2,x_3,\cdots,x_n,\cdots\}$. P be a property on U deviding U into positive sets and inverse sets, function $f:+\alpha\to-\alpha$ is a bijection on U, $U=+\alpha\bigcup-\alpha$,

set of positive terms $+\alpha=\{x\mid P(x)\}$, $+\alpha=\{x_1,x_2,\cdots,x_i,\cdots\}$

set of fixed terms $e=\{x_P\}$,

set of inverse terms $-\alpha=\{x\mid\neg P(x)\}$, $-\alpha=\{f(x_1),f(x_2),\cdots,f(x_i),\cdots\}$;

Function: $f_1,\ f_2,\ f_3,\cdots,\ f_n$ is unary function, binary function,…n-ary function respectively.

Predicate: $A_1,B_1,C_1......$, $A_2,B_2,C_2,......$, $A_n,B_n,C_n,......$ is unary function, binary function,…n-ary function respectively.

Connective: $\neg,\to,\forall$, bracket (,);

Formula: $P_i^{\alpha}\ (i=1,2,3,\cdots)$ formula, if A,B are formulas, then $A\to B,\neg A,\forall(x_i)A$

are formulas;

If $x = f(x) = x_P$, then $P(x_P)$ is an extra-field proposition, $P(x_P)$ has no definition on U.

2. Definition

$A \vee B = def\ \neg A \to B$; $A \wedge B = def\ \neg(\neg A \to B)$;

$A \leftrightarrow B = def\ (A \to B) \wedge (B \to A)$; $\exists(x_i)A = def\ \neg\forall(x_i)\neg A$.

3. Axiom

***SK*1** $x_i \in U \ \vdash A \to (B \to A)$;

***SK*2** $x_i \in U \ \vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$;

***SK*3** $x_i \in U \ \vdash (\neg A \to \neg B) \to (B \to A)$;

***SK*4** $x_i \in U \ \vdash \forall(x_i)A \to A$;

***SK*5** $x_i \in U \ \vdash \forall(x_i)A(x_i) \to A(t)$;

***SK*6** $x_i \in U \ \vdash \forall(x_i)(A \to B) \to (A \to \forall(x_i)B)$;

***SK*0** $x_i \in U \vdash P(x_i) \leftrightarrow \neg P(f(x_i))$;

4. Inference rule inside the system

$R1$**:** Rule of detachment: if $x_i \in U \vdash A$ and $x_i \in U \vdash A \to B$, then $x_i \in U \vdash B$;

$R2$**:** Comprehension principle: if $x_i \in U \vdash A$, then $x_i \in U \vdash \forall x_i A$

Definition 2.1.1 System *SK*

We call the axiom system composed of part 1, part 2, part 3 and part 4 as system ***SK***.

Theorem 2.1.1 Validity of classical logic theorem on the closed field

In system ***SK***, proposition of same sets $+\alpha$ and $-\alpha$, all classical logic theorems and calculus modes are valid.

Theorem 2.1.2 Paradox theorem

In system ***SK***, when $x_i = f(x_i) = x_P$, $\neg(x_P \in U)$, that is x_P is an extra-field term.

Proof: P is a symmetrical partition on $+\alpha$ and $-\alpha$, according to the duality transformation axiom for the positive set and the inverse set,

$$x_i \in U \vdash \vdash P(x_i) \leftrightarrow \neg P(f(x_i));$$

When $x_i = f(x_i) = x_P$

$$x_P \in U \vdash P(x_P) \leftrightarrow \neg P(x_P),$$

$$\vdash \neg(x_P \in U).$$

That is x_P is an extra-field term, this is a paradox. The paradox term is the extra-field term and the paradox proposition is the extra-field proposition.

Below we will give accurate definition of "***SK***" semantic model:

Definition 2.1.2 *SK* model

A "***SK***" model is an order two-tuples (D,V), D is called as a term set, V is the assignment with D as term set, it is the function satisfying following two conditions:

1. There is $V(x) \in D$ for every term x in the system;

2. For a n-ary predicate A_n (n =1,2…..) in the system, there is $V(A_n) \subseteq D^n$, namely $V(A_n)$ is a subset of D^n, it is a n-ary relation on D.

Definition 2.1.3 i-equivalence

(D,V) and (D,V') are two "***SK***" models with same term set, if system assignments V and V' satisfy following two conditions, it is called that V and V' are i-equivalent (i is a natural number).

1. For every predicate A_n in system, there is $V(A_n) = V'(A_n)$;

2. When $t \neq x_i$, there is $V(t) = V'(t)$, at most, there is $V(x_i) \neq V'(x_i)$.

Definition 2.1.4 *SK* Assignment

System "***SK***" model (D,V), the assignment V satisfies the recursion definition of formula A of "***SK***".

If V satisfies formula A, denote as $V(A)=1$, if V does not satisfy formula A, denote as $V(A)=0$.

1. If A is the original sub-formula P_i^{α}, $V(P_i^{\alpha})=0$ or $V(P_i^{\alpha})=1$;

2. If A is formula $B \to C$, $V(B \to C)=1$, if and only if $V(B)=0$ or $V(C)=1$, $V(B \to C)=0$ if and only if $V(B)=1$ and $V(C)=0$;

3. If A is formula $B \wedge C$, $V(B \wedge C)=1$; if and only if $V(B)=1$ and $V(C)=1$, $V(B \wedge C)=0$ if and only if $V(B)=0$ or $V(C)=0$;

4. If A is formula $B \vee C$, $V(B \vee C)=1$, if and only if $V(B)=1$ or $V(C)=1$, $V(B \vee C)=0$ if and only if $V(B)=0$ and $V(C)=0$;

5. If A is formula $\neg B$, $V(\neg B)=1$ if and only if $V(B)=0$, $V(\neg B)=0$ if and only if $V(B)=1$;

6. If A is formula $\forall(x_i)B$, $V(\forall(x_i)B)=1$ if and only if there is a V' which is i equivalent with V, make $V'(B)=1$; $V(\forall(x_i)B)=0$ if and only if there is a V' which is i equivalent with V, make $V'(B)=0$.

7. If A is formula $P(x)$ or $P(f(x))$,

$V(P(x))=1$ if and only if $V(P(f(x)))=0$, $V(P(x))=0$ if and only if $V(P(f(x)))=1$.

8. If $x=f(x)=x_p$, $P(x_p)$ is an extra-field term proposition and undefined term, $V(P(x_p))=i$, undetermined true or false.

Theorem 2.1.3 *SK* equivalent system

In system ***SK***, all calculus formulas excluding extra-field term $P(x_p)$ can be translated to formulas in classical predicate calculus system, that is: system ***SK*** excluding extra-field term $P(x_p)$ is equivalent with classical predicate calculus system K.

Proof:

$U=+\alpha \cup -\alpha=\{x_1,x_2,\cdots,x_n,\cdots\}$, fixed term x_P does not take part in predicate calculus, system ***SK*** can be regarded as system K.

So system ***SK*** is equivalent with classical predicate calculus system K.

The consistency, undecidability and completeness these metatheorems of system K are valid in system ***SK***;

Proposition $P(x_P)$ is an undefined proposition in system ***SK***;

In predicate calculus, if $x_i = f(x_i) = x_P$, then $P(x_P)$ is an extra-field proposition; x_P and $P(x_P)$ have no definition on U.

Theorem 2.1.6 $P(x_P)$ **is an undecidable proposition**

Proposition $P(x_P)$ is an undecidable proposition in system ***SK*** (the proof is omitted, because that paradox proposition is undecidable.)

2.2 Proposition calculus system *SL*

The proposition calculus system of extra-field terms can be easily established.

Set $U = \{X_1, X_2, \cdots, X_i, \cdots\}$, the following is the logic system ***SL*** including extra-field proposition calculus.

1.Definition

$A \vee B = def \neg A \to B$; $A \wedge B = def \neg(\neg A \to B)$;

$A \leftrightarrow B = def (A \to B) \wedge (B \to A)$.

2. Axiom

SL **1** $X \in U \vdash A \to (B \to A)$;

SL **2** $X \in U \vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$;

SL **3** $X \in U \vdash (\neg A \to \neg B) \to (B \to A)$;

SL **0** $X \in U \vdash P^{+\alpha} \leftrightarrow \neg P^{-\alpha}$;

.3. Deduction rule in system

$R1$**:** Rule of detachment: if $X \in U \vdash A$ and $X \in U \vdash A \to B$, then $X \in U \vdash B$;

Theorem 2.2.1 Paradox theorem

In system ***SL***, when $X \leftrightarrow \neg X$, if system ***SL*** is consistent, $\neg(X \in U)$, that is : X is an extra-field.

Proof: According to ***SL*** **0 axiom,**

$$X \in U \vdash P^{+\alpha} \leftrightarrow \neg P^{-\alpha};$$

When $X \leftrightarrow P^{+\alpha} \leftrightarrow P^{-\alpha}$, is making self-reference,

$$X \in U \vdash X \leftrightarrow \neg X,$$

$$\vdash \neg(X \in U).$$

Namely, X is an extra-field term, that is the paradox.

Un-closed terms and extra-field terms can be interpreted as the third value i other than logic true and logic false;

¬	
T	F
F	T
i	i

→	T	F	i
T	T	F	i
F	T	T	i
i	i	i	i

Other truth tables of $\wedge, \vee, \leftrightarrow$, can be derived from definitions,

$A \vee B = def\ \neg A \to B$;

$A \wedge B = def \neg(\neg A \to B)$;

$A \leftrightarrow B = def\,(A \to B) \wedge (B \to A)$.

∨	T	F	i
T	T	F	i
F	T	T	i
i	i	i	i

∧	T	F	i
T	T	F	i
F	T	T	i
i	i	i	i

↔	T	F	i
T	T	F	i
F	T	T	i
i	i	i	i

According to the above truth table, the logic metatheorem can be proved, and the classical logic metatheorem is still true and would not change.

Generally:

(1) In Logical system ***SL*** and ***SK***, when $X \in U$, $x_i \in U$ are true propositions, that is: when $V(X \in U) = 1, V(x_i \in U) = 1$, $X \in U$, $x_i \in U$ can be omitted, "logic system ***SL***, ***SK***" turns into classical logic system L and system K";

(2) "Classical logic system L, system K" are "logic system ***SL*** and ***SK***," when $X \in U$,

$x_i \in U$ is the special case of true proposition.

Classical logic system calculus cannot be true unconditionally but must be true in the closed definition domain U and classical logic system cannot deal with un-closed terms, when there is unclosed calculus, a paradox is generated. Therefore, the classical logic system must be modified. The significance of "logic system S" is to transform the classical logic closed system into an open logic system in which the paradox is an unclosed term of logic calculus.

The paradox is an either true or false undecidable proposition in logical calculus. If classical logic system is regarded as “either true or false” logic, fuzzy logic is regarded as “both true and false” logic, and then “logic system S” is a “neither true nor false” logic system other than “classical logic “and “fuzzy logic”.

Inference of system SK(I) –terms constructed by “diagonal proof of method” are unclosed terms. Nature of a proposition constructed by diagonal method of proof is a paradox, so it is an unclosed term and an extra-field proposition. There are numbers in addition to real numbers, while the diagonal numbers of all real numbers are hyperreal numbers other than real numbers and the set of real numbers is not uncountable, so Cantor’s diagonal method of proof is false.

Inference of system SK(II)- Gödel’s un-decidable propositions are unclosed terms .Gödel's undeterminable proposition U (m) is also an unclosed term, which is a paradox in nature and an extra-field term. The extra-field term U (m) is an undeterminable proposition, which is independent of the determinability of the propositions in the natural number system PA, and will not affect the completeness of the system PA.

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