δ-small Subobjects and Upwards Closure in ∞-Categories

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Abstract

This paper develops the notion of δ-smallness as proposed by Barwick and Haine. This allows the author to investigate pointlike topological spaces from a category-theoretic perspective, by considering manifolds of negative dimension as cardinally inaccessible k-subobjects

Paper

Let σ denote a strongly inaccessible cardinal, and S the interval [0,1], here homeomorphic to any one-dimensional curve of infinite length. Classically, this would represent the reals ($\mathcal{R}^1$), although technically one may extend this construction to a special subset of $\mathbf{C}\mathscr{P}^1$ consisting of all complex numbers with identical argument to one another. Denote by ε the minimum possible measurable value of a compact segment in S, and $S\pm\varepsilon = |\sigma_1|$ the smallest strongly inaccessible cardinal. We can identify σ_0 with S, and $S\backslash|\sigma_k| = \{\varnothing\}$ for all values k>0. Following Barwick and Haine [PO], with slight modification, we will call an object of $|\sigma_0|$ "small" and an object of $|\sigma_1|$ "tiny." In general, an object $\sigma_\delta \in \sigma_k$ will be called δ-small, a subobject (δ+1)-small, and k+1 will always be equal to δ. Initially we are interested only in the case of S-ε; later, we will examine the corollary scenario, and extrapolate its results to dramatic effect.

A. Foundation Axiom

Set theoretically, in some sense, sub-point compact sets are forbidden by the *foundation axiom* of ZFC, because they are infinitely recursive; however, non-well-founded set theory provides a capable toolset for describing varieties of functions (streams), and by extension, graphs and their set-order analogues, which satisfy exactly the desirable sorts of forbidden structures which we aim to make use of.

The axiom of foundation (FA) states that for every set **A** containing as members {a, B}, where B is again a set containing some members {b, c … z}, that subsets of **A** (more precisely, subsets of the *powerset* of A) may be decomposed only into the elements A contains.

Therefore, sets such as A = {A} are strictly forbidden, and inductively, all recursive sets of this kind must be forbidden as well, and further, infinite sets are ruled out by virtue of the fact that subsets of A must be finite in cardinality, and therefore the terminal element is always the null set, which by definition excludes all sets inheriting any "interesting" fundamental structure from A. This is because FA requires that sets are constituted only by "substantive" elements, i.e., for a set A = {∅, {∅}, a, b, c}, by the requirement that each element be uniquely identifiable, a subset {a, b} may not be reduced to a set {∅, {∅}}.

Correspondingly, FA also prohibits atlases which contain an infinite number of charts on topological spaces, and demands that if *p* is the set of charts on an atlas $\mathcal{A}$, that if $\mathcal{A}$ be empty, p must be null. So, if the dimension of a manifold is taken to be the number of linearly independent curves along which differentiation may be performed[1], then the atlas at or on a point would be entirely empty. By reverting to our previous assignment of the interval *S*, one could identify any element $(\lambda \geq 1) \notin S$ with some strongly inaccessible cardinal, δ, and obtain easily $\delta \cap S = \{S\} \vee \{\emptyset\}$. The result is S when δ is taken to be a superset of S, and ∅ if it is defined strictly to lie outside the interval S. We will assume the latter of the two in this writing, namely because to define δ as a superset of S is to mistake the *universe* $\mathbf{V}^*$ in which δ is valued in for the value of δ itself. The reverse of this assignment is also valid; write δ^{-1} for some $\lambda \leq 1$, and we have $\delta^{\pm k} \stackrel{\text{def}}{=} (\lambda \gtreqless S)$ representing the field of universes which are inaccessible from S, which we will call $\mathbf{V}^*_{\underline{\omega}}$.

B. Inaccessibility

We define our notion of inaccessibility as follows: the infimum (resp. supremum) of a closed interval is the minimal (resp. maximal) element of a set *S*, i.e., the elements which are excluded when S is open. Inductively, a *weakly inaccessible cardinal* is an endpoint which is included only under the closure of an open set S′. A *strongly inaccessible cardinal*, then, becomes an element λ which is excluded from some universe $\mathbf{V}^*$, where $\mathbf{V}^*$ is a proper superset of $\mathbf{V}$, the universe of S and S′. We have that $(\lambda \cap S) \notin S'$, and for every object {∅} in S′, there is a closure $\{a, b\} \in \lambda \subset S$. For some λ which is not an element in S, or in other words is not k-valued in

[1]This definition is lifted from the vector spaces of linear algebra; however, topological spaces belong to a different category than vector spaces, and so the conventions are modified somewhat in order to accommodate these differences.

zero, we write δ for the smallest such possible item, and we call it "tiny," or 2-small.[2] Canonically, this is a subobject of S.

When S is taken to be the category of topological manifolds, objects of S can be viewed as manifolds of one lower dimension, such as lines contained within a plane, or points within a line, and subobjects a further reduction of dimension. If we restrict our study to 1-dimensional objects, then a tiny object would necessarily have a dimension of negative one, and so would be compact within a single point. From the outset, it is not exactly clear how one should seek to embed a dimension, negative or otherwise, within a point. If one is to take the absolute value, $|-\sigma_1|$ is obtained. By analogy with the extension of a single point to a line, one may conceive of a line contained within a pointlike atlas which encloses it as a boundary.

While it may seem a matter of cultural perception or limitation, this raises just a single significant issue: namely, it is unspecified in which direction this line is supposed to run. If points were represented as infinitesimal lines, this would make sense, however, in order to preserve their structure as circular objects, we require that at least two negative dimensions be induced upon them in order to specify the position of the first; in other words, to force the δ-small subobjects to conform to charts. This has the additional benefit of allowing us to consider individual points as chart-preserving atlases, while the lines they populate are in and of themselves co-charts of their auto-atlas.

Large Cardinal Axioms

Large cardinal axioms allow mathematicians to reason with strongly inaccessible cardinals in the context of set theory proper. According to Blass, et al. (see [Inacc.]), "every natural[3] set-theoretic axiom system is equiconsistent with ZFC + L," where L stands for some "large cardinal axiom." The specific form of such an axiom comes in several flavors, but for our purposes we will be working here with Grothendieck's axiom of universes (UA). A standard variant of the axiom is written as follows: for any set χ, there must exist a universe $U|\chi$ which is itself a set.[4] Accordingly, $\boldsymbol{P}(\chi)$ is well-ordered, and contained within U, and for some strongly inaccessible cardinal $\sigma_\lambda \notin U$, there is some $U|\lambda \supset U|\chi$ to which it belongs.

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[2] Items which are valued in δ shall be called 1-small, or simply "small."
[3] The term "natural" remains undefined here as well as in the source material.
[4] The reader is referred to [UC] for more information.

Given this information, one then proceeds to define the notion of *U-smallness* with regards to the scope of a particular universe. Low[5] defined a **U-small category** as a category $\mathbb{C}$ such that ob$\mathbb{C}$ and mor$\mathbb{C}$ are U-sets; a *locally U-small category* $\mathcal{D}$ is adapted slightly: the requirement U-set requirement is lifted to a *U-class* requirement, while requiring the hom-set $\mathcal{D}(x,y)$ of all objects x and y in $\mathcal{D}$ are U-sets. There, **U-set** refers to *members* of U, and **U-class** refers to *subsets* of U. Our context favors the notion of local small categories. It should be emphasized that our notion of δ-smallness *does not* correspond to Low's U-smallness; while the former is a characterization of *geometric* size, the latter is a notion of *universal size*. We will refer to this as **V-smallness** in order to avoid any confusion between the two; for instance, **V** is **V***-small, and every $\mathbf{V}_\kappa$* is **V***ẉ-small (for short we will call this ẉ*-small)[6] but not vice versa, as the relationship is transitive and antisymmetric.

THEOREM 1.01 Every δ-small subobject is **V***ẉ-small

PROOF Since every element of **V** has k=0, and because δ=k+1, **V** is the universal category of every 1-small object. As it stands, λ, our closure parameter, is finitely contained within the monoidal universe of size ẉ, because ẉ $\equiv \sigma^\sigma$, where $\sigma = \lambda_\delta$(k) is the closure of an open k-set. For **V**, we have $\{\lambda_\delta(k)\} \cup (k)$. It follows that ẉ encloses every inaccessible cardinal σ, such that any $\mathbf{V}_\kappa$*-small universe is contained in **V***ẉ.

COROLLARY 1.01a Every σ_λ-category is a λ-small object in $\mathbf{V}_{\lambda+1}$*.

PROOF Follows from 1.01, if one lets $\lambda_ẉ(\mathbf{V}_\kappa*)$ denote the set of all finite closures for a countably infinite set of universes. It follows that for every specific σ_κ, there is a universe $\mathbf{V}_\kappa$* in which it is contained. We conclude by writing $\sigma_\kappa \subset (\text{ob}(\mathbf{V}_\kappa*) \cup \text{mor}(\mathbf{V}_\kappa*))$.

COROLLARY 1.01b Every k-small object is contained within $\mathbf{V}_\delta$*.

PROOF By letting $\mathbf{V} = \mathbf{V}_0^*$, we recall that $\lambda_\delta \in \mathbf{V}_\delta^*$. It is then trivial to show that $\mathbf{V}_0^* \subset \mathbf{V}_1^*$, since $(\delta - k) = 1$.

Finally, this allows us to write:

COROLLARY 1.01c Every ẉ-small object is **V***ẉ*-small.

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[5] See [UC], definition 1.12

[6] An asterisk is included to allow the reader to easily distinguish **V***ẉ*-smallness from δ-smallness; i.e., δ-smallness when δ=ẉ.

The proof of which is trivial.

In words, we have a nested tower of universes, the minimal element of each being constant across all of them. For each universe, we have a local, weakly inaccessible cardinal λ serving as the boundary, which is included only under the closure of the universe $\mathbf{V}_\lambda^*$, and a corresponding cardinal σ which is accessible only from the open set $\mathbf{V}_{(\lambda+1)}^*$. Corollary 1.01c, while apparently trivial given the previous statements, is quite profound: essentially, it is a statement about how topologically minimal spaces encode information about the higher-dimensional manifolds in which they are embedded. This is essentially an approximation Heisenberg's "uncertainty principle;" that is to say, perfect, or infinitely precise data (in this case represented by infinitesimal points) models perfect information about the global systems in which they appear.

References

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