The Problem of the Causality in the Atomic World

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Abstract:

Interaction theories are usually based on a relativistically invariant Lagrange function. This function is generally known and accepted for the electromagnetic interaction. The variation of that Lagrangian leads to the system of the coupled Maxwell-Dirac equations. It contains a non-linear term. If you neglect this term, you obtain the well-known linear Dirac equation and rules for determining the correct values of the spectral lines of atoms. However, one cannot describe the radiation process and has to introduce the quantum hypothesis. But, if the non-linear term is also taken into account, there are solutions of the system what describe the emission of "quantum jumps" in space and time with correct frequencies. This is demonstrated in the presented work for hydrogen and helium atoms. It explains the entangled eigenfunctions in the context of a classical near-field theory. Further problems like diffraction effects, photo effects and relativistic transformation of the field tensor are discussed. Aim of the work is a proposal of an alternative to the statistical interpretation of the quantum theory in context of a classical near-field theory.

Contents

Chapter 1

Introduction

Many authors discuss the interpretation of the quantum theory. The here mentioned papers are only a small selection of these articles [1-12]. In particular the problem of the causality requires an examination. This question can be better explained by introducing a definition of the causality. In general, a system behaves causal if its behaviour in time is determined by fixed rules. These rules must be formulated mathematically for a physical system. For example, if you apply Newton mechanics, elasticity theory, thermodynamics, fluid mechanics and the Maxwell's equations, you can certainly predict the behaviour of a system in the macro world, unless it is too complex. Prerequisites are the knowledge of its state at the beginning and proper mathematical methods. Then you can speak from a causal process in a mathematical framework. However, it does not mean that an equivalent system in the real physical world shows the same behaviour. This has to be checked by experimental methods. If the measured start system, the interim systems and the end system agree with the mathematical results, one speaks from a causal physical process. Such successful tests took centuries and led to the formulation of the known physical laws of the macro world, what is named as classical physics. However, these laws are not able to explain the experimental results of the micro world. In particular the radiation effects of atoms are typical examples for the dynamics in the micro world.

The assumptions of the tested atomic model are the following: The electron is a point charge that revolves around an oppositely charged nucleus. The origin of the electromagnetic field is the motion of these charges. The applied laws are the Newtons laws of point mechanics and Maxwell's equations. It is well-known that the results of calculations using these equations do not agree with the related experimental results. That allows two different interpretations. Either the model is wrong or the dynamics is not causal in the atomic world. But, here the classical mechanics is reduced to point mechanics. Then a causal description is not possible.

However, an alternative model is obtained if the electron is described by an expanded field. A general definition of the causal behaviour of such fields was formulated by A. Einstein [13, 14]. He describes the system with a four-dimensional energy-momentum tensor of classical field functions. Such a system behaves causally in all volume elements if the four-dimensional divergences of its lines are zero. Three lines describe the momentum laws and the fourth line the energy law in all spatial elements. The laws of angular momentum follow from combinations. The starting point for such consideration must be a relativistically invariant Lagrange function. This Lagrange function is generally known and accepted for the electromagnetic interaction [15-18]. It also serves as model for other interaction theories [17]. One obtains by variation of the Lagrange function the system of the coupled Maxwell-Dirac equations. This system contains a non-linear term. The non-linearity is a consequence of the self-field, which must be taken into account due to the variation procedure. If one forms the energy-momentum tensor using the solutions of these differential equations, one gets a system what behaves causally in a mathematical framework. In addition, the system is relativistically invariant. But, it must be shown that the mathematical results agree with linked experimental results. Then you can say that the dynamics of the real atomic world also behaves causally.

First we discuss the motivation for this investigation. It is usually practice to neglect the self-field, which leads to linear equations [15, 18]. The consequence is the statistical interpretation of the quantum theory. Here we give not a report on the problems of this hypothesis. We only mention several central points. A linear system has many advantages from a mathematical point of view. With this theory, the correct frequencies of the spectral-lines are obtained according to special rules. In addition, the eigenfunctions of the basic equations can be applied to characterise the elements. In the quantum-field-theory (QFT) eigenfunctions of the Dirac-equation stand for the electrons and eigenfunctions of the Maxwell's equations in free space stand for the photons [15-18]. These particles are exchangeable and have no individual properties. It is a central point of the interpretation. This assumption explains the entangled eigenfunctions and the multi-electron spectrum. The problem is widely discussed in the literature with the result that no alternative interpretation is possible [5-7, 10]. Examples are the discussions about hidden parameters, the Bell's unequation and the EPR paradox [1, 2]. However, all of these considerations assume that the linear system is valid. Essential elements of this picture are states, which can only be left by timeless jumps. The theory only defines the probability of the transition from one state to another. It does not include pictures of this transition. The problem of the causality in the context of such model will not be discussed here.

The non-linear classical field theory allows the introduction of individual properties for the electron due to the non-zero rest-mass. Prerequisites are individual laws of conservation for the linked energy-momentum tensor like the charge conservation. The shape of such field is usually not static. In this paper we try to solve such non-linear systems and hope that finally all experimental results can be explained with the aid of these solutions. Of course, in a first step one must restrict this task to a few important problems. It is a purely mathematical approach. In particular, it is necessary to define a classical field with properties of a single particle.

The law of charge conservation requires a normalising procedure of the solutions of the Dirac equation what leads to an individual charge distribution in space. Therefore, we can a normalised field define as particle. In this context the eigenfunctions lose its independent meaning for the characterisation of particles. In addition, the current $J_\mu = J_\mu^1 + J_\mu^2$ of two electrons must now fulfils two separate law of charge conservation $\frac{\partial}{\partial x_\mu} J_\mu^1 = 0$ and $\frac{\partial}{\partial x_\mu} J_\mu^2 = 0$. That means, a particle is not given by a particular shape, but by a separating mechanism. The central demand for fields of particles is the local validity of separable laws of conservation for each particle field, independent on the extension. This claim are replaced by the hypothesis of the point charge in the linear theory, that means an assumption about the form of the charge distribution. In the non-linear interaction theory the eigenfunctions are only describing elements of the whole field. We consider the interaction between an electron and the electromagnetic field as in the complete time-space continuum describable process.

The introduction of individual characteristics for a share of an electromagnetic field is impossible in this picture, due to the vanishing rest-mass. If an atom absorbs such a "photon" from an electromagnetic field, the source of this share of the electromagnetic field is normally not clear (one or more radiation sources). In the present context, its particle properties can be detected only by interaction with an electron. In addition, the classical field theory must also show that the energy exchange between the electron and the electromagnetic field amounts to $h\nu$.

A general aspect is the question, where and when the Maxwell's equations are fulfilled as well in a quasi-static state and also during a radiation. In the non-linear theory the electromagnetic field must fulfil Maxwell's equations at all times, even during a transition state ("quantum leaps"). Nevertheless, the time-space integrals over the electromagnetic field must interact with the electron like a particle.

Then the introduction of particle properties to characterise the electromagnetic field is not really necessary. However, one must describe the "quantum leap" by a causal process with the known result. That means, the dynamics of the charge distributions should explain the radiation properties. Radiation and absorption processes of the electromagnetic fields must be describable in all local details. Another advantage of the hypothesis is the following: Since the basic equations reacts invariantly to a Lorentz transformation, the exact solutions of these equations also fulfil all demands of the special theory of relativity. This point alone justifies such investigations.

It should be mentioned that this is not a new attempt. The problem of the explanation of the electron properties by classical fields was first formulated by G-Mie [16-19]. Further developments of the topic, however, lead to a renunciation of this opinion as a result of the functional-analytical foundation of the quantum theory. In the established formulation of the quantum theory, the problems of the shape of particles and of the transition states are not considered explicitly [1, 2, 5-10]. Consequently, this point of view forces to apply a statistical interpretation.

The solution of coupled non-linear partial differential equations is the common mathematical root. You can also ask what is the proper method for the solution of such equations, regardless of the physical interpretation. The functional-analytical methods neglect the self-field and the local field properties. Consequently, the eigenfunction of linear operators and of which Green functions are the central topics. Applying the non-linear theory, one has to solve the system exactly with the aid of numerical methods. Another aspects are the followings: The statistical interpretation was developed for the explanation of the effects of radiation and impact. These effects are usually measured in far-field and a mathematical description should be possible by asymptotic terms. Such terms are small in comparison with other near-field terms. An example is the Hertz-dipole radiation. In addition, all dynamic processes in a condensed phase must be covered by quantum-leaps if you apply the functional-analytical methods. That is difficult to understand.

Therefore, the introduction of an alternative concept for the definition of particles is reasonable. The non-linear classical field theory allows a more detailed description of the interaction between particles due to the separate divergence equations for the exchange of energy and momentum at all points of the field. Therefore a remote reaction is excluded in all cases. In addition, this interaction is not strongly linked with the motion of the mass centre of the electron field. The mathematical difficulties of this theory are great because the hypothesis should cover the influence of all external fields. The success of such non-linear theories depends strongly on the state of the theory of partial differential equations. Not only the non-linear terms are the problem but, one must deal with the numerics of oscillating functions too. That requires special methods. However, the power of modern computers has improved the situation and leads to the hope that a better proof of the hypothesis should be possible.

It is well-known that the Dirac (Schrödinger) equation and the Maxwell's equations are originally linear equations. Proposed supplementations of the Dirac (Scrödinger) equation by non-linear, additional terms appear artificially [20]. Whereas the consequence of the Lagrange procedure seems plausible. Especially because it ensures the compatibility with the relativity theory. Although the Dirac equation has lost its linear nature as a result of the coupling with the electromagnetic self-field, we expect that the frequency of an emitting share of the charge is independent of the self-field. Then one obtains correct spectral lines. These points lead to the following hypothesis:

"All effects of the interaction between the electron and the electromagnetic field can be described by solutions of the system of the coupled Maxwell-Dirac equations without any contradiction to the principles of the classical, causal field theory."

This hypothesis corresponds to the discussed Lagrange procedure. It can be considered as a purely mathematical construct where its solution has no linking to the physical discussion. These are only complementary considerations. The complete description of the physical reality is not a claim of the hypothesis until a generally convincing proof is available. Here, we investigate only the possibility of a mathematical approximation to the physical reality by special solutions of the basic equations within the framework of a classical near-field theory [20-23]. The generally accepted prohibition of the description of non-measurable quantities should not be an obstacle for such an investigation since the mathematical questions are very interesting. Our central topic is the study of the energy-momentum tensor of the electromagnetic interaction in the context of classical fields. The focus is on the radiation of bound transition states. We regard the radiation as a dynamic process and not as an indescribable jump between two static states. The statistic behaviour of the start time of transition states is in such context a consequence of the variety of dynamic solutions and not an "a priori" property. An atom should not be considered as a rigid particle. That means, non-radiating dynamical states must be possible too.

The different interpretation of the atomic world can be described in the following points:

The first possibility (linear model):

1. One rejects the validity of the laws of the classical physics in the atomic world. This is justified by the uncertainty principle, which does not allow the introduction of exact values for coordinates and momenta of a particle at the same time, due to experimental reasons. However, here the classical mechanics is reduced to the point mechanics or the mechanics of rigid bodies.

2. One avoids a definition of the electron. The question , "What is a particle ?", is replaced by the question, "Which properties does a particle have?". Therefore, it is not necessary to derive all properties of the electron from a theoretical model.
3. The basic equations are linear.
4. The demand of the validity of the laws of causality in the micro world, in particular in an atom, is replaced by the quantum hypothesis. That requires a statistical interpretation of the solutions.

The second possibility (non-linear model):
1. The basic equations of the electromagnetic interaction result from the variation of the well-known Lagrange function. That means, one has to solve the coupled Maxwell-Dirac equations. It is a non-linear problem.
2. With the normalised solutions of the Dirac equation and the associated electromagnetic field, one can define an extended energy-momentum tensor what fulfils locally all laws of conservation. These laws guarantee the individuality of the electron field.
3. If one regards this tensor field as an electron, one has to show that all experimentally proven qualities of the electron can be explained by this model [23].

This requires solving of many mathematical problems. However, it is a worthwhile goal to show that the definition of the particle requires only basic equations and normalising conditions and not additional assumptions. In the presented paper, some of the related problems are investigated and open questions are discussed. The following topics are described in the single chapters:

Chapter 2:

Here the basic equations of the fields and of which relationships are presented in the applied form. All equations are well-known.

Chapter 3:

The influence of the self-field on the solution of the Schrödinger equation of a hydrogen atom is investigated. It is shown that the self-field leads to vibration of the electron field, where the intensity of which depends on the total energy of the system. An energy minimum is associated with a static solution. Only special current dynamics causes radiation effects. In these cases the dynamic solution must form a radiation moment what oscillates with the known spectral frequency. Such moments are built up by solutions that contain transition states between different excitation levels. The interact of these atoms with the electromagnetic field can cause spectral lines. However, the influence of this field on the charge distribution is small. If one neglects the radiation effect, one can show there are separate laws of charge conservation for each level. That means, the exchange of charge between the levels is very slow compared to the reciprocal frequency. If several atoms form radiation moments, one has conditions like in a field of macroscopic antennas. This is a basic for a causal description of the emission or the absorption of "photons". The frequency must meet the known quantum condition and the final energy exchange has to amount $h\nu$.

Besides, a strong vibration of the normalised electron field can lead to its escape from the atom. It is only a hypothesis, but it shows a way, how the photo effect can be explained in the framework presented, because an external radiation field changes the energy and the momentum of the system. However, an exact proof has not yet been attempted.

Chapter 4

The purpose of this chapter is the test of the radiation conditions of "quantum-leaps". Therefore the properties of simple transition states are investigated. These have stable dynamic forms and contain shares which fulfil the radiation conditions. Furthermore, the divergences of the energy-momentum tensor do not injure the causality conditions.

Chapter 5 and chapter 6

These chapters correspond to the chapters 3 and 4. Here the same problems are discussed for two electron systems (helium problem). One can show that the known frequency conditions of the entangled eigenfunctions are also fulfilled in the new picture. That means, no remote interaction is necessary for the explanation of the spectral lines. Several studies on the fine structure are added.

Chapter 7

A few properties of a free solution, influenced by its self-field and by an external field, are presented. The results can be applied for a plausible consideration of the diffraction effects and the path of the electron in a tube. But, an exact description is still lacking.

Chapter 2

The basic equations

The formulas given in this section are general known. There are some alternative descriptions, but no different meanings. We use the sum convention in the form

$$x_\mu x_\mu := \sum_1^4 x_\mu x_\mu \tag{2.1}$$

or

$$x_{\bar{\mu}} x_{\bar{\mu}} := \sum_1^3 x_\mu x_\mu \tag{2.2}$$

and the following notations of the coordinates:

$x_1 = x$, $x_2 = y$, $x_3 = z$ and $x_4 = i\acute{c}t$.

These notations have an advantage over the normal covariance convention when other types of indices and of which positions are more important. Generally an over lined index of a vector indicates its space components. We start from the well-known Lagrange density for the electromagnetic interaction, which in the Gaussian system is given by [16, 21]

$$\begin{aligned} L = &-\tfrac{1}{8\pi}\tfrac{\partial A_\mu}{\partial x_\nu}\tfrac{\partial A_\mu}{\partial x_\nu} - \tfrac{c}{2}\bar{U}(\gamma_\mu \hbar \tfrac{\partial}{\partial x_\mu} - i\tfrac{\acute{e}}{\acute{c}}\gamma_\mu A_\mu + \acute{M}\acute{c})U \\ &+\tfrac{c}{2}\bar{U}(\gamma_\mu \hbar \tfrac{\overleftarrow{\partial}}{\partial x_\mu} + i\tfrac{\acute{e}}{\acute{c}}\gamma_\mu A_\mu - \acute{M}\acute{c})U. \end{aligned} \tag{2.3}$$

The relation $\bar{U}(\mathcal{R}_1,t)\frac{\overleftarrow{\partial}}{\partial x_\nu} = \frac{\partial \bar{U}(\mathcal{R}_1,t)}{\partial x_\nu}$ explains the meaning of the arrows. By varying the fields U, $\bar{U}$ or A_μ independently, one obtains the Dirac equation, the conjugated Dirac equation or the wave equation in the forms [16-18]

$$(\gamma_\mu(\hbar\frac{\partial}{\partial x_\mu} - i\frac{\acute{e}}{\acute{c}}A_\mu) + \acute{M}\acute{c})U = 0, \tag{2.4}$$

$$\bar{U}(\gamma_\mu(\hbar\frac{\overleftarrow{\partial}}{\partial x_\mu} + i\frac{\acute{e}}{\acute{c}}A_\mu) - \acute{M}\acute{c}) = 0 \tag{2.5}$$

and

$$(\triangle + \frac{\partial^2}{(\partial x_4)^2})A_\mu = -4\pi J_\mu \tag{2.6}$$

with

$$J_\mu = i\acute{e}\bar{U}\gamma_\mu U,$$

$$U = \begin{pmatrix} b_1 \\ b2 \\ b_3 \\ b4 \end{pmatrix}, \ \bar{U} = (b_1^* \ \ b_2^* \ -b_3^* \ -b_4^*). \tag{2.7}$$

The variation of a slightly modified Lagrangian leads directly to the second pair of Maxwell's equations [20, 21]. From the eqs.(2.4/2.5) result the continuity equation

$$\frac{\partial}{\partial x_\mu}J_\mu = 0. \tag{2.8}$$

The wave eqs.(2.6) follow from the condition

$$\frac{\partial}{\partial x_\mu}A_\mu = 0 \tag{2.9}$$

for the vector potentials. This approach are useful to solve the Maxwell's equations of the vacuum, what reads

$$\nabla \times \mathbf{E} = -\frac{1}{\acute{c}}\dot{\mathbf{H}}, \tag{2.10}$$

$$\nabla \times \mathbf{H} = \frac{1}{\acute{c}}\dot{\mathbf{E}} + 4\pi\mathbf{J} \tag{2.11}$$

with the additional conditions

$$\nabla\mathbf{E} = -4\pi i J_4 \tag{2.12}$$

and

$$\nabla\mathbf{H} = 0. \tag{2.13}$$

Using the solutions of the eqs.(2.6), the field strengths can be described by

$$\mathbf{H} = \nabla \times \mathbf{A} \tag{2.14}$$

and

$$\mathbf{E} = \nabla(iA_4) - \frac{1}{\acute{c}}\dot{\mathbf{A}}. \tag{2.15}$$

These vectors satisfy the Maxwell's eqs.(2.10/2.11) if the potentials fulfil the wave eq.(2.6). With the notations

$$
\begin{array}{c}
F_{ik} = \frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k},\ \tilde{F}_{\nu\nu} = 0, \\
\tilde{F}_{12} = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2},\ \tilde{F}_{13} = \frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3},\ \tilde{F}_{14} = \frac{\partial A_4}{\partial x_1} - \frac{\partial A_1}{\partial x_4}, \\
\tilde{F}_{21} = -F_{12},\quad F_{23} = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3},\ \tilde{F}_{24} = \frac{\partial A_4}{\partial x_2} - \frac{\partial A_2}{\partial x_4}, \\
\tilde{F}_{31} = -\tilde{F}_{13},\ ,\ \tilde{F}_{32} = -\tilde{F}_{13},\ \tilde{F}_{14} = \frac{\partial A_4}{\partial x_1} - \frac{\partial A_1}{\partial x_4}, \\
\tilde{F}_{41} = -\tilde{F}_{14},\ \tilde{F}_{42} = -\tilde{F}_{14},\ \tilde{F}_{43} = -\tilde{F}_{14}
\end{array}
\tag{2.16}
$$

we can introduce the known field tensors $\mathbf{F} = [F_{ik}]$ and

$$
\tilde{\mathbf{F}} = [\tilde{F}_{ik}] = \begin{bmatrix} 0 & H_3 & -H_2 & -iE_1 \\ -H_3 & 0 & H_1 & -iE_2 \\ H_2 & -H_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{bmatrix}. \tag{2.17}
$$

This allows the following fourth dimensional description of the second pair of the Maxwell's equations

$$
\mathbf{DIV}\ \tilde{\mathbf{F}} = -4\pi\mathbf{J}. \tag{2.18}
$$

A fourth dimensional description of the first pair of Maxwell's equations is also possible by introducing of additional notations [20]. The Lagrange density eq.(2.3) leads to an energy-momentum tensor of the form

$$
\begin{array}{c}
T^a_{\nu\mu} = \frac{1}{4\pi}\left(\frac{\partial A_\sigma}{\partial x_\nu}\frac{\partial A_\sigma}{\partial x_\mu} - \frac{1}{2}\delta_{\nu\mu}\frac{\partial A_\sigma}{\partial x_\lambda}\frac{\partial A_\sigma}{\partial x_\lambda}\right) \\
+\frac{\acute{c}\hbar}{2}\left(\bar{U}\gamma_\mu\frac{\partial U}{\partial x_\nu} - \frac{\partial \bar{U}}{\partial x_\nu}\gamma_\mu U\right).
\end{array}
\tag{2.19}
$$

It is an asymmetric tensor and we add the terms

$$\triangle T_{\nu\mu} = \triangle T^b_{\nu\mu} + \triangle T^m_{\nu\mu} \tag{2.20}$$

with

$$\begin{array}{c}\triangle T^b_{\nu\mu} = -\frac{1}{4\pi}(\frac{\partial}{\partial x_\sigma}(A_\nu F_{\mu\sigma}) + \frac{\partial}{\partial x_\sigma}(A_\mu \frac{\partial A_\sigma}{\partial x_\nu}) \\ -\frac{1}{2}\delta_{\nu\mu}\frac{\partial}{\partial x_\sigma}(A_\lambda \frac{\partial A_\sigma}{\partial x_\lambda}))\end{array} \tag{2.21}$$

and

$$\begin{array}{c}\triangle T^m_{\nu\mu} = \frac{\acute{c}\hbar}{4}\frac{\partial}{\partial x_\sigma}(\bar{U}\gamma_\nu\gamma_\sigma\gamma_\mu U) \\ \times(1-\delta_{\nu\mu})(1-\delta_{\sigma\mu})(1-\delta_{\sigma\nu}).\end{array} \tag{2.22}$$

You can show that the divergences of the tensors eqs.(2.21/2.22) and eq.(2.22) are zero [16]. In addition, the relations

$$\int \triangle T_{4\mu} dV = 0 \tag{2.23}$$

are valid and therefore the symmetry correction has no influence on the energy balance. After adding eq.(2.20) to eq.(2.19) one obtains the following symmetrical energy-momentum tensor

$$\begin{array}{c}T_{\nu\mu} = \frac{1}{4\pi}(F_{\nu\sigma}F_{\mu\sigma} - \frac{1}{4}\delta_{\nu\mu}F_{\sigma\lambda}F_{\sigma\lambda}) \\ +\frac{\acute{c}}{4}(\bar{U}(\hbar\frac{\partial}{\partial x_\nu} - i\frac{\acute{e}}{\acute{c}}A_\nu)\gamma_\mu U - \bar{U}\gamma_\mu(\hbar\frac{\overleftarrow{\partial}}{\partial x_\nu} + i\frac{\acute{e}}{\acute{c}}A_\nu)U \\ +\bar{U}(\hbar\frac{\partial}{\partial x_\mu} - i\frac{\acute{e}}{\acute{c}}A_\mu)\gamma_\nu U - \bar{U}\gamma_\nu(\hbar\frac{\overleftarrow{\partial}}{\partial x_\mu} + i\frac{\acute{e}}{\acute{c}}A_\mu)U).\end{array} \tag{2.24}$$

$T_{\nu\,\mu}$ can be split into

$$T_{\nu\,\mu} = \bar{T}^d_{\nu\mu} + \bar{T}^w_{\nu\mu} + T^e_{\nu\mu} \tag{2.25}$$

or

$$T_{\nu\,\mu} = T^d_{\nu\mu} + T^w_{\nu\mu} + \Delta\, T^m_{\nu\mu} + T^e_{\nu\mu}. \tag{2.26}$$

Here are

$$\bar{T}^d_{\nu\,\mu} = \frac{\acute{c}\hbar}{4}(\bar{U}\gamma_\mu\frac{\partial U}{\partial x_\nu} - \frac{\partial \bar{U}}{\partial x_\nu}\gamma_\mu U + \bar{U}\gamma_\nu\frac{\partial U}{\partial x_\mu} - \frac{\partial \bar{U}}{\partial x_\mu}\gamma_\nu U), \tag{2.27}$$

$$\bar{T}^w_{\nu\mu} = -\frac{1}{2}(A_\nu\, J_\mu + A_\mu\, J_\nu) \tag{2.28}$$

and

$$T^e_{\nu\mu} = \frac{1}{4\pi}(F_{\nu\sigma}F_{\mu\sigma} - \frac{1}{4}\delta_{\nu\mu}F_{\sigma\lambda}F_{\sigma\lambda}), \tag{2.29}$$

respectively

$$T^d_{\nu\,\mu} = \frac{\acute{c}\hbar}{2}(\bar{U}\gamma_\mu\frac{\partial U}{\partial x_\nu} - \frac{\partial \bar{U}}{\partial x_\nu}\gamma_\mu U) \tag{2.30}$$

and

$$T^w_{\nu\mu} = -A_\nu\, J_\mu. \tag{2.31}$$

The investigation of these tensors, using the methods of the classical field theory, is the main topic of the presented paper. Therefore, we need following divergences (see eqs.(2.8/2.9)

$$\frac{\partial}{\partial x_\mu}\bar{T}^d_{\nu\mu} = J_\mu(\frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu}) + \frac{1}{2}(A_\mu\frac{\partial J_\nu}{\partial x_\mu} + \frac{\partial A_\nu}{\partial x_\mu}J_\mu), \tag{2.32}$$

$$\frac{\partial}{\partial x_\mu}\bar{T}^w_{\nu\mu} = -\frac{1}{2}(A_\mu\frac{\partial J_\nu}{\partial x_\mu} + \frac{\partial A_\nu}{\partial x_\mu}J_\mu) \tag{2.33}$$

and

$$\frac{\partial}{\partial x_\mu} T^e_{\nu\mu} = -J_\mu (\frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu}) \tag{2.34}$$

or

$$\frac{\partial}{\partial x_\mu} T^d_{\nu\mu} = J_\mu (\frac{\partial A_\mu}{\partial x_\nu}), \tag{2.35}$$

$$\frac{\partial}{\partial x_\mu} T^w_{\nu\mu} = -J_\mu (\frac{\partial A_\nu}{\partial x_\mu}). \tag{2.36}$$

For the validity of these relations the basic differential equations (2.4)-(2.6) must be fulfilled by U, $\bar{U}$ and A_μ. Then, one obtains

$$\frac{\partial}{\partial x_\mu}(T_{\nu\mu}) = \frac{\partial}{\partial x_\mu}(\bar{T}^d_{\nu\mu} + \bar{T}^w_{\nu\mu} + T^e_{\nu\mu}) = 0 \tag{2.37}$$

and

$$\frac{\partial}{\partial x_\mu}(T_{\nu\mu}) = \frac{\partial}{\partial x_\mu}(T^d_{\nu\mu} + T^w_{\nu\mu} + T^e_{\nu\mu}) = 0. \tag{2.38}$$

After the space integration, it reads

$$\frac{d}{i\acute{c}dt} \int T_{\nu\, 4}\, dV \; = \; - \int \frac{\partial}{\partial x_{\bar{\mu}}} (T_{\nu\bar{\mu}}) dV \tag{2.39}$$

and therefore is

$$\begin{aligned} &\textstyle\int T_{\nu\, 4}\,(t_1) dV \; - \; \int T_{\nu\, 4}\,(t_0) dV \\ &= \; -i\acute{c} \textstyle\int_{t_0}^{t_1} \int R^2 (\frac{x_1}{R} T_{\nu 1} + \frac{x_2}{R} T_{\nu 2} + \frac{x_3}{R} T_{\nu 3}) sin\vartheta\, d\vartheta\, d\varphi dt. \end{aligned} \tag{2.40}$$

These equations represent the integral forms of the conservation laws of momenta ($\nu = \bar{\nu}$) and energy ($\nu = 4$). Singularities have to be excluded. The right hand side of eq.(2.40) describes for $R \to \infty$ the radiation losses. The equations are valid for a combination of a field of one electron with the electromagnetic field caused by this electron. If an external electromagnetic field, described by A_μ^K, influences this system, A_μ must be replaced by $A_\mu + A_\mu^K$ in the eqs.(2.4/2.5/2.24), but not in eq.(2.6). Then eq.(2.3) obtains the form

$$\begin{aligned} L = -\tfrac{1}{8\pi}\tfrac{\partial A_\mu}{\partial x_\nu}\tfrac{\partial A_\mu}{\partial x_\nu} - \tfrac{\acute{c}}{2}\bar{U}(\gamma_\mu \hbar \tfrac{\partial}{\partial x_\mu} - i\tfrac{\acute{e}}{\acute{c}}\gamma_\mu(A_\mu + A_\mu^K) + \acute{M}\acute{c})U \\ +\tfrac{\acute{c}}{2}\bar{U}(\gamma_\mu \hbar \tfrac{\overleftarrow{\partial}}{\partial x_\mu} + i\tfrac{\acute{e}}{\acute{c}}\gamma_\mu(A_\mu + A_\mu^K) - \acute{M}\acute{c})U. \end{aligned} \tag{2.41}$$

That means, the external field is strongly separated and has no influence on the variation procedure. Therefore one obtains modified eqs.(2.4/2.5/2.24). This is in case of a nucleus field A_μ^K possible if the related charge is not singular and fixed. In addition, as long as the sources of the external field are outside of the considered area, the equations can be also applied.

Chapter 3

The one-electron systems with self-field

3.1 *General properties*

In the first part we are dealt with the properties of the one-electron field be bound by a nucleus. The known system of eigenfunctions is given by [15, 24]

$$\chi(n,j,m) = \begin{pmatrix} f_n^1(r)\sqrt{\frac{j-m+1}{2j+2}}Y_{j+1/2\ m-1/2}(\vartheta,\varphi) \\ f_n^1(r)\sqrt{\frac{j+m+1}{2j+2}}Y_{j+1/2\ m+1/2}(\vartheta,\varphi) \\ -ig_n^1(r)\sqrt{\frac{j+m}{2j}}Y_{j-1/2\ m-1/2}(\vartheta,\varphi) \\ ig_n^1(r)\sqrt{\frac{j-m}{2j}}Y_{j-1/2\ m+1/2}(\vartheta,\varphi) \end{pmatrix} \tag{3.1}$$

and

$$\phi(n,j,m) = \begin{pmatrix} if_n^2(r)\sqrt{\frac{j+m}{2j}}Y_{j-1/2\ m-1/2}(\vartheta,\varphi) \\ -if_n^2(r)\sqrt{\frac{j-m}{2j}}Y_{j-1/2\ m+1/2}(\vartheta,\varphi) \\ g_n^2(r)\sqrt{\frac{j-m+1}{2j+2}}Y_{j+1/2\ m-1/2}(\vartheta,\varphi) \\ g_n^2(r)\sqrt{\frac{j+m+1}{2j+2}}Y_{j+1/2\ m-1/2}(\vartheta,\varphi) \end{pmatrix}. \tag{3.2}$$

n stands for the principal quantum number, $j = l + m_s$ for the momentum quantum number, m for the magnet quantum number, l for the path quantum number and $m_s = \pm 1/2$ for the spin quantum number. The radial functions are solutions of the equations

$$\begin{aligned} &\tfrac{1}{\hbar \acute{c}}(\bar{E}^1_{nj} + \tfrac{\acute{e}^2}{r} - \acute{M}\acute{c}^2) f^1_n + (\tfrac{dg^1_n}{dr} - (j-1/2)\tfrac{g^1_n}{r} = 0, \\ &\tfrac{1}{\hbar \acute{c}}(\bar{E}^1_{nj} + \tfrac{\acute{e}^2}{r} + \acute{M}\acute{c}^2) g^1_n - (\tfrac{df^1_n}{dr} + (j+3/2)\tfrac{f^1_n}{r} = 0, \end{aligned} \tag{3.3}$$

or

$$\begin{aligned} &\tfrac{1}{\hbar \acute{c}}(\bar{E}^2_{nj} + \tfrac{\acute{e}^2}{r} - \acute{M}\acute{c}^2) f^2_n + (\tfrac{dg^2_n}{dr} + (j+3/2)\tfrac{g^2_n}{r} = 0, \\ &\tfrac{1}{\hbar \acute{c}}(\bar{E}^2_{nj} + \tfrac{\acute{e}^2}{r} + \acute{M}\acute{c}^2) g^2_n - (\tfrac{df^2_n}{dr} + (j-1/2)\tfrac{f^2_n}{r} = 0. \end{aligned} \tag{3.4}$$

$\bar{E}^1_{nj}$ and $\bar{E}^2_{nj}$ represent the eigenvalues of the time independent Dirac equation and $\chi exp(-i\bar{E}^1_{nj}t)$ respectively $\phi exp(-i\bar{E}^2_{nj}t)$ the complete eigenfunctions.
The first eq.(3.3) leads to following equation $(generally : E = \bar{E} - \acute{M}\acute{c}^2)$

$$\begin{aligned} &(E_{n\,l} + \tfrac{\hbar^2}{2\acute{M}}((1/r^2)\partial/\partial r(r^2\partial/\partial r) \\ &-l(l+1)/(r^2)) + \tfrac{\acute{e}^2}{r}) f^1_n(r) \begin{pmatrix} Y_{l\,m-1/2} \\ Y_{l\,m+1/2} \end{pmatrix} = 0 \end{aligned} \tag{3.5}$$

if one uses the approximation

$$g^1_n \cong \frac{\hbar}{2\acute{M}\acute{c}}(\frac{df^1_n}{dr} + (j+3/2)\frac{f^1_n}{r}) \tag{3.6}$$

and sets l=j+1/2. Analogously one gets for the first eq.(3.4) also the eq.(3.5), because l=j-1/2 and

$$g_n^2 \cong \frac{\hbar}{2\acute{M}\acute{c}}((\frac{df_n^2}{dr} - (j-1/2)\frac{f_n^2}{r}). \qquad (3.7)$$

Then the relativistic effects are neglected in the eigenvalue $E_{n\,l}$. You can see that the eq.(3.5) has the form of the linked Schrödinger equation, where $f_n^1(r)(Y_{l\,m-1/2}\ over\ Y_{l\,m+1/2})$ is replaced by $f_n^1(r)Y_{l\,m}$.

Here we consider only the special cases with $m = \pm 1/2$. That leads for $j = l+1/2$ in eq.(3.1) to $m_s = j - l = -1/2 :=\downarrow$ and to following spinor

$$\chi'(\mathcal{R}, n, l = j+1/2, m = 1/2, \downarrow) = \begin{pmatrix} F_n(r)\sqrt{\frac{l}{2l+1}}\,Y_{l\,0} \\ F_n(r)\sqrt{\frac{l+1}{2l+1}}\,Y_{l\,1} \\ -iG_n(r)\sqrt{\frac{l}{2l-1}}Y_{l-1\,0} \\ iG_n(r)\sqrt{\frac{l-1}{2l-1}}Y_{l-1\,1} \end{pmatrix} \qquad (3.8)$$

and in eq.(3.2) to $m_s = j - l = 1/2 :=\uparrow$ and

$$\phi'(\mathcal{R}, n, l = j+1/2, m = 1/2, \uparrow) = \begin{pmatrix} iF_n(r)\sqrt{\frac{l+1}{2l+1}}\,Y_{l\,0} \\ -iF_n(r)\sqrt{\frac{l}{2l+1}}\,Y_{l\,1} \\ G_n(r)\sqrt{\frac{l+1}{2l+3}}Y_{l+1\,0} \\ G_n(r)\sqrt{\frac{l+2}{2l+3}}Y_{l+1\,1} \end{pmatrix}.$$

With the values $j = l - 1/2, m = -1/2$ is $m_s = j - l = -1/2 :=\downarrow$. It leads in eq.(3.1) to

$$\chi'(\mathcal{R}, n, l = j + 1/2, m = -1/2, \downarrow) = \begin{pmatrix} F_n(r)\sqrt{\frac{l+1}{2l+1}}\, Y_{l\ -1} \\ F_n(r)\sqrt{\frac{l}{2l+1}}\, Y_{l\ 0} \\ -iG_n(r)\sqrt{\frac{l+1}{2l-1}} Y_{l-1\ -1} \\ iG_n(r)\sqrt{\frac{l+1}{2l-1}} Y_{l-1\ 0} \end{pmatrix}$$

and in eq.(3.2) because of $m_s = j - l = 1/2 :=\uparrow$ to

$$\phi'(\mathcal{R}, n, l = j - 1/2, m = -1/2, \uparrow) = \begin{pmatrix} iF_n(r)\sqrt{\frac{l}{2l+1}}\, Y_{l\ -1} \\ -iF_n(r)\sqrt{\frac{l+1}{2l+1}}\, Y_{l\ 0} \\ G_n(r)\sqrt{\frac{l+2}{2l+3}} Y_{l+1\ -1} \\ G_n(r)\sqrt{\frac{l+1}{2l+3}} Y_{l+1\ 0} \end{pmatrix}.$$

Using these spinors, we form the following sum functions of a single electron system:

$$\begin{aligned} U(\mathcal{R}, t, \uparrow) &= \sum_{n\,l} (b'_{n\,l}(t)\ \tilde{\xi}_{n\,l}(\mathcal{R}, \uparrow), \\ V(\mathcal{R}, t, \uparrow) &= \sum_{n\,l} b''_{n\,l}\ \tilde{\eta}_{n\,l}(\mathcal{R}, \uparrow), \\ U(\mathcal{R}, t, \downarrow) &= \sum_{n\,l} (d'_{n\,l}(t)\ \tilde{\xi}_{n\,l}(\mathcal{R}, \downarrow), \\ V(\mathcal{R}, t, \downarrow) &= \sum_{n\,l} d''_{n\,l}\ \tilde{\eta}_{n\,l}(\mathcal{R}, \downarrow). \end{aligned} \tag{3.9}$$

Here we have introduced the expressions

$$\tilde{\xi}_{n\,l}(\mathcal{R}, 1/2) = \begin{pmatrix} \hat{\xi}_{n\,l}(\mathcal{R}, 1/2) \\ \check{\xi}_{n\,l}(\mathcal{R}, 1/2) \end{pmatrix},\ \tilde{\eta}_{n\,l}(\mathcal{R}, 1/2) = \begin{pmatrix} \hat{\eta}_{n\,l}(\mathcal{R}, 1/2) \\ \check{\eta}_{n\,l}(\mathcal{R}, 1/2) \end{pmatrix}$$

and

$$\tilde{\xi}_{n\,l}(\mathcal{R},-1/2)=\begin{pmatrix}\hat{\xi}_{n\,l}(\mathcal{R},-1/2)\\ \check{\xi}_{n\,l}(\mathcal{R},-1/2)\end{pmatrix},\;\tilde{\eta}_{n\,l}(\mathcal{R},-1/2)=\begin{pmatrix}\hat{\eta}_{n\,l}(\mathcal{R},-1/2)\\ \check{\eta}_{n\,l}(\mathcal{R},-1/2)\end{pmatrix}$$

with

$$\begin{aligned}\tilde{\xi}_{n\,l}(\mathcal{R},1/2)= &\sqrt{\tfrac{l}{2l+1}}\chi'(\mathcal{R},n,l=j+1/2,1/2,\downarrow)\\ &-i\sqrt{\tfrac{l+1}{2l+1}}\,\phi'(\mathcal{R},n,l-1/2,1/2,\uparrow),\end{aligned}\tag{3.10}$$

$$\begin{aligned}\tilde{\xi}_{n\,l}(\mathcal{R},-1/2)= &\sqrt{\tfrac{l}{2l+1}}\chi'(\mathcal{R},n,l=j+1/2,-1/2,\downarrow)\\ &+i\sqrt{\tfrac{l+1}{2l+1}}\,\phi'(\mathcal{R},n,l=j-1/2,-1/2,\uparrow),\end{aligned}\tag{3.11}$$

$$\begin{aligned}\tilde{\eta}_{n\,l}(\mathcal{R},1/2)= &\sqrt{\tfrac{l+1}{2l+1}}\chi'(\mathcal{R},n,l=j+1/2,1/2,\downarrow)\\ &+i\sqrt{\tfrac{l}{2l+1}}\,\phi'(\mathcal{R},n,l=j-1/2,1/2,\uparrow)\end{aligned}\tag{3.12}$$

or

$$\begin{aligned}\tilde{\eta}_{n\,l}(\mathcal{R},-1/2)= &\sqrt{\tfrac{l+1}{2l+1}}\chi'(\mathcal{R},n,l=j+1/2,-1/2,\downarrow)\\ &-i\sqrt{\tfrac{l}{2l+1}}\,\phi'(\mathcal{R},n,l=j-1/2,\uparrow).\end{aligned}\tag{3.13}$$

The arrows indicate the orientation of the spin ($m_s=1/2:=\uparrow,\ m_s=-1/2:=\downarrow$) and the spinors of the eqs.(3.10-3.13) are given by

$$\tilde{\xi}_{n\,l}(\mathcal{R},1/2)=\begin{pmatrix}F_nY_{l\,0}\\ 0\\ -iG^1_n\frac{l}{\sqrt{4l^2-1}}Y_{l-1\,0}-iG^2_n\frac{l+1}{\sqrt{(2l+1)(2l+3)}}Y_{l+1\,0}\\ iG^1_n\frac{\sqrt{l(l-1)}}{\sqrt{4l^2-1}}Y_{l-1\,1}-iG^2_n\frac{\sqrt{(l+1)(l+2)}}{\sqrt{(2l+1)(2l+3)}}Y_{l+1\,1}\end{pmatrix},\tag{3.14}$$

$$\tilde{\xi}_{n\,l}(\mathcal{R},-1/2)=\begin{pmatrix} 0 \\ F_nY_{l\,0} \\ -iG^1_n\frac{\sqrt{l(l-1)}}{\sqrt{4l^2-1}}Y_{l-1\,-1}+iG^2_n\frac{\sqrt{(l+1)(l+2)}}{\sqrt{(2l+1)(2l+3)}}Y_{l+1\,-1} \\ iG^1_n\frac{l}{\sqrt{4l^2-1}}Y_{l-1\,0}+iG^2_n\frac{l+1}{\sqrt{(2l+1)(2l+3)}}Y_{l+1\,0} \end{pmatrix},$$

$$\tilde{\eta}_{n\,l}(\mathcal{R},1/2)=\begin{pmatrix} 0 \\ F_nY_{l\,1} \\ -iG^1_n\frac{\sqrt{l(l+1)}}{\sqrt{4l^2-1}}Y_{l-1\,0}+iG^2_n\frac{\sqrt{l(l+1)}}{\sqrt{(2l+1)(2l+3)}}Y_{l+1\,0} \\ iG^1_n\frac{\sqrt{l^2-1}}{\sqrt{4l^2-1}}Y_{l-1\,1}+iG^2_n\frac{\sqrt{l(l+2)}}{\sqrt{(2l+1)(2l+3)}}Y_{l+1\,1} \end{pmatrix}$$

and

$$\tilde{\eta}_{n\,l}(\mathcal{R},-1/2)=\begin{pmatrix} F_nY_{l\,-1} \\ 0 \\ -iG^1_n\frac{\sqrt{l^2-1}}{\sqrt{4l^2-1}}Y_{l-1\,-1}-iG^2_n\frac{\sqrt{l(l+2)}}{\sqrt{(2l+1)(2l+3)}}Y_{l+1\,-1} \\ iG^1_n\frac{\sqrt{l(l+1)}}{\sqrt{4l^2-1}}Y_{l-1\,0}-iG^2_n\frac{\sqrt{l(l+1)}}{\sqrt{(2l+1)(2l+3)}}Y_{l+1\,0} \end{pmatrix}.$$

This description corresponds to the representation in [15]. $\tilde{\xi}_{n\,l}(\mathcal{R},m=1/2)$ (analogously: $\tilde{\xi}_{n\,l}(\mathcal{R},-1/2)$, $\tilde{\eta}_{n\,l}(\mathcal{R},1/2)$, $\tilde{\eta}_{n\,l}(\mathcal{R},-1/2)$) satisfies the eigenvalue equation

$$(\gamma_{\bar{\mu}}\hbar\frac{\partial}{\partial x_{\bar{\mu}}}-\frac{\acute{e}^2}{\acute{c}r}\gamma_4+\acute{M}\acute{c})\tilde{\xi}_{n\,l}(\mathcal{R},1/2)=\gamma_4\bar{E}_{n\,l}\tilde{\xi}_{n\,l}(\mathcal{R},1/2). \tag{3.15}$$

One finds in [15] the the normalising factors. It becomes 1 if one neglects the small influence of G_n^k. This enables the use of a relation to the related Schrödinger equation. We can write

$$\hat{\xi}_{n\,l}(\mathcal{R},1/2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xi_{n\,l},\ \hat{\xi}_{n\,l}(\mathcal{R},-1/2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xi_{n\,l} \tag{3.16}$$

and

$$\hat{\eta}_{n\,l}(\mathcal{R},1/2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \eta_{n\,l}\ ,\ \hat{\eta}_{n\,l}(\mathcal{R},-1/2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \eta'_{n\,l} \tag{3.17}$$

where the expressions

$\xi_{n\,l} = F_n(r) Y_{l\,0}(\grave{\vartheta},\grave{\varphi}),$

$\eta_{n\,l} = F_n(r) Y_{l\,1}(\grave{\vartheta},\grave{\varphi})$

and

$\eta'_{n\,l} = F_n(r) Y_{l\,-1}(\grave{\vartheta},\grave{\varphi})$

are eigenfunctions of the following Schrödinger equation

$$\begin{gathered}(E_{n\,l} + \hbar^2/(2\acute{M})((1/r^2)\partial/\partial r(r^2\partial/\partial r) \\ -l(l+1)/(r^2)) + \acute{e}^2/r)F_n(r)Y_{l\,0} = 0\ ,\ \mathit{or}\ (....)F_n(r)Y_{l\,\pm 1} = 0\ .\end{gathered} \tag{3.18}$$

Here the spin orientation has no influence on $E_{n\,l}$. This influence can be added what is described in [18]. Using these solutions, the functions of eq.(3.9) are

$$\begin{aligned}
\hat{U}(\mathcal{R},t,1/2) &= \sum_{n\,l}(b'_{n\,l}(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xi_{n\,l}, \\
\hat{V}(\mathcal{R},t,-1/2) &= \sum_{n\,l} b''_{n\,l} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xi_{n\,l}, \\
\check{U}(\mathcal{R},t,1/2) &= \sum_{n\,l} d'_{n\,l}(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \eta_{n\,l}, \\
\check{V}(\mathcal{R},t,-1/2) &= \sum_{n\,l} d''_{n\,l}(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \eta'_{n\,l}.
\end{aligned} \tag{3.19}$$

For the linked solution of the Schrödinger equation one obtains such as the sum

$$u = \sum_{n,l} b'_{nl}(t)\xi_{nl}. \tag{3.20}$$

In the following we replace n and l by one number k, with a fixed relation k=k(n,l), and the functions of the eqs.(3.9) by the sum

$$U = \sum_{k} c_k(t) H_k \tag{3.21}$$

with

$$H_k = \begin{pmatrix} \hat{h}_k \\ \check{h}_k \end{pmatrix}. \tag{3.22}$$

Using the linked eigenfunctions h_k of the eq.(3.18), it becomes

$$u = \sum_k a_k(t) h_k. \tag{3.23}$$

with $c_k(t) = a_k(t) \exp(-i \acute{M} \acute{c}^2 t/\hbar)$. As an example we insert in the eqs.(3.20/3.21) the expressions

$$\hat{h}_k = \begin{pmatrix} 1 \\ 0 \end{pmatrix} h_k \tag{3.24}$$

and $\bar{a}_{n\,l} = a_k$. This requires a relativistic correction of the eigenvalue which is described in [15, 18]. The formulas of $\check{h}_k$ and the spin correction are presented in section 6.3.

In a transition state the functions are divided into two or more excitation levels. For these cases we introduce a number z which indicates the excitation level. It is

$$U^z = \sum_k c_k^z(t) H_k^z. \tag{3.25}$$

Using the new notation of the new quantum numbers, the eigenvalue eqs.(2.4/2.5) read now

$$(\gamma_{\bar{\mu}} \hbar \frac{\partial}{\partial x_{\bar{\mu}}} - \frac{\acute{e}^2}{\acute{c} r} \gamma_4 + \acute{M} \acute{c}) H_k^z(\mathcal{R}) = \frac{1}{\acute{c}} \gamma_4 \bar{E}_k^z H_k^z(\mathcal{R}) \tag{3.26}$$

and

$$-\hbar \frac{\partial \bar{H}_p^z(\mathcal{R})}{\partial x_{\bar{\mu}}} \gamma_{\bar{\mu}} - \bar{H}_p^z(\mathcal{R})(\gamma_4 \frac{\acute{e}^2}{\acute{c}r} - \acute{M}\acute{c}) = \frac{1}{\acute{c}} \bar{H}_p^z(\mathcal{R})\gamma_4 \bar{E}_p^z \tag{3.27}$$

where H_k^z and $\bar{H}_p^z$ represent the eigenfunctions and $\bar{E}_k^z$ and $\bar{E}_p^z$ the eigenvalues. With eq.(3.21) the current

$$J_\mu(\mathcal{R}, t) = i\acute{e}\bar{U}\gamma_\mu U \tag{3.28}$$

becomes

$$J_{\bar{\mu}} = \acute{e}(\hat{u}^*\sigma_{\bar{\mu}}\breve{u} - \hat{u}^*\sigma_{\bar{\mu}}\hat{u}) \tag{3.29}$$

and with the approximation

$$\breve{u} \cong -(i\hbar/(2\acute{M}\acute{c}))\sigma_{\bar{\nu}}(\partial\,\hat{u}/\partial x_{\bar{\nu}}) \tag{3.30}$$

one obtains

$$\begin{aligned} J_{\bar{\mu}} = \acute{e}\hbar/(2\acute{M})(- i((\hat{u}^*(\partial\,\hat{u}/\partial x_{\bar{\mu}}) - (\partial \hat{u}^*/\partial x_{\bar{\mu}})\hat{u}) \\ +(rot(\hat{u}^*\overrightarrow{\sigma}\,\hat{u})). \end{aligned} \tag{3.31}$$

Here the relation

$$\begin{aligned} \hat{u}^*\overrightarrow{\sigma}\,\overrightarrow{\sigma}(\partial\,\hat{u}/\partial x_{\bar{\bar{\nu}}}) - (\partial\,\hat{u}^*/\partial x_{\bar{\nu}})\overrightarrow{\sigma}\,\overrightarrow{\sigma}\,\hat{u}) \\ = \hat{u}^*(\nabla\hat{u}) - (\nabla\hat{u}^*)u + irot(\hat{u}^*\overrightarrow{\sigma}\,\hat{u}) \end{aligned} \tag{3.32}$$

is applied [18] . The last term in eq.(3.32) has a vanishing divergence.

A relation between the solution of the Dirac equation and the Schrödinger equation can be derived in a similar way [18]. It starts with

$$(i\frac{\hbar\partial}{\acute{c}\partial t}+i\frac{\acute{e}}{\acute{c}}A_4-\acute{M}\acute{c})\hat{U}-(\sigma_{\bar{\mu}}(-i\hbar\frac{\partial}{\partial x_{\bar{\mu}}}-\frac{\acute{e}}{\acute{c}}A_{\bar{\mu}}))\check{U}=0, \tag{3.33}$$

and

$$(i\frac{\hbar\partial}{\acute{c}\partial t}+i\frac{\acute{e}}{\acute{c}}A_4+\acute{M}\acute{c})\check{U}-(\sigma_{\bar{\mu}}(-i\hbar\frac{\partial}{\partial x_{\bar{\mu}}}-\frac{\acute{e}}{\acute{c}}A_{\bar{\mu}}))\hat{U}=0. \tag{3.34}$$

Using the relation $i\frac{\hbar\partial}{\acute{c}\partial t}U=(\acute{M}\acute{c}+i\frac{\hbar\partial}{\acute{c}\partial t}\exp(i\acute{M}\acute{c}^2t/\hbar))U$, one obtains the approximation

$$\check{U}\cong\frac{1}{2\acute{M}\acute{c}}(\sigma_{\bar{\mu}}(-i\hbar\frac{\partial}{\partial x_{\bar{\mu}}}-\frac{\acute{e}}{\acute{c}}A_{\bar{\mu}}))\hat{U}. \tag{3.35}$$

Hence it becomes

$$(i\frac{\hbar\partial}{\acute{c}\partial t}+i\frac{\acute{e}}{\acute{c}}A_4-\acute{M}\acute{c})\hat{U}-\frac{1}{2\acute{M}\acute{c}}(i\hbar\frac{\partial}{\partial x_{\bar{\mu}}}+\frac{\acute{e}}{\acute{c}}A_{\bar{\mu}})^2\hat{U}+\frac{\acute{e}\hbar}{2\acute{M}\acute{c}^2}(\overrightarrow{\sigma}rot_{\bar{\mu}}A_{\bar{\mu}})\hat{U}=0 \tag{3.36}$$

or (see eq.2.14))

$$(i\frac{\hbar\partial}{\partial t}+i\acute{e}A_4-\frac{1}{2\acute{M}}(i\hbar\frac{\partial}{\partial x_{\bar{\mu}}}+\frac{\acute{e}}{\acute{c}}A_{\bar{\mu}})^2+\frac{\acute{e}\hbar}{2\acute{M}\acute{c}}(\overrightarrow{\sigma}\mathbf{H}))\hat{U}'=0 \tag{3.37}$$

with $\hat{U}'=\exp(i\acute{M}\acute{c}^2t/\hbar))\hat{U}$. Eq.(3.37) is the non-relativistic approximation of the Dirac equation in a magnetic field. It is named Pauli equation. $\hat{U}'$ has two components. It can be replaced by a normal function u if the term $\frac{\acute{e}\hbar}{2\acute{M}\acute{c}}\sigma_{\bar{\mu}}\mathbf{H}_{\bar{\mu}}$ is zero. That explains the relation between the Dirac and the Schrödinger equation with the mentioned difference.

All presented expressions serve the preparation of the following topics. We start with the influence of the self-fields on simple solutions of the Schrödinger equation. The self-fields are given by (see eq.2.6)

$$A_\mu(\mathcal{R},t) = \int \frac{1}{r_{01}}(J_\mu(\mathbf{R}_1, t - |\mathbf{R}_1 - \mathbf{R}|/c)dV_1. \qquad (3.38)$$

Then, the Dirac equation reads

$$[\gamma_\mu(\hbar\frac{\partial}{\partial x_\mu} - i\frac{\acute{e}}{\acute{c}}(A_\mu^K + A_\mu)) + \acute{M}\acute{c}]U = 0 \qquad (3.39)$$

where the potentials of the nucleus are given by

$A_4^K = -i\acute{e}/r, \quad A_{\bar{\mu}}^K = 0$

and the potentials A_μ are caused by the currents of the electron. The Schrödinger equation takes the form

$$(i\hbar\frac{\partial}{\partial t} + (\hbar^2/(2\acute{M}))\Delta + \acute{e}^2/r + i\acute{e}A_4)u(\mathcal{R},t) = 0 \qquad (3.40)$$

if the potentials $A_{\bar{\nu}}$ are neglected and only the quasi-static approximation

$$A_4(\mathcal{R},t) = i\acute{e}\int \frac{1}{r_{01}}u(\mathcal{R}_1,t)^*\, u(\mathcal{R}_1,t)dV_1 \qquad (3.41)$$

of the self-field is used. That means, a few contributions of the order $\acute{\alpha}^2$ have no influence.

First we consider the static solution of the basic level z=0 of a s-state. Then the coefficients in eq.(3.23) have the form

$$a_k^0(t) = |a_k^0| \exp(i\phi_k^0) \exp(-iE_0 t/\hbar) \tag{3.42}$$

and the eq.(3.40) leads to the system

$$(E_0 - \; E_k^0) a_k^0 - \sum_{m,k',m'} a_m^0 (a_{k'}^0)^* a_{m'}^0 ({}^{0\,0}_{k\,m} M^{0\,0}_{k'\,m'}) = 0 \tag{3.43}$$

with

$$({}^{0\,0}_{k\,n} M^{0\,0}_{k'\,m'}) \; = \acute{e}^2 \int \frac{1}{r_{01}} (h_k^0(\mathcal{R}))^* \, h_m^0(\mathcal{R}) (h_{k'}^0(\mathcal{R}_1))^* \, h_{m'}^0(\mathcal{R}_1) dV_1 dV. \tag{3.44}$$

E_k^0 represent the eigenvalues of the Schrödinger equation. The system eq.(3.43) can be solved numerically by varying E_0 and a_k. Fig.1 and Fig.2 show a few of such examples. Static solutions are only possible for $E_0 < 0$. An eigenfunction can be used as first approximation and as main term of a solution u. Therefore we characterise these solution by the quantum number of the main term. The energy

E_0 is less negative than the eigenvalue of the main term.

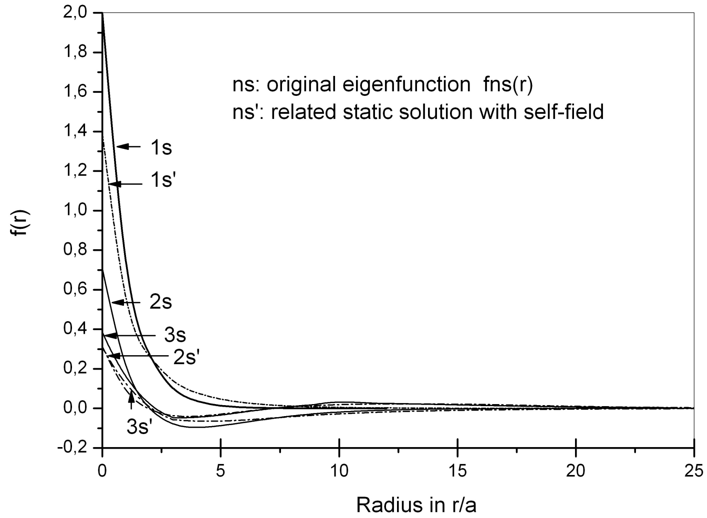

Fig. 1: Comparison of static solutions

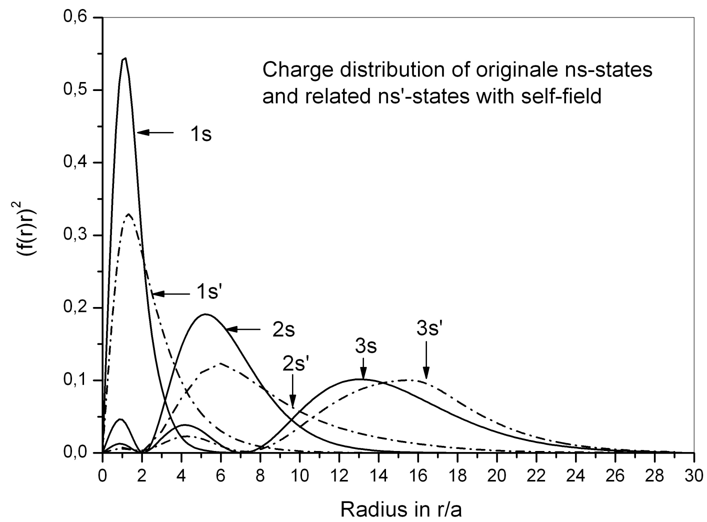

Fig. 2: Comparison of charge distributions of static solutions

3.2 *Dynamic solutions of pure s-states*

In dynamic cases the coefficients $a_k^0(t)$ are functions of time and eq.(3.43) takes the form

$$i\hbar\frac{\partial}{\partial t}a_k^0(t) = E_k^0 a_k^0(t) + \sum_{m,k',m'} a_m^0(t)a_{k'}^0(t)^* a_{m'}^0(t)({}^{0\,0}_{k\,m}M^{0\,0}_{k'\,m'}), \tag{3.45}$$

as long as only one excitation level z=0 is involved. We apply for numerical calculations the following time definition $t' = \acute{\alpha}(\acute{c}/\acute{a})t$ with the unit $\Delta t = \acute{a}/(\acute{c}\,\acute{\alpha}) = 0.24167 * 10^{-16}s$. $\acute{a} = \hbar^2/(\acute{M}\acute{e}^2) = 5.292 * 10^{-9}cm$ describes the Bohr-radius and the used energy unit is $\acute{e}^2/\acute{a} = \acute{\alpha}^2\,\acute{M}\,\acute{c}^2 = 27.21eV$. That leads to $\hbar/\Delta t = \acute{e}^2/\acute{a}$.

The total energy of the system is determined by an assumed initial distribution of $u^0(\mathcal{R}, t'_0)$, as long as the loss by radiation can be neglected. Such fields are not static, unless the energy is a minimum. In eq.(3.18) the expression $Y_{l\,0}$ can be separated which results in

$$\tfrac{\acute{e}^2}{\acute{a}}(i\tfrac{\partial}{\partial t'} + \tfrac{1}{2}\tfrac{\acute{a}^2}{r^2}\,\tfrac{\partial}{\partial r}r^2\tfrac{\partial}{\partial r}\; + \tfrac{\acute{a}}{r} - \tfrac{1}{2}l(l+1)\tfrac{\acute{a}^2}{r^2}\; - \;\acute{a}A'_4(r,t))f_{l\,0}^0(r,t') = 0\ . \tag{3.46}$$

Here the mentioned dimensions are introduced. That means, it is $(\hbar/\Delta t)\frac{\Delta t\partial}{\partial t} = (\acute{e}^2/\acute{a})\frac{\partial}{\partial t'}$, $\hbar^2/(\acute{a}^2\acute{M}) = (\acute{e}^2/\acute{a})$, $\hbar\omega = \frac{\hbar 2\pi}{T} = (\acute{e}^2/\acute{a})\frac{2\pi}{T'} = (\acute{e}^2/\acute{a})\omega'$ and $-i\acute{e}A_4 = (\acute{e}^2/\acute{a})\acute{a}A'_4$.

We consider first a purely rotationally symmetrical solution (l=0, s-state). The solution of the Schrödinger equation has the form:

$$u^0 = f_{0\,0}^0(r,t')Y_{0\,0} \tag{3.47}$$

with

$$f^0_{0\,0}(r,t') = \sum_k a^0_k(t')\, h^0_k(r). \tag{3.48}$$

Due to eq.(3.18) and l=0, the functions $h^0_k(r)$ are the eigenfunctions of the equation

$$\frac{é^2}{á}((E^0_k)' + \frac{1}{2}\frac{á^2}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{á}{r})h^0_k(r) = 0\ . \tag{3.49}$$

These can be used to solve the equation

$$\frac{é^2}{á}(i\frac{\partial}{\partial t'} + \frac{1}{2}\frac{á^2}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{á}{r} - áA^{0\,0}_{0\,0}(r,t'))f^0_{0\,0}(r,t') = 0\ . \tag{3.50}$$

In preparation for the description of transition fields, the high index 0 is introduced to characterise the excitation level. Hence the charge density is given by

$$J^{0\,0}_4 = ié\rho^{0\,0}_{0\,0}(r,t')\, Y_{0\,0}Y_{0\,0} \tag{3.51}$$

with

$$\rho^{0\,0}_{0\,0} = (f^0_{0\,0}(r,t')^* f^0_{0\,0}(r,t'). \tag{3.52}$$

The quasi-static component of the vector potential has the form

$$A_{0\,0}^{0\,0}(r,t') = (I_{0\,0}^{0\,0}(\infty,t') - \int\limits_0^r 1/y^2 \int\limits_0^y x^2 \rho_{0\,0}^{0\,0}(x,t'))dx\,dy) \tag{3.53}$$

which results from the integration of

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\, A_{0\,0}^{0\,0}(r,t') = \rho_{0\,0}^{0\,0}(r,t'). \tag{3.54}$$

Eq.(3.50) therefore leads to

$$i\tfrac{d}{dt'} a_k^0(t') = (E_k^0)' a_k^0(t') + \,{}_{k}^{0\,0}M^{0\,0}(t') \tag{3.55}$$

with

$${}_{k}^{0\,0}M^{0\,0}(t') \;=\; \int_0^\infty \; r^2 (h_k^0(r))^* \, f_{0\,0}^0(r,t') á \, A_{0\,0}^{0\,0}(r,t')\, dr\,. \tag{3.56}$$

Here is $\frac{é^2}{á}\,{}_{k}^{0\,0}M^{0\,0}(t') \;=\; \Sigma_{m,k',m'}\, a_m^0(t') a_{k'}^0(t')^* a_{m'}^0(t') ({}_{k\,m}^{0\,0}M_{k'\,m'}^{0\,0})$.

The Fourier transformation of this expression reads

$$({}_{k}^{0\,0}mx_l^{0\,0}) = \tfrac{1}{T'} \int_{t_0'}^{t_0'+T'} \exp(il\omega'\,(t'-t_0')\,{}_{k}^{0\,0}M^{0\,0}(t')\,dt'. \tag{3.57}$$

With this equation and the relation

$$a_k^0(t') = \sum_l {}^0d_k^l \, \exp(-\,i\,l\,\omega'\,(t' \,-\, t_0')), \tag{3.58}$$

the transformation of eq.(3.50) results in

$$\begin{aligned} \Sigma_l\ \omega'\,l\,d_k^l\,\exp(-i\ l\,\omega'\ (t'\ -t_0')) = \ (E_k^0)'\,\Sigma_l\ {}^0d_k^l\ \ \exp(-i\,l\,\omega'\ (t'\ -t_0')) \\ +\ \Sigma_l\ ({}_k^{0\,0}mx_l^{0\,0})\,\exp(-\,i\,l\,\omega'\ (t'\ -t_0')\,). \end{aligned} \tag{3.59}$$

The eq.(3.59) for the coefficients of the eigenfunction h_k^0 must be valid for all l (here not a quantum number) in the interval $t_0' \ \le\ t' \ \le\ T' + t_0'$. If the time functions $f_{0\,0}^0(\mathcal{R}, t')$, respectively $a_k^0(t')$, are obtained by numerical integration of eq.(3.50/3.55)), the coefficients ${}^0d_k^l$ and $({}_k^{0\,0}mx_l^{0\,0})$ can be calculated using the Fourier transformation.

For the solution of eq.(3.50) or eq.(3.55) one needs a complete system of eigenfunctions $h_k^0(r)$ which also contains energy values $(E_k^0)' > 0$. The known methods for determining such functions cannot be used, since they are only in range $0 < r < \infty$ orthogonal. Numerical calculations are not possible for such ranges. Therefore a program was developed that allows to calculate eigenfunctions h_k^z in the range $0 < r < RR$ for values $(E_k^z)' < 4$ and large RR. For low energy $(E_k^z)' < 0$ the functions h_k^z take on forms similar to the known radial eigenfunctions of the s-states. A same program exists for p-states. The algorithms are described in the appendices C1 and C2. Results of several calculations are documented in the Fig.1-Fig.3 and in the Table 1 - Table 5. Fig.1 shows the radial functions $f_k(r)$ of a few static s-states with and without self-field and Fig. 2 the linked charge distributions $r^2 f_k(r)^* f_k(r)$. You can see that the form of the functions with and without self-field are similar. The number of zero-points is the same, however the radial charge distributions are expanded. In static cases the Fourier coefficients are reduced to

$$a_k^0(t') \cong \ {}^0d_k^{\bar{l}} \ \exp(-\,i\,\bar{l}\omega'(t' - t_0')), \quad E' = \bar{l}\omega'\,,$$
$${}^0d_k^l \ = \ 0 \ \ for \ \ l \neq \bar{l}, \ \ \omega' = 2\pi/T'. \tag{3.60}$$

The eigenfunction of the dominant coefficient a_k^0 is the eigenfunction of the linked solution without self-field. For example:
$|{}^0d_1^{\bar{l}}| < 1$ dominates the sum of $1s'$, (${}^0d_1^{\bar{l}}$=1 in 1s), $|{}^0d_2^{\bar{l}}|$ in 2s (${}^0d_2^{\bar{l}} = 1$ in 2s), etc. The solution and the charge oscillate around a mean distribution if the total energy eg exceeds the static minimum. This behaviour can be characterised as a fixation of the electron field in an energy trough. The Fig. 3 shows the average charge distribution for various total energies eg. Characteristic parameters of the vibrations around the average distributions are documented in Tabl.2 and Tabl.3. These follow from

$$ed(t') = \tfrac{\acute{e}^2}{\acute{a}} \textstyle\sum_k (a_k^0(t'))^* a_k^0(t')(E_k^0)',$$
$$ea(t') = \tfrac{\acute{e}^2}{2\acute{a}} \textstyle\sum_{k,m,k',m'} (a_k^0(t'))^* a_m^0(t')(a_{k'}^0(t'))^* a_{m'}^0(t') ({}^{0\,0}_{k\,m}M^{0\,0}_{k'\,m'}),$$
$$eh(t') = ed(t') + 2ea(t'),$$
$$eg(t') = ed(t') + ea(t'). \tag{3.61}$$

$eh \ = \ E'$ corresponds to the Hamilton energy in static cases. In addition, the mean radii $R/\acute{a}$ of charge distributions and the dimensionless potentials $A(0)$ for r=0 are presented. Magnitudes of vibrations are characterised by mean deviation Δ of the documented parameters. Tabl. 3 shows such dynamic parameters of solution functions for several total energies eg. One can see that a rising value of the total energy eg leads to stronger vibration, whereas in static limit the dynamics is negligible. An external radiation field has influence on this vibration due to the interaction terms. Maybe that explains the photo effect.

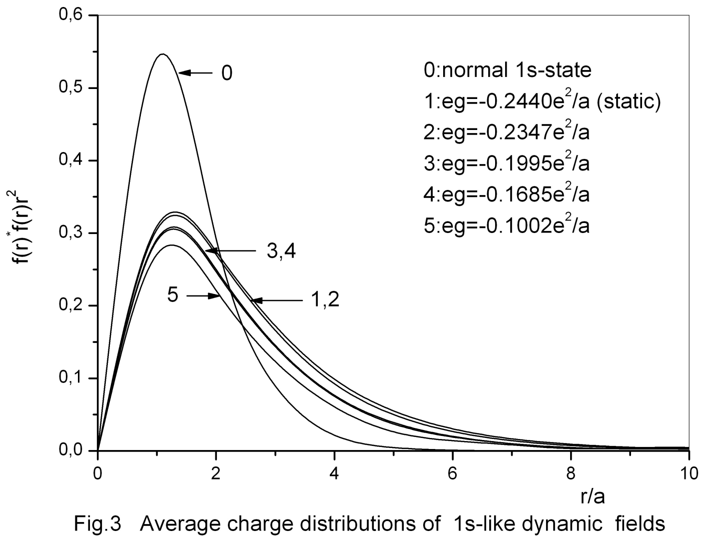

Fig. 3: Average charge distributions of dynamic 1s-like states

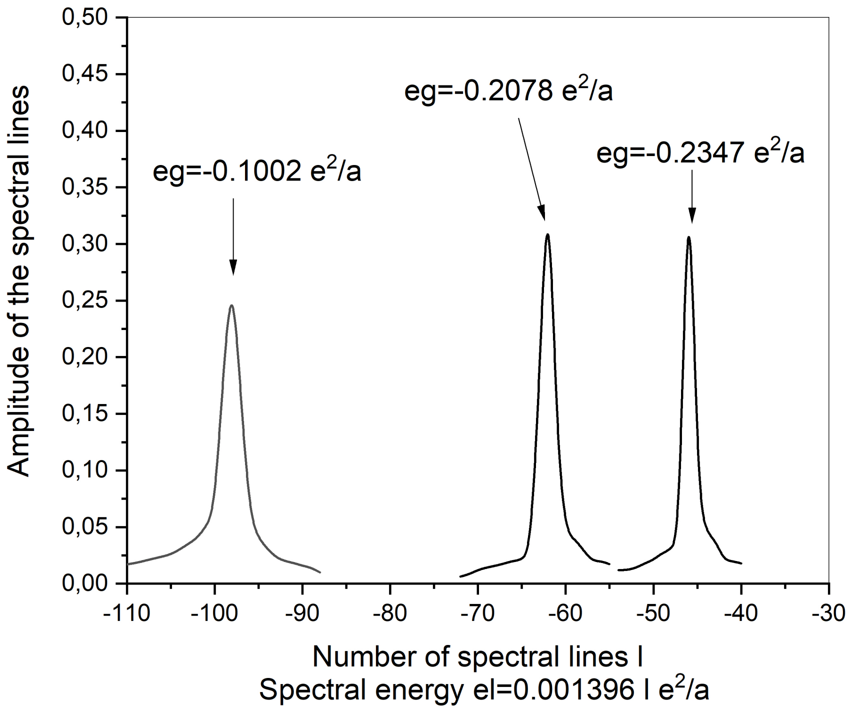

Fig. 4: Spectral distributions of dynamic 1s-like states

Table 1 (RR=100$\acute{a}$, energy parameter of static s-states in $\acute{e}^2/\acute{a}$)

1s:

eg	eh	ed	ea	A(0)	$R/\acute{a}$
-0.2440	-0.0476	-0.4403	0.1963	0.7247	2.525

2s:

eg	eh	ed	ea	A(0)	$R/\acute{a}$
-0.06262	-0.0072	-0.1161	0.05347	0.1564	8.74

3s:

eg	eh	ed	ea	A(0)	$R/\acute{a}$
-0.02416	0.0062	-0.05442	0.03026	0.09534	14.83

Table 2 (RR=100$\acute{a}$, parameter of dynamic 1s-states)

1.: eg=-0.2440$\acute{e}^2/\acute{a}$

eg	eh	ed	ea	$A(0)$	$R/\acute{a}$
Δeg	Δeh	Δed	Δea	$\Delta A(0)$	$\Delta R/\acute{a}$
-0.2440	-0.0476	-0.4403	0.1963	0.7247	2.525
0.0	0.0	0.0	0.0	0.0	0.0

2.:eg=-0.2347$\acute{e}^2/\acute{a}$

eg	eh	ed	ea	$A(0)$	$R/\acute{a}$
Δeg	Δeh	Δed	Δea	$\Delta\bar{A}(0)$	$\Delta R/\acute{a}$
-0.2347	-0.0478	-0.4216	0.1869	0.7067	4.45
0.0	0.0058	0.0058	0.0059	0.0170	0.66

3.:eg=-0.2078$\acute{e}^2/\acute{a}$

eg	eh	ed	ea	$A(0)$	$R/\acute{a}$
Δeg	Δeh	Δed	Δea	$\Delta A(0)$	$\Delta R/\acute{a}$
-0.1995	-0.0867	-0.3689	0.1611	0.6645	8.10
0.0	0.0099	0.0102	0.0100	0.035	1.48

4.:eg=-0.1685$\acute{e}^2/\acute{a}$

eg	eh	ed	ea	$A(0)$	$R/\acute{a}$
Δeh	Δea	Δed	ΔE_a	$\Delta\bar{A}(0)$	$\Delta R/\acute{a}$
-0.1685	-0.0107	-0.3263	0.1578	0.6589	10.05
0.0	0.0088	0.0090	0.0090	0.0377	0.79

5.:eg=-0.1363$\acute{e}^2/\acute{a}$

eg	eh	ed	ea	$A(0)$	$R/\acute{a}$
Δeg	Δeh	Δed	Δea	$\Delta A(0)$	$\Delta R/\acute{a}$
-0.1363	0.0057	-0.2783	0.1420	0.6293	13.12
0.0	0.0095	0.010	0.0098	0.0449	0.72

6.:eg=-0.1003$\acute{e}^2/\acute{a}$

eg	eh	ed	ea	$\bar{A}(0)$	$R/\acute{a}$
Δeg	Δeh	Δed	Δea	$\Delta\bar{A}(0)$	$\Delta R/\acute{a}$
-0.1003	0.0313	-0.2319	0.1316	0.6080	14.78
0.0	0.0089	0.094	0.0091	0.0462	1.72

Table 3:
Fluctuations $\Delta FF(r)$ of dynamic 1s-states ($FF(r) = r^2 f^*(r) f(r)$)

1.: eg=-0.2440$\acute{e}^2/\acute{a}$

$FF(1)$	$FF(2)$	$FF(3)$	$FF(4)$	$FF(5)$
$\Delta FF(1)$	$\Delta FF(2)$	$\Delta FF(3)$	$\Delta FF(4)$	$\Delta FF(5)$
0.5413	0.2931	0.0892	0.0093	0.0045
0.0	0.0	0.0	0.0	0.0

2.:eg=-0.2347$\acute{e}^2/\acute{a}$

$FF(1)$	$FF(2)$	$FF(3)$	$FF(4)$	$FF(5)$
$\Delta FF(1)$	$\Delta FF(2)$	$\Delta FF(3)$	$\Delta FF(4)$	$\Delta FF(5)$
0.3050	0.2706	0.1560	0.0924	0.050
0.0113	0.0083	0.0032	0.0032	0.072

3.:eg=-0.2078$\acute{e}^2/\acute{a}$

$FF(1)$	$FF(2)$	$FF(3)$	$FF(4)$	$FF(5)$
$\Delta FF(1)$	$\Delta FF(2)$	$\Delta FF(3)$	$\Delta FF(4)$	$\Delta FF(5)$
0.2918	0.2502	0.1453	0.0765	0.0390
0.0241	0.0132	0.0110	0.0148	0.0126

4.:eg=-0.1685$\acute{e}^2/\acute{a}$

$FF(1)$	$FF(2)$	$FF(3)$	$FF(4)$	$FF(5)$
$\Delta FF(1)$	$\Delta FF(2)$	$\Delta FF(3)$	$\Delta FF(4)$	$\Delta FF(5)$
0.2894	0.2470	0.1435	0.0750	0.0370
0.0280	0.0119	0.0108	0.0169	0.0101

5.:eg=-0.1363$\acute{e}^2/\acute{a}$

$FF(1)$	$FF(2)$	$FF(3)$	$FF(4)$	$FF(5)$
$\Delta FF(1)$	$\Delta FF(2)$	$\Delta FF(3)$	$\Delta FF(4)$	$\Delta FF(5)$
0.2789	0.2320	0.1312	0.0665	0.0307
0.0339	0.0094	0.0248	0.0201	0.0095

6.:eg=-0.1003$\acute{e}^2/\acute{a}$

$FF(1)$	$FF(2)$	$FF(3)$	$FF(4)$	$FF(5)$
$\Delta FF(1)$	$\Delta FF(2)$	$\Delta FF(3)$	$\Delta FF(4)$	$\Delta FF(5)$
0.2708	0.2216	0.1224	0.0605	0.0263

0.0353	0.0115	0.0115	0.0198	0.0072

3.3 *Dynamic solutions of pure p-states*

Due to the influence of the potentials, a solution of p-cases can be described by

$$u^1 = f^1_{1\,0}(r,t')Y_{1\,0} + f^1_{3\,0}(r,t')Y_{3\,0} + \ ... \ . \tag{3.62}$$

We put z=1 to prepare for the transition states. The function $f^1_{3\,0}$ should be small, because it is only excited by A_4. Therefore we consider the following charge distribution

$$J_4^{1\,1} = i\acute{e}(\rho^{1\,1}_{0\,0}(r,t')\,Y_{0\,0}Y_{0\,0} + \rho^{1\,1}_{2\,0}(r,t')\,Y_{2\,0}Y_{0\,0}) \tag{3.63}$$

with

$$\rho^{1\,1}_{0\,0} = f^1_{1\,0}(r,t')^* f^1_{1\,0}(r,t') \tag{3.64}$$

and

$$\rho^{1\,1}_{2\,0} = \frac{2}{\sqrt{5}}\, f^1_{1\,0}(r,t')^* f^1_{1\,0}(r,t'). \tag{3.65}$$

These relations are a consequence of

$$(Y_{1\,0})^2 \;=\; (Y_{0\,0})^2 \;+\; \frac{2}{\sqrt{5}}\,Y_{2\,0}Y_{0\,0}. \tag{3.66}$$

Therefore the quasi-static component of the vector potential A_4 is given by:

$$-i\acute{e}A_4(\mathcal{R},t') = (\acute{e}^2/\acute{a})(\acute{a}\, A^{1\,1}_{0\,0}(r,t') + \frac{2\acute{a}}{\sqrt{5}}\,(Y_{2\,0}/Y_{0\,0})\, A^{1\,1}_{2\,0}(r,t')). \qquad (3.67)$$

Here the functions are

$$A^{1\,1}_{0\,0}(r,t') = (I^{1\,1}_{0\,0}(\infty,t') - \int\limits_0^r 1/y^2 \int\limits_0^y x^2 \rho^{1\,1}_{0\,0}(x,t'))dx\,dy) \qquad (3.68)$$

and

$$A^{1\,1}_{2\,0}(r,t') = \; r^2\,(I^{1\,1}_{2\,0}(\infty,t') - \int\limits_0^r 1/y^6 \int\limits_0^y x^4 \rho^{1\,1}_{2\,0}(x,t'))dx\,dy). \qquad (3.69)$$

Eq.(3.69) follows from the integration of

$$(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} \;-\; \frac{6}{r^2})\, A^{1\,1}_{2\,0}(\mathcal{R},t') = \rho_{2\,0}(r,t'). \qquad (3.70)$$

If the other components of the vector potential can be neglected, the Schrödinger equation decompose into

$$\begin{aligned} i\tfrac{\partial}{\partial t'} f^1_{1\,0}(r,t')Y_{1\,0} = \; -(\tfrac{\acute{a}^2}{2}\tfrac{\partial}{r^2\partial}\tfrac{r^2\partial}{\partial r} - \tfrac{\acute{a}^2}{r^2} + \tfrac{\acute{a}}{r}) f^1_{1\,0}(r,t')Y_{1\,0} \\ +\acute{a}A^{1\,1}_{0\,0}(r,t) f^1_{1\,0}(r,t')Y_{1\,0} \;+\; \tfrac{2\acute{a}}{\sqrt{5}}\, A^{1\,1}_{2\,0}(r,t') f^1_{1\,0}(r,t')Y_{1\,0} + ... \end{aligned} \qquad (3.71)$$

and

$$i\frac{\partial}{\partial t'}f^1_{30}(r,t')Y_{30} = -\left(\frac{\acute{a}^2}{2}\frac{\partial}{r^2\partial}\frac{r^2\partial}{\partial r} - \frac{6\acute{a}^2}{r^2} + \frac{\acute{a}}{r}\right)f^1_{30}(r,t')Y_{30}$$
$$+\acute{a}A^{1\,1}_{0\,0}(r,t)f^1_{30}(r,t')Y_{30} + 3\acute{a}\sqrt{\frac{3}{35}}\,A^{1\,1}_{2\,0}(r,t')f^1_{10}(r,t')Y_{30} + ..\ . \tag{3.72}$$

We describe $f^1_{10}(r,t')$ by the sum

$$f^1_{10}(r,t') = \sum_k a^1_k(t')\,h^1_k(r) \tag{3.73}$$

where $h^1_k(r)$ are eigenfunctions of

$$\frac{\acute{e}^2}{\acute{a}}((E^1_k)' + \frac{1}{2}\frac{\acute{a}^2}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{\acute{a}}{r} - \frac{\acute{a}^2}{r^2})h^1_k(r) = 0\ . \tag{3.74}$$

If one neglects the influence of $f^1_{30}Y_{30}$, the transformation of eq.(3.71) leads to

$$i\frac{d}{dt'}a^1_k(t') = (E^1_k)'a^1_k(t') + {}^{1\,1}_{k}M^{1\,1}(t'). \tag{3.75}$$

Therefore the dynamic behaviour of the coefficient $a^1_k(t'_0)$ results from the following relation

$$a^1_k(t'_0 + \bar{\Delta}t') = a^1_k(t'_0)$$
$$-i\,\bar{\Delta}t'\,((E^1_k)'\,a^1_k(t'_0) + {}^{1\,1}_{k}M^{1\,1}(t'_0)), \tag{3.76}$$

what can be solved by integration. The Fourier transformation of eq.(3.76) yields

$$
\begin{aligned}
&\textstyle\sum_l \omega' \, l1 \; {}^1d_k^{l1} \exp(-i \; l1 \, \omega' \; (t' \; - t_0')) \\
&= \; (E_k^1)' \textstyle\sum_{l1} \; {}^1d_k^{l1} \; \exp(-i \, l1 \, \omega' \; (t' \; - t_0')) \\
&+ \textstyle\sum_{l1} \, ({}_k^{1\,1}mx_{l1}^{1\,1}) \exp(- \, i \, l1 \, \omega' \; (t' \; - t_0') \,).
\end{aligned}
\tag{3.77}
$$

Here the notations

$$
a_k^1(t') = \sum_{l1} \, {}^1d_k^{l1} \; \exp(- \, i \, l1 \, \omega' \, (t' \; - t_0')), \tag{3.78}
$$

$$
{}_k^{1\,1}M^{1\,1}(t') = \; á \int_0^R r^2 \, (h_k^1(r))^* f_{1\,0}^1(r,t') \, (A_{0\,0}^{1\,1}(r,t') + (2/\sqrt{5}) \, A_{2\,0}^{1\,1}(r,t')) \, dr \tag{3.79}
$$

and

$$
({}_k^{1\,1}mx_{l1}^{1\,1}) = \; \tfrac{1}{T'} \textstyle\int_{t_0'}^{t_0'+T'} \exp(il1\omega' \, (t' - t_0')) \; {}_k^{1\,1}M^{1\,1}(t') \, dt' \tag{3.80}
$$

are introduced. It is also possible to describe ${}_k^{1\,1}M^{1\,1}(t')$ as function of $a_k^1(t')$. However, the eq.(3.79) was applied for numerical calculations. One has to calculate the functions $f_{1\,0}^1(r,t')$ in interval between t_0' and $T' + t_0'$ by numerical integration of the eqs.(3.75). Then, the coefficients ${}^1d_k^{l1}$ and $({}_k^{1\,1}mx_{l1}^{1\,1})$ can be determined using Fourier transformations. The results must satisfy the eq.(3.77) for all l1. The energy of the Dirac field ed is given by

$$
ed(t') \; = \frac{é^2}{á} \sum_k (E_k^1)' \, a_k^1(t')^* a_k^1(t') \tag{3.81}
$$

and the quasi-static part of the electromagnetic energy $ea(t) \cong -i\frac{\acute{e}}{2}\int A_4 f^1_{1\,0} Y_{1\,0} dV$ by

$$ea(t') \cong \frac{\acute{e}^2}{2\acute{a}} \int_0^R r^2(\acute{a}A^{0\,0}_{0\,0}(r,t') + (4/5)\,\acute{a}A^{1\,1}_{2\,0}(r,t'))(f^1_{1\,0}(r,t'))^* \; f^1_{1\,0}(r,t')dr. \qquad (3.82)$$

In eq.(3.82) the value 4/5 results from the integral $\int (Y_{1\,0}(\vartheta')^2 Y_{2\,0)}(\vartheta')/Y_{0\,0})$ $\times \sin(\vartheta')d\vartheta' d\varphi' = 4/5$. The laws of conservations built with these expressions remain valid within the context of the applied exactness. That has to be proved by numerical calculations. The results of such calculations for p-examples are documented in Fig.7- Fig.10 and Table 4 - Table 6. All expressions have the same meaning as in the s-cases.

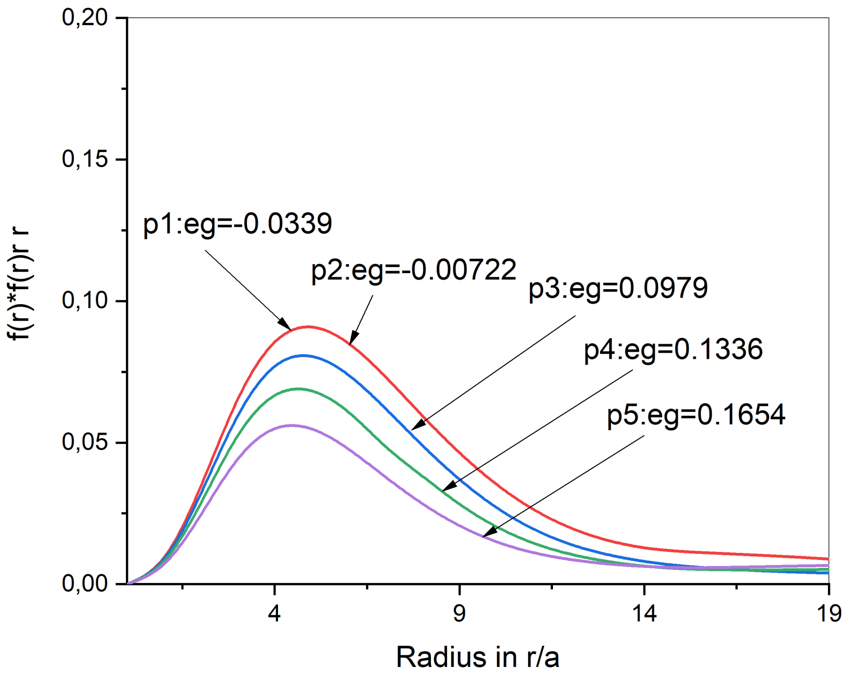

Fig. 5: Average charge distributions of dynamic 2p-like states

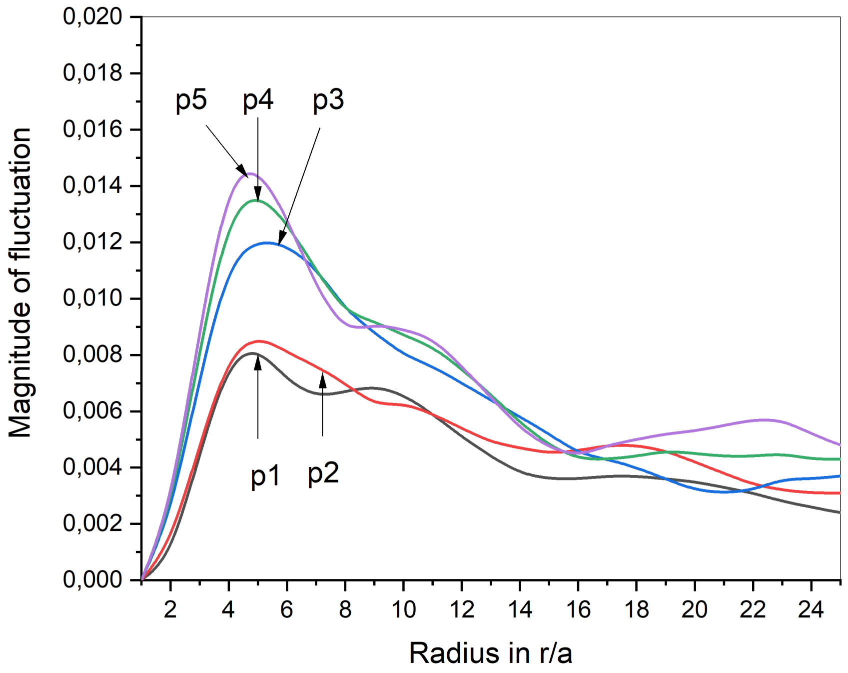

Fig. 6: Relative fluctuation of charge distributions of dynamic 2p-like states

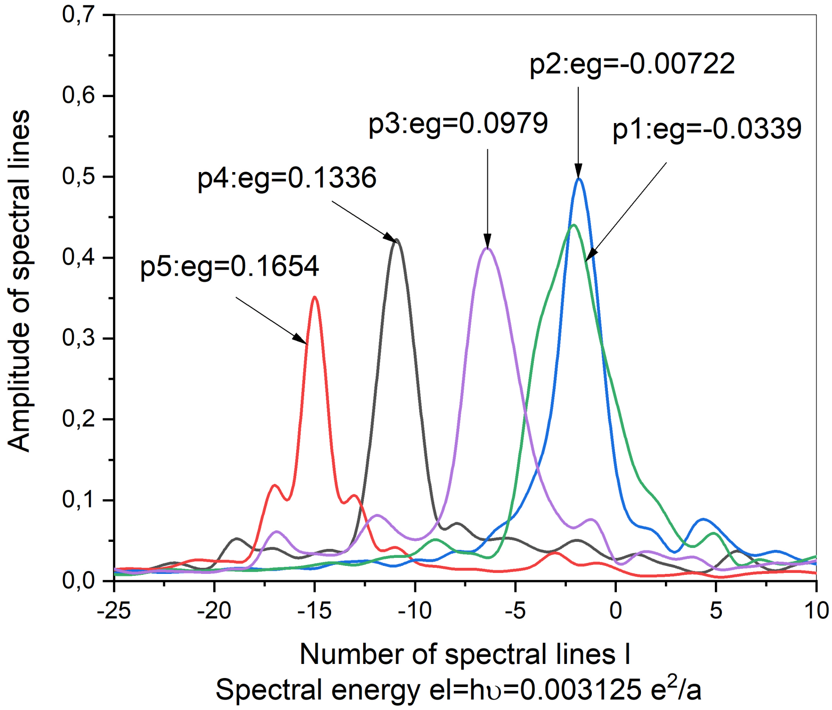

Fig. 7: Spectral distributions of dynamic 2p-like states

Table 4 (RR=200$\acute{a}$, parameter of dynamic 2p-like-states)

p1.: eg=-0.0339$\acute{e}^2/\acute{a}$

eg	ed	ea	$A(0)$	$R/\acute{a}$
Δeg	Δed	ΔE_a	$\Delta A(0)$	$\Delta R/\acute{a}$
-0.0339	-0.0739	0.0400	0.1400	34.52
0.0	0.0026	0.0036	0.0072	0.98

p2.: eg=-0.00722$\acute{e}^2/\acute{a}$

eg	ed	ea	$A(0)$	$R/\acute{a}$

Δeg	Δed	ΔE_a	$\Delta A(0)$	$\Delta R/á$
-0.0072	-0.0491	0.0419	0.1430	32.0
0.0	0.0020	0.0029	0.0067	1.61

p3.: eg=0.0979$é^2/á$

eg	ed	ea	$A(0)$	$R/á$
Δeg	Δed	ΔE_a	$\Delta A(0)$	$\Delta R/á$
0.0979	0.0661	0.0318	0.126	44.61
0.0	0.0025	0.0025	0.0089	2.80

p4.: eg=0.1336$é^2/á$

eg	ed	ea	$A(0)$	$R/á$
Δeg	Δed	ΔE_a	$\Delta A(0)$	$\Delta R/á$
0.1336	0.1091	0.0245	0.1109	52.07
0.0	0.0022	0.0022	0.0101	1.17

p5.: eg=0.1654$é^2/á$

eg	ed	ea	$A(0)$	$R/á$
Δeg	Δed	ΔE_a	$\Delta A(0)$	$\Delta R/á$
0.1654	0.1473	0.0182	0.0950	60.86
0.0	0.0021	0.0021	0.0111	3.37

3.4 *Solutions of transition states*

3.4.1 *General characteristics of a transition solution*

Now we consider solutions formed from two excitation levels. Several of the above results can be applied to solve the Schrödinger equation using the approaches

$$u^0(\mathbf{R}, t') = (Y_{0\,0} f^0_{0\,0}(r, t') + Y_{2\,0} f^0_{2\,0}(r, t') + \dots\,, \tag{3.83}$$

and

$$u^1(\mathbf{R}, t') = Y_{1\,0} f^1_{1\,0}(r, t') + Y_{3\,0} f^1_{3\,0}(r, t') + \dots\,. \tag{3.84}$$

Then one obtains the following expression of the charge distribution

$$\begin{aligned} J_4^{0\,0} + J_4^{1\,1} + J_4^{1\,0} + J_4^{0\,1} = i\acute{e}((\rho^{0\,0}_{0\,0}(r, t') \,+ \rho^{1\,1}_{0\,0}(r, t'))\, Y_{0\,0} Y_{0\,0} \\ +\rho^{1\,0}_{1\,0}(r, t')\, Y_{1\,0} Y_{0\,0} + \rho^{0\,1}_{1\,0}(r, t')\, Y_{1\,0} Y_{0\,0} + (2/\sqrt{5})\rho^{1\,1}_{2\,0}(r, t')\, Y_{2\,0} Y_{0\,0}). \end{aligned} \tag{3.85}$$

This is justified by numerical results. The related quasi-static component of the vector potential A_4 has the form

$$\begin{aligned} -i\acute{e} A_4(\mathcal{R}, t') = \acute{e}^2 (A^{0\,0}_{0\,0}(r, t') \,+ A^{1\,1}_{0\,0}(r, t') \\ +A^{1\,0}_{1\,0}(r, t')\, Y_{1\,0}/Y_{0\,0} + A^{0\,1}_{1\,0}(r, t')\, Y_{1\,0}/Y_{0\,0} + (2/\sqrt{5}) A^{1\,1}_{2\,0}(r, t')\, Y_{2\,0}/Y_{0\,0}). \end{aligned} \tag{3.86}$$

$A^{1\,0}_{1\,0}(r, t') = (A^{0\,1}_{1\,0}(r, t'))^*$ results from the following integral

$$A^{1\,0}_{1\,0}(r, t') = \; r(I^{1\,0}_{1\,0}(\infty, t') - \int_0^r 1/y^4 \int_0^y x^3 \rho^{1\,0}_{1\,0}(x, t') dx\, dy), \tag{3.87}$$

due to the special potential equation

$$(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{2}{r^2})\, A^{1\,0}_{1\,0}(r,t') = \rho_{1\,0}(r,t'). \tag{3.88}$$

The solutions can be described by eigenfunctions of eq.(3.49) or eq.(3.74). We use the notations

$$f^{0}_{0\,0}(r,t') = \sum_q a^0_q(t')\, h^0_q(r) \tag{3.89}$$

and

$$f^{1}_{1\,0}(r,t') = \sum_p a^1_p(t')\, h^1_p(r). \tag{3.90}$$

These functions must satisfy the following equations

$$\begin{aligned} i\tfrac{\acute{e}^2}{\acute{a}}\tfrac{\partial}{\partial t'}f^{0}_{0\,0}(r,t')Y_{0\,0} = \tfrac{\acute{e}^2}{\acute{a}}(-\tfrac{1}{2}(\tfrac{\acute{a}}{r})^2\tfrac{\partial}{\partial r}\tfrac{r^2\partial}{\partial r} - \tfrac{\acute{a}}{r} \\ + \acute{a}(A^{0\,0}_{0\,0}(r,t') + A^{1\,1}_{0\,0}(r,t'))f^{0}_{0\,0}(r,t)Y_{0\,0} + \acute{a}A^{1\,0}_{1\,0}(r,t')f^{1}_{1\,0}(r,t')Y_{0\,0}) \end{aligned} \tag{3.91}$$

and

$$\begin{aligned} i\tfrac{\acute{e}^2}{\acute{a}}\tfrac{\partial}{\partial t'}f^{1}_{1\,0}(r,t')Y_{1\,0} = \tfrac{\acute{e}^2}{\acute{a}}(-\tfrac{1}{2}(\tfrac{\acute{a}}{r})^2\tfrac{\partial}{\partial r}\tfrac{r^2\partial}{\partial r} + (\tfrac{\acute{a}}{r})^2 - \tfrac{\acute{a}}{r} \\ + \acute{a}(A^{0\,0}_{0\,0}(r,t') + A^{1\,1}_{0\,0}(r,t'))f^{1}_{1\,0}(r,t')Y_{1\,0}) \\ + \tfrac{\acute{e}^2}{\acute{a}}(\tfrac{2\acute{a}}{\sqrt{5}}A^{1\,1}_{2\,0}(r,t')f^{1}_{1\,0}(r,t')Y_{1\,0} + \acute{a}A^{0\,1}_{1\,0}(r,t')f^{0}_{0\,0}(r,t')Y_{1\,0}) \end{aligned} \tag{3.92}$$

if the influence of $f^1_{3\,0}Y_{3\,0}$ is neglected. Here the charge sharing can be depicted by

$$C_0 \; = \; \sum_q (a^0_q(t'))^* a^0_q(t'), \tag{3.93}$$

$$C_1 \; = \; \sum_p (a^1_p(t'))^* a^1_p(t'), \tag{3.94}$$

where $C_0 + C_1 \cong 1$. This is an approximation of the law of charge conservation, which generally reads $\sum_n C_n = 1$. The conditions $C_0 = const.$ and $C_1 = const.$ are fulfilled when the radiation is negligible, which will be shown in the next section. In these cases the coefficients are solutions of

$$i\tfrac{d}{dt'} a^0_q(t') = \; a^0_q(t')(E^0_q)' + {}^{0\,0}_{\;q}M^{0\,0}(t') + {}^{0\,1}_{\;q}M^{1\,0}(t') \tag{3.95}$$

and

$$i\tfrac{d}{dt'} a^1_p(t') \; = \; a^1_p(t')(E^1_p)' + {}^{1\,1}_{\;p}M^{1\,1}(t') + {}^{1\,0}_{\;p}M^{0\,1}(t'). \tag{3.96}$$

Here ${}^{0\,0}_{\;q}M^{0\,0}(t')$ and ${}^{1\,1}_{\;q}M^{1\,1}(t')$ are given by the eqs.(3.56/3.79), while ${}^{0\,1}_{\;q}M^{1\,0}(t')$ and ${}^{1\,0}_{\;p}M^{0\,1}(t')$ result from

$${}^{0\,1}_{\;q}M^{1\,0}(t') \; = \; \acute{a} \int_0^R r^2 \, (h^0_q(r))^* \, f^1_{1\,0}(r,t')(A^{1\,0}_{1\,0}(r,t')) \; dr \tag{3.97}$$

or

$${}^{1\,0}_{\;p}M^{0\,1}(t') \; = \; \acute{a} \int_0^R r^2 \; (h^1_p(r))^* f^0_{0\,0}(r,t')(A^{0\,1}_{1\,0}(r,t')) \; dr. \tag{3.98}$$

Due to the eqs.(3.95/3.96), the dynamic behaviour of the coefficients follows from

$$a_q^0(t_0' + \bar{\Delta}t') \;=\; a_q^0(t_0') - i\,\bar{\Delta}t'\,(\,a_q^0(t_0')(E_q^0)' \,+\, {}_q^{0\,0}M^{0\,0}(t_0') + {}_q^{0\,1}M^{1\,0}(t_0')) \qquad (3.99)$$

or

$$a_p^1(t_0' + \bar{\Delta}t') \;=\; a_p^1(t_0') - i\,\bar{\Delta}t'\,(\,a_q^0(t_0')(E_p^1)' \,+\, {}_p^{1\,1}M^{1\,1}(t_0') \,+\, {}_p^{1\,0}M^{0\,1}(t_0')). \qquad (3.100)$$

Final results of such calculations show that these fields also vibrate in an energy valley.

3.4.2 *Spectral properties of a transition solution*

We describe the Fourier coefficients of $a_q^0(t')$ and $a_p^1(t')$ by

$$\begin{aligned} a_q^0(t') \;&= \textstyle\sum_l \, {}^0d_q^l \, exp(-il\,\omega'\,(t' - t_0')), \\ a_p^1(t') \;&= \textstyle\sum_{l1} \, {}^1d_p^{l1} \, exp(il1\,\omega'(t' - t_0')). \end{aligned} \qquad (3.101)$$

Hence the transformations of eqs.(3.83/3.84) are given by

$$u^0 \,=\, \textstyle\sum_{l,\,q} \, {}^0d_q^l \; exp(-i\,l\,\omega'\,(\,t' \,-\, t_0'))h_q^0, \qquad (3.102)$$

$$u^1 \,=\, \textstyle\sum_{l1,\,p} \, {}^1d_p^{l1} \, exp(i(\; exp(-i\,(l1\,\omega'\,(\,t' \,-\, t_0'))h_p^1 \qquad (3.103)$$

with

$${}^0d_q^l = \frac{1}{T'} \int_{t_0'}^{t_0'+T'} \exp(il\,\omega'\,(t' - t_0'))a_q^0(t')dt' \qquad (3.104)$$

and

$$ {}^{1}d_p^{l1} = \frac{1}{T'} \int_{t_0'}^{t_0'+T'} \exp(il1\,\omega'\,(t'-t_0'))\, a_p^1(t)' dt'. \tag{3.105} $$

Besides, we apply the following notations

$$ ({}_q^{0\,0}mx_l^{0\,0}) = \tfrac{1}{T'} \int_{t_0'}^{t_0'+T'} \exp(il\omega'\,(t'-t_0'))\,{}_q^{0\,0}M^{0\,0}(t')dt', \tag{3.106} $$

$$ ({}_q^{0\,1}mx_l^{1\,0}) = \tfrac{1}{T'} \int_{t_0'}^{t_0'+T'} \exp(il\omega'\,(t'-t_0'))\,{}_q^{0\,1}M^{1\,0}(t')dt', \tag{3.107} $$

$$ ({}_p^{1\,0}mx_{l1}^{0\,1}) = \tfrac{1}{T'} \int_{t_0'}^{t_0'+T'} \exp(-il\omega'\,(t'-t_0'))\,{}_p^{1\,0}M^{1\,0}(t')dt' \tag{3.108} $$

and

$$ ({}_p^{1\,1}mx_{l1}^{1\,1}) = \tfrac{1}{T'} \int_{t_0'}^{t_0'+T'} \exp(-il\omega'\,(t'-t_0'))\,{}_p^{1\,1}M^{1\,1}(t')dt'. \tag{3.109} $$

If we put the eqs.(3.101-3.109) into the eqs.(3.95/3.96), we obtain

$$ \begin{aligned} l\,\omega'\,{}^0d_q^l \exp(-i\,l\,\omega'\,(t'-t_0')) \;=\; & \exp(-i\,l\,\omega'\,)(t'\,-\,t_0'))\,((E_q^0)'\,({}^0d_q^l) \\ & +({}_q^{0\,0}mx_l^{0\,0}) + ({}_q^{0\,1}mx_l^{1\,0})) \end{aligned} \tag{3.110} $$

and

$$ \begin{aligned} l1\,\omega'\,{}^0d_p^{l1} \exp(-i\,l\,\omega'\,(t'-t_0')) \;=\; & \exp(-i\,l\,\omega'\,)(t'\,-\,t_0'))\,((E\,{}_p^1)'(d_p^{l1}) \\ & +({}_p^{1\,1}mx_{l1}^{1\,1}) + ({}_p^{1\,0}mx_{l1}^{0\,1})). \end{aligned} \tag{3.111} $$

Several results of these calculations are shown in Fig. 7, Fig 8 and Table 9.

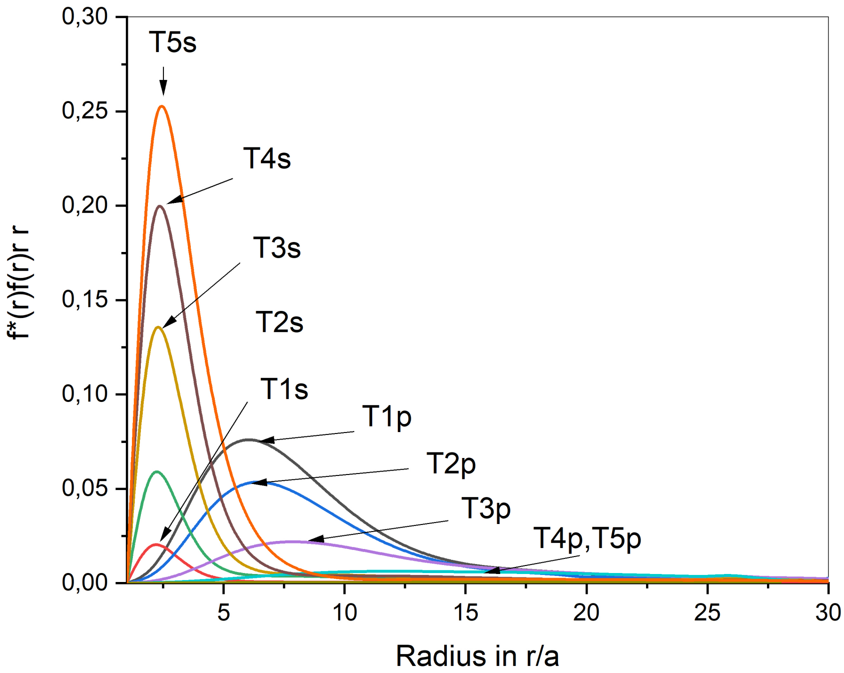

Fig. 7: 2p-1s transition states: charge shares (mean distributions) of s-states and p-states

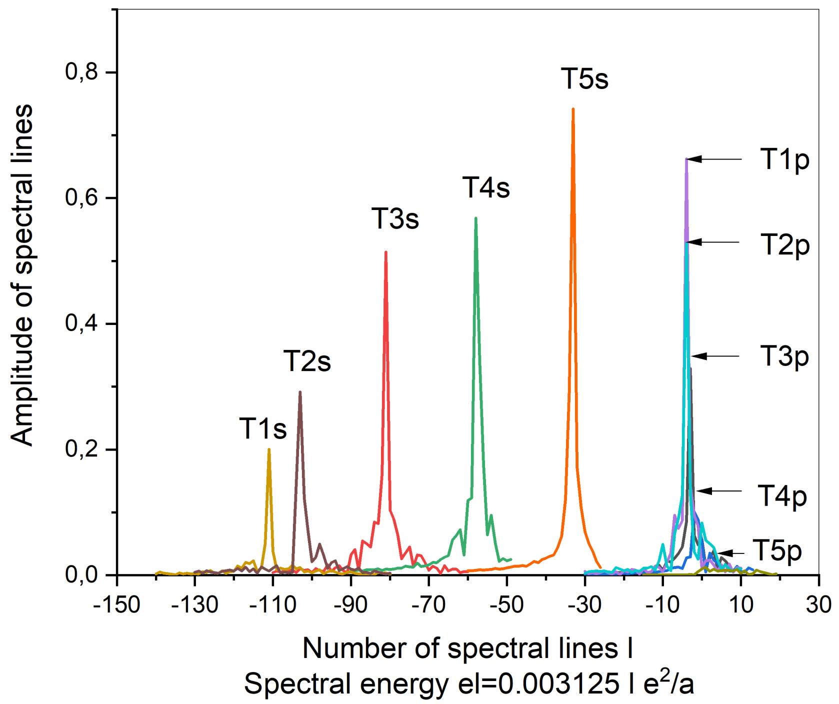

Fig. 8: Spectral distributions of 2p-1s transition states

T1p: $\varrho_p = 0.9$, T1s: $\varrho_s = 0.1$;

T2p: $\varrho_p = 0.7$, T2s: $\varrho_s = 0.3$;

T3p: $\varrho_p = 0.5$, T3s: $\varrho_s = 0.5$;

T4p: $\varrho_p = 0.3$, T4s: $\varrho_s = 0.7$;

T5p: $\varrho_p = 0.1$, T5s: $\varrho_s = 0.9$;

Table 9 (RR=200$\acute{a}$, energy parameter of 2p-s transition states in $\acute{e}^2/\acute{a}$)

T1. $\varrho_p = 0.9$, $\varrho_s = 0.1$, $T' = 2000$

eg	edg	eag	$A(0)$	$\Delta A(0)$
0.2036	0.1665	0.0371	0.1715	0.0093

egp	edp	eap	ωlp	$R/\acute{a}$
Δegp	Δedp	Δeap		$\Delta R/\acute{a}$
0.1678	0.1351	0.0327	-0.0125	41.91
0.0001	0.00218	0.0020		1.33

egs	eds	eas	ωls	$R/\acute{a}$
Δegs	Δeds	Δeas		$\Delta R/\acute{a}$
0.0358	0.0314	0.0044	-0.0125	26.63
0.0001	0.00023	0.00023		1.46

T2. $\varrho_p = 0.7,\ \varrho_s = 0.3,\ T' = 2000$

eg	edg	eag	$A(0)$	$\Delta A(0)$
0.1356	0.0963	0.0394	0.2384	0.0182

egp	edp	eap	ωlp	$R/\acute{a}$
Δegp	Δedp	Δeap		$\Delta R/\acute{a}$
0.1003	0.0765	0.0238	-0.0125	40.56
0.0001	0.0046	0.0019		1.89

egs	eds	eas	ωls	$R/\acute{a}$
Δegs	Δeds	Δeas		$\Delta R/\acute{a}$
0.0353	0.0197	0.0156	-0.0125	24.11
0.0001	0.0011	0.00011		2.34

T3. $\varrho_p = 0.5,\ \varrho_s = 0.5,\ T' = 2000$

eg	edg	eag	$A(0)$	$\Delta A(0)$
0.0418	-0.0158	0.0576	0.3818	0.0245

egp	edp	eap	ωlp	$R/á$
Δegp	Δedp	Δeap		$\Delta R/á$
0.0911	0.0789	0.0122	-0.0125	57.07
0.0002	0.0087	0.0013		3.89

egs	eds	eas	ωls	$R/á$
Δegs	Δeds	Δeas		$\Delta R/á$
-0.0493	-0.0947	0.0454	-0.0125	12.80
0.0001	0.0032	0.0032		1.34

T4. $\varrho_p = 0.3,\ \varrho_s = 0.7,\ T' = 2000$

eg	edg	eag	$A(0)$	$\Delta A(0)$
-0.0494	-0.1427	0.0933	0.5154	0.0245

egp	edp	eap	ωlp	$R/á$
Δegp	Δedp	Δeap		$\Delta R/á$
0.0699	0.0649	0.0050	-0.0125	74.96
0.0005	0.0040	0.0005		1.04

egs	eds	eas	ωls	$R/á$
Δegs	Δeds	Δeas		$\Delta R/á$
-0.1193	-0.2076	0.0883	-0.0133	9.43
0.0001	0.0051	0.0051		0.56

T5. $\varrho_p = 0.1, \ \varrho_s = 0.9, \ T' = 2000$

eg	edg	eag	$A(0)$	$\Delta A(0)$
-0.1556	-0.3017	0.1461	0.6379	0.0201

egp	edp	eap	ωlp	$R/á$
Δegp	Δedp	Δeap		$\Delta R/á$
0.0303	0.0294	0.0010	-0.0125	101.92
0.0005	0.0026	0.0001		5.04

egs	eds	eas	ωls	$R/á$
Δegs	Δeds	Δeas		$\Delta R/á$
-0.1859	-0.3311	0.1451	-0.0133	6.85
0.0001	0.0055	0.0055		0.38

3.4.3 *The law of charge conservation of a transition state*

If you want to fully characterise the dynamics of fields, it is necessary to investigate the tensor defined in eq.(2.30). For this purpose we avoid specifying the dimensions and apply solutions of the Dirac equation divided into different excitation levels. The linked tensor can be used to verify the law of charge conservation in separate excitation levels. With the functions

$$U^{z-s}(\mathbf{R},t) = \, exp(-i\acute{M}\acute{c}^2(t-t_0)/\hbar) \sum_q a_q^{z-s}(t) \, H_q^{z-s}(\mathbf{R}) \tag{3.112}$$

and

$$U^z(\mathbf{R},t) = \, exp(-i\acute{M}\acute{c}^2(t-t_0)/\hbar) \, \sum_p a_p^1(t) \, H_p^z(\mathbf{R}), \tag{3.113}$$

the divergence of the tensor

$${}^{z\,z-s}T^d_{\nu\,\mu} \, = \frac{\acute{c}\,\hbar}{2} \, (\bar{U}^z \gamma_\mu \, \frac{\partial \, U^{z-s}}{\partial \, x_\nu} \, - \, \frac{\partial \, \bar{U}^z}{\partial \, x_\nu} \, \gamma_\mu \, U^{z-s}) \tag{3.114}$$

reads

$$\frac{\partial \, {}^{z\,z-s}T^d_{\nu\,\mu}}{\partial \, x_\mu} \, = \frac{\acute{c}\,\hbar}{2} \, \frac{\partial}{\partial \, x_\mu} \, \{\bar{U}^z \gamma_\mu \, \frac{\partial \, U^{z-s}}{\partial \, x_\nu} \, - \, \frac{\partial \, \bar{U}^z}{\partial \, x_\nu} \, \gamma_\mu \, U^{z-s}\}. \tag{3.115}$$

The functions H_p^z and H_q^{z-s} are eigenfunctions of eq.(3.5), which correspond to the solutions h_p^z and h_q^{z-s} of the Schrödinger equation (see eq.(3.18) and eq.(3.46)). Due to $\mid da_n^z/dt \mid << \acute{M}\acute{c}_2/\hbar$ and $c_n^z(t) \cong a_n^z(t) \; \exp(-i\,\acute{M}\,\acute{c}^2(t-t_0)/\hbar)$, it becomes

$$\hbar \frac{\partial \, U^z}{\partial \, x_4} \, \cong \, - \, \acute{M} \, \acute{c} \, U^z. \tag{3.116}$$

Therefore eq.(3.115) describes for $v = 4$ the continuity equation

$$\frac{\partial}{\partial\, x_\mu}\,{}^{z\,z-s}T^{d}_{4\,\mu} \,\cong\, -\,\acute{M}\,\acute{c}^2\frac{\partial}{\partial\, x_\mu}\,\{\bar{U}^z\gamma_\mu\; U^{z\,-\,s}\}. \qquad (3.117)$$

The integral over the divergence should be zero for all values of z and s=0 in the considered approximation. To prove this remark we apply eq.(2.4) or eq.(2.5) in the following reduced form

$$[\gamma_\mu\hbar\frac{\partial}{\partial x_\mu} - i\frac{\acute{e}}{\acute{c}}\bar{A}_\mu\gamma_\mu + \,\acute{M}\,\acute{c}]U^0 - i\frac{\acute{e}}{\acute{c}}A^{1\,0}_\mu)\gamma_\mu U^1 = 0, \qquad (3.118)$$

$$\bar{U}^0[\gamma_\mu\hbar\frac{\overleftarrow{\partial}}{\partial x_\mu} + i\frac{\acute{e}}{\acute{c}}\bar{A}_\mu\gamma_\mu - \,\acute{M}\,\acute{c}] + i\frac{\acute{e}}{\acute{c}}\bar{U}^1\gamma_\mu A^{0\,1}_\mu = 0. \qquad (3.119)$$

$\bar{A}_\mu(\mathbf{R}, t)$ describes the common potential formed from the charges $J^{0\,0}_\mu + J^{1\,1}_\mu$ and the charge of the nucleus. All potentials have quasi-static character. Therefore the integral over the separated continuity equation reads

$$\hbar \int \tfrac{\partial}{\partial\, x_\mu}\,\{\bar{U}^0\gamma_\mu\; U^0\}dV = i\tfrac{\acute{e}}{\acute{c}}\int(\bar{U}^0\gamma_\mu U^1\; A^{1\,0}_\mu - \bar{U}^1\gamma_\mu U^0\; A^{0\,1}_\mu)dV \qquad (3.120)$$

and, because of

$$\begin{aligned} i\acute{e}\int(\bar{U}^0\gamma_\mu U^1\; A^{1\,0}_\mu dV &= \int \tfrac{1}{r_{0\,1}}J^{0\,1}_\mu(\mathcal{R})J^{1\,0}_\mu(\mathcal{R}_1)dV_1 dV,\\ i\acute{e}\int(\bar{U}^1\gamma_\mu U^0\; A^{0\,1}_\mu dV &= \int \tfrac{1}{r_{0\,1}}J^{1\,0}_\mu(\mathcal{R})J^{0\,1}_\mu(\mathcal{R}_1)dV_1 dV, \end{aligned} \qquad (3.121)$$

it becomes in this approximation

$$-i\,\frac{\hbar}{\acute{c}}\sum_n \frac{\partial}{\partial t}((a_n^0(t))^* a_n^0(t)) = 0. \qquad (3.122)$$

The prerequisite of eq.(3.122) is the application of the quasi-static potentials. That means, the radiation is neglected. Then the eq.(3.120) leads to eq.(3.122). A similar consideration yields $-i\frac{\hbar}{\acute{c}}\sum_n \frac{\partial}{\partial t}((a_n^1)^* a_n^1)) = 0$.

This result remains valid for more than two excitation levels. That is, all values $z = 0, \pm 1, \pm 2, ..$ are possible. In the eqs.(3.118/3.119) the test is restricted to $z = 0$ and $s = 0, -1$. Separate laws of charge conservation can be proved for all excitation levels.

Chapter 4

Radiation properties

4.1 *The solution functions*

It is convenient to use the Dirac equation to determine the currents. The solutions must satisfy the following equations:

$$\gamma_4\hbar\frac{\partial}{\partial x_4}U^0 = [-\gamma_{\bar{\mu}}\hbar\frac{\partial}{\partial x_{\bar{\mu}}} + i\frac{\acute{e}}{\acute{c}}(A_\mu + A^K_\mu)\gamma_\mu) - \acute{M}\acute{c}]U^0 + i\frac{\acute{e}}{\acute{c}}A^{1\,0}_\mu\gamma_\mu U^1, \tag{4.1}$$

$$\hbar\frac{\partial\bar{U}^0}{\partial x_4}\gamma_4 = [-\hbar\frac{\partial\bar{U}^0}{\partial x_{\bar{\mu}}}\gamma_{\bar{\mu}} - i\frac{\acute{e}}{\acute{c}}(A_\mu + A^K_\mu)\bar{U}^0\gamma_\mu + \acute{M}\acute{c}\bar{U}_0] - i\frac{\acute{e}}{\acute{c}}A^{0\,1}_\mu\bar{U}^1\gamma_\mu, \tag{4.2}$$

$$\gamma_4\hbar\frac{\partial}{\partial x_4}U^1 = [-\gamma_{\bar{\mu}}\hbar\frac{\partial}{\partial x_{\bar{\mu}}} + i\frac{\acute{e}}{\acute{c}}(A_\mu + A^K_\mu)\gamma_\mu) - \acute{M}\acute{c}]U^1 + i\frac{\acute{e}}{\acute{c}}A^{0\,1}_\mu\gamma_\mu U^0 \tag{4.3}$$

and

$$\hbar\frac{\partial\bar{U}^1}{\partial x_4}\gamma_4 = [-\hbar\frac{\partial\bar{U}^1}{\partial x_{\bar{\mu}}}\gamma_{\bar{\mu}} - i\frac{\acute{e}}{\acute{c}}(A_\mu + A^K_\mu)\bar{U}^1\gamma_\mu + \acute{M}\acute{c}\bar{U}^1] - i\frac{\acute{e}}{\acute{c}}A^{1\,0}_\mu\bar{U}^0\gamma_\mu. \tag{4.4}$$

The functions can be described by

$$U^0(\mathbf{R},t) = \, exp(-i\acute{M}\acute{c}^2(t-t_0)/\hbar) \sum_q a^0_q(t)\, H^0_q(\mathbf{R}), \tag{4.5}$$

$$U^1(\mathbf{R},t) = \, exp(-i\acute{M}\acute{c}^2(t-t_0)/\hbar) \sum_p a^1_p(t)\, H^1_p(\mathbf{R}) \tag{4.6}$$

and of which sum by

$$U^0 + U^1 = \, exp(-i\acute{M}\acute{c}^2(t-t_0)/\hbar) \begin{pmatrix} \hat{u}^0 \\ \check{u}^0 \end{pmatrix} + \, exp(-i\acute{M}\acute{c}^2(t-t_0)/\hbar) \begin{pmatrix} \hat{u}^1 \\ \check{u}^1 \end{pmatrix} \tag{4.7}$$

with

$$\hat{u}^0(\mathbf{R},t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} u^0 \tag{4.8}$$

and

$$\hat{u}^1(\mathbf{R},t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} u^1. \tag{4.9}$$

Here is

$$u^0 = Y_{0\,0} f^0_{0\,0}(r,t) + ... \tag{4.10}$$

and

$$u^1 = Y_{1\,0} f^1_{1\,0}(r,t) + ... \; . \tag{4.11}$$

The functions u^0 and u^1 can be approximated by the solution of the Schrödinger equation (see eqs.(3.91/3.92)). Then, the potential $-i\acute{e}A_4(\mathcal{R},t)$ is given by eq.(3.86).

4.1.1 *Coefficient dynamics of a transition solution*

We apply again the dimension introduced in section 3.2. That means, $\hbar\frac{d}{dt}$ and the energy values E are replaced by $\frac{é^2}{á}\frac{d}{dt'}$ and $E' = E/(é^2/á)$. After such transformation, we multiply eq.(4.1) with $\bar{H}^0_q$ and integrate over the volume V. This leads to

$$\begin{aligned} i\tfrac{é^2}{á}\tfrac{d}{dt'}a^0_q(t') = & \tfrac{é^2}{á}(E^0_q)'a_q(t') + \tfrac{é^2}{}\textstyle\sum_\mu \left({}^{0\,0}_{\;q}M^{0\,0}_\mu(t') + {}^{0\,0}_{\;q}M^{1\,1}_\mu(t') \right. \\ & \left. + {}^{0\,1}_{\;q}M^{1\,0}_\mu(t')\right) + \tfrac{é^2}{á}\,á({}^{0\,1}_{\;q}S^{1\,0}) \end{aligned} \tag{4.12}$$

and an analogous procedure with $\bar{H}^0_q$ results in

$$\begin{aligned} i\tfrac{é^2}{á}\tfrac{d}{dt'}(a^0_q(t'))^* = & -\tfrac{é^2}{á}(E^1_q)'(a^1_q(t'))^* - \tfrac{é^2}{}\textstyle\sum_\mu \left({}^{0\,0}_{\;q}\bar{M}^{0\,0}_\mu(t') + {}^{0\,0}_{\;q}\bar{M}^{1\,1}_\mu(t') \right. \\ & \left. + {}^{0\,1}_{\;q}\bar{M}^{1\,0}_\mu(t')\right) - \tfrac{é^2}{á}({}^{1\,0}_{\;q}\bar{S}^{0\,1}). \end{aligned} \tag{4.13}$$

Here are

$$\textstyle\sum_\mu {}^{0\,0}_{\;q}M^{0\,0}_\mu(t') = -\frac{iá}{é}\,exp(i(t'-t'_0)/ά^2)\int \bar{H}^0_q\gamma_\mu U^0(t')\,A^{0\,0}_\mu(t')dV, \tag{4.14}$$

$$\textstyle\sum_\mu {}^{0\,0}_{\;q}M^{1\,1}_\mu(t') = -\frac{iá}{é}\,exp(i(t'-t'_0)/ά^2)\int \bar{H}^0_q\gamma_\mu U^1(t')A^{1\,1}_\mu(t')dV, \tag{4.15}$$

$$\textstyle\sum_\mu {}^{0\,1}_{\;q}M^{1\,0}_\mu(t') = -\frac{iá}{é}\,exp(i(t'-t'_0)/ά^2)\int \bar{H}^0_q\gamma_\mu U^1(t')A^{1\,0}_\mu(t')dV, \tag{4.16}$$

$$\textstyle\sum_\mu {}^{0\,0}_{\;q}\bar{M}^{0\,0}_\mu(t') = -\frac{iá}{é}\,exp(-i(t'-t'_0)/ά^2)\int \bar{U}^0\gamma_\mu H^0_q\,A^{0\,0}_\mu(t')dV, \tag{4.17}$$

$$\textstyle\sum_\mu {}^{0\,0}_{\;q}\bar{M}^{1\,1}_\mu(t') = -\frac{iá}{é}\,exp(-i(t'-t'_0)/ά^2)\int \bar{U}^0\gamma_\mu H^0_q\,A^{1\,1}_\mu(t')dV \tag{4.18}$$

and

$$\Sigma_\mu \; {}^{1\,0}_{\;q}\bar{M}^{0\,1}_\mu(t') = -\tfrac{i\acute{a}}{\acute{e}}\, exp(-i(t'-t'_0)/\acute{\alpha}^2) \int \bar{U}^1\gamma_\mu H^0_q \, A^{0\,1}_\mu(t')dV. \qquad (4.19)$$

All potentials have quasi-static character, while $({}^{0\,1}_{\;q}S^{1\,0})$ and $({}^{1\,0}_{\;q}\bar{S}^{0\,1})$ describe the influence of radiation. That follows from the difference between the retarded potential, given in eq.(3.38), and the linked quasi-static potential. Details are presented in section 4.1.2. In addition, the potentials $A^{0\,0}_{\bar{\mu}}(t')$, etc. are not be taken into account there, because the magnitude of the integral ${}^{0\,0}_{\;q}M^{0\,0}_{\bar{\mu}}(t')$ has an order of $\acute{\alpha}^2$. Its influence on the quasi-static field is small. That means, the sum in the eqs.(4.12/4.13) can be reduced to the terms with μ=4. This is applied, for example, in the eqs.(3.95/3.96) where radiation is also neglected. The Fourier transformations of the functions of the eqs.(4.5/4.6) are given by

$$U^0 = \Sigma_{l,\,q} \; {}^0d^l_q \; exp(-i(1/\acute{\alpha}^2) + l\omega')(t' - t'_0))H^0_q \qquad (4.20)$$

or

$$\bar{U}^0 = \Sigma_{l,\,q} \, ({}^0d^l_q)^* \, exp(+i(1/\acute{\alpha}^2) + l\omega')(t' - t'_0))H^0_q. \qquad (4.21)$$

Here we use the relations

$${}^0d^l_q = \frac{1}{T'}\int_{t'_0}^{t'_0+T'} \exp(il\omega'\,(t'-t'_0))^0a^1_q(t')dt' \qquad (4.22)$$

and

$$({}^0d^l_q)^* = \frac{1}{T'}\int_{t'_0}^{t'_0+T'} \exp(-il\omega'\,(t'-t'_0))({}^0a^1_q(t)')^*dt'. \qquad (4.23)$$

Similar expressions can be introduced for U^1 and $\bar{U}^1$. Besides, we apply following notations

$$({}_q^{0\,0}mx_l^{0\,0})_\mu = \frac{1}{T'}\int_{t'_0}^{t'_0+T'} \exp(il\omega\,(t'-t'_0))\,{}_q^{0\,0}M_\mu^{0\,0}(t')dt', \tag{4.24}$$

$$({}_q^{0\,0}m\bar{x}_l^{0\,0})_\mu = \frac{1}{T'}\int_{t'_0}^{t'_0+T'} \exp(-il\omega\,(t'-t'_0))\,{}_q^{0\,0}\bar{M}_\mu^{0\,0}(t')dt' \tag{4.25}$$

and

$$({}_q^{0\,1}sm_l^{1\,0}) = \frac{1}{T'}\int_{t'_0}^{t'_0+T'} \exp(il\omega'\,(t'-t'_0))\,({}_q^{0\,1}S^{1\,0}(t'))dt' \tag{4.26}$$

for the Fourier transformations of ${}_q^{0\,0}M_\mu^{0\,0}(t')$, ${}_q^{0\,0}\bar{M}_\mu^{0\,0}(t')$ and $({}_q^{0\,1}S^{1\,0}(t'))$. Analogous expressions follow from the Fourier transformations of ${}_q^{1\,1}M_\mu^{1\,1}(t')$, etc. The terms $({}_q^{0\,1}sm_l^{1\,0})$ and $({}_q^{0\,1}s\bar{m}_l^{1\,0})$ cause small changes in the magnitude ${}^0d_q^l(t')$ and ${}^1d_p^{l1}(t')$ when special conditions are fulfilled. Then the separated laws of charge conservation of the excitation levels are violated. Because of

$$\begin{aligned} i\frac{\acute{e}^2}{\acute{a}}\frac{d}{dt'}a_q^0(t') &= (\acute{e}^2/\acute{a})\sum_l (l\,\omega'\,{}^0d_q^l \\ &+ i\frac{d\,({}^0d_q^l)}{d\,t'})\exp(-i(\,l\,\omega'\,)(t'-t'_0)) \end{aligned} \tag{4.27}$$

and

$$(\acute{e}^2/\acute{a})\gamma_{\bar{\mu}}\acute{c}\frac{\partial\,H_q^0}{\partial x_{\bar{\mu}}} = (\acute{e}^2/\acute{a})(\frac{\acute{a}}{r} + (E_q^0)')\gamma_4 H_q^0, \tag{4.28}$$

the eqs.(4.12/4.13) yield

$$
\begin{array}{c}
(\acute{e}^2/\acute{a})(l\,\omega'\,)\,{}^0d^l_q + i\frac{d\,({}^0d^l_q)}{d\,t'}) \;=\; (\acute{e}^2/\acute{a})(\,(E^0_q)'\;{}^0d^l_q \\
+\,\Sigma_\mu(({}^{0\,0}_{q}mx^{0\,0}_{l})_\mu + ({}^{0\,0}_{q}mx^{1\,1})_\mu + ({}^{0\,1}_{q}mx^{1\,0}_{l})_\mu) + \acute{a}\,({}^{0\,1}_{q}sm^{1\,0}_{l}))
\end{array}
\tag{4.29}
$$

respectively

$$
\begin{array}{c}
(\acute{e}^2/\acute{a})(-l\,\omega'\,({}^0d^l_q) \;+\; i\frac{d\,({}^0d^l_q)}{d\,t'}) = (\acute{e}^2/\acute{a})(\,-\;(E^0_q)'\,{}^0d^l_q \\
-\;\Sigma_\mu(({}^{0\,0}_{q}m\bar{x}x^{0\,0}_{l})_\mu + ({}^{0\,0}_{q}m\bar{x}x^{1\,1}_{l})_\mu + ({}^{1\,0}_{q}m\bar{x}x^{0\,1}_{l})_\mu)) - \acute{a}({}^{0\,1}_{q}s\bar{m}^{1\,0}_{l})).
\end{array}
\tag{4.30}
$$

$(E^0_q)' = (\bar{E}^0_q)' - 1/\acute{\alpha}^2$ in the eqs.(4.29/4.30) contains the small difference between the integrals $\int \bar{H}HdV$ and $\int \bar{H}\gamma_4 HdV$. The eq.(4.29)) can be reduced for quasi-static potentials to

$$
\begin{array}{c}
l\,\omega'\,({}^0d^l_q) \;=\; (E^0_q)'\,{}^0d^l_q \\
+\acute{a}\;\Sigma_\mu(({}^{0\,0}_{q}mx^{0\,0}_{l})_\mu + ({}^{0\,0}_{q}mx^{1\,1}_{l})_\mu + ({}^{0\,1}_{q}mx^{1\,0}_{l})_\mu).
\end{array}
\tag{4.31}
$$

An analogous relation is valid for eq.(4.30).

This result is a consequence of the separate laws of conservation for the linked excitation levels in these cases, what is shown in section 3.4.3. However, a charge transfer from one level to another can be caused by radiation. We assume that the relations

$$
\begin{array}{rl}
i\frac{d\,({}^0d^l_q)}{d\,t'} & = \acute{a}\,({}^{0\,1}_{q}sm^{1\,0}_{l}), \\
i\frac{d\,({}^0d^l_q)^*}{d\,t'} & = -\acute{a}\,({}^{1\,0}_{q}s\bar{m}^{0\,1}_{l})
\end{array}
\tag{4.32}
$$

and analogously

$$
\begin{aligned}
i\frac{d\,({}^1d_p^l)}{d\,t'} &= \acute{a}\,({}_p^{1\,0}sm_l^{0\,1}),\\
i\frac{d\,({}^1d_p^l)^*}{d\,t'} &= -\acute{a}\,({}_p^{0\,1}s\bar{m}_l^{1\,0})
\end{aligned}
\tag{4.33}
$$

describe this effect. The small values of $({}_q^{0\,1}sm_l^{1\,0})$, ect. have an influence on the solution of the eq.(4.29/4.30) only during a long time $((t' - t_0' >> 1/\varpi')$.

4.1.2 *The single radiation effect*

In this section we study radiation effects of a transition state. Such an effect is only possible if the field includes a "freely" oscillating radiation moment. "Freely" oscillating means, an external radiation field of a specific frequency can influence the phase position of this oscillation. We assume that such a moment is formed from the eigenfunctions H_q^0 and H_p^1 and consider the sum

$$
\begin{aligned}
U^{1a} + U^{0a} = {} & a_p^1(t')\,exp(-i(t' - t_0')/\acute{\alpha}^2)\begin{pmatrix}\hat{h}_p^1\\ \check{h}_p^1\end{pmatrix}\\
& + (a_q^0(t')\,exp(-i(t' - t_0')/\acute{\alpha}^2)\begin{pmatrix}\hat{h}_q^0\\ \check{h}_q^0\end{pmatrix}
\end{aligned}
\tag{4.34}
$$

with

$$
\hat{h}_p^1(\mathbf{R}) = \begin{pmatrix}1\\0\end{pmatrix} Y_{1\,0}h_p^1(r) \tag{4.35}
$$

and

$$
\hat{h}_q^0(\mathbf{R}) = \begin{pmatrix}1\\0\end{pmatrix} Y_{0\,0}h_q^0(r). \tag{4.36}
$$

The linked solution of the Schrödinger equation is

$$u^{1a}(r,t') + u^{0a}(r,t') = \, a_p^1(t') Y_{1\,0} h_p^1(r) \\ + \, a_q^0(t') Y_{0\,0} h_q^0(r). \tag{4.37}$$

Using the Fourier transformation $a_q^0(t') = \sum_l \, {}^0d_q^l \exp(-il\omega'(t'-t_0'))$ and $(a_p^1(t'))^* = \sum_{l1} \, ({}^1d_p^{l1})^* \exp(il1\omega'(t'-t_0'))$ you get the current

$${}^{1\,0}_{p\,q}J_\mu(\mathbf{R},t') = \, i\acute{e} \, \textstyle\sum_{l,l1} \, ({}^1d_p^{l1})^* \, ({}^0d_q^l)(j_{p\,q}^{1\,0})_\mu(\mathbf{R}) \exp(i\,(l1-l)\omega'\,(t'-t_0')). \tag{4.38}$$

The sum includes terms like

$$D_{p\,q}^{1\,0} = \sum_l ({}^1d_p^{l+L})^* \, ({}^0d_q^l). \tag{4.39}$$

This share of eq.(4.38) describes an oscillating dipole moment with the frequency $\varpi'/(2\pi\Delta t) = L\omega/(2\pi) = \varpi/(2\pi)$. It fulfils separately the continuity equation for frequencies $\hbar\varpi = ((E_p^1)' - (E_q^0)')\acute{e}^2/\acute{a}$. Then the related energy-momentum tensor oscillate quasi "freely". Such "freely" oscillating current is the prerequisite for a radiation effect, as the theory of antennas shows [26/27]. The other terms have no influence on the far field. Here we consider the following current components (see eq.(3.31), $\hbar/(\acute{M}\acute{c}) = \acute{\alpha}\acute{a}$)

$${}^{1\,0}_{p\,q}\tilde{J}_\mu(\mathbf{R},t') = \, i\acute{e}\, D_{p\,q}^{1\,0} \, (j_{p\,q}^{1\,0})_\mu(\mathbf{R}) \exp(i\,\varpi'\,(t'-t_0')), \tag{4.40}$$

and

$${}^{0\,1}_{q\,p}\tilde{J}_\mu(\mathbf{R},t') = \, i\acute{e} D_{q\,p}^{0\,1} \, (j_{q\,p}^{0\,1})_\mu(\mathbf{R}) \exp(-i\,\varpi'\,(t'-t_0')), \tag{4.41}$$

with

$$
\begin{aligned}
(j^{1\,0}_{p\,q})_\mu &= \bar{H}^1_p \gamma_\mu H^0_q, \\
(j^{1\,0}_{p\,q})_3 &= -0.5\acute{c}\acute{\alpha}\acute{a}\, Y_{0\,0}\,(Y_{1\,0}(h^1_p(r))^* \,\tfrac{\partial\, h^0_q(r)}{\partial\, x_3} - \tfrac{\partial\,(Y_{1\,0}\, h^1_p(r))^*}{\partial\, x_3}\, h^0_q(r)), \\
(j^{0\,1}_{q\,p})_3 &= = -0.5\acute{c}\acute{\alpha}\acute{a}\, Y_{0\,0}\,((h^0_q(r))^* \,\tfrac{\partial\, Y_{1\,0}\, h^1_p(r)}{\partial x_3} - (\tfrac{\partial\, h^0_q(r)}{\partial x_3})^* Y_{1\,0}\, h^1_p(r)), \\
(j^{1\,0}_{p\,q})_4 &= Y_{1\,0} Y_{0\,0}\, \varrho^{1\,0}_{p\,q}(r) = Y_{1\,0} Y_{0\,0}\,(h^1_p(r))^* h^0_q(r), \\
(j^{0\,1}_{q\,p})_4 &= Y_{1\,0} Y_{0\,0}\, \varrho^{0\,1}_{q\,p}(r) = Y_{1\,0} Y_{0\,0}\,(h^0_q(r))^* h^1_p(r).
\end{aligned}
\tag{4.42}
$$

For a simple description we can assume that $D^{1\,0}_{p\,q}$ is given by the maximum of all products $|({}^1d^{l+L}_p)^*\,({}^0d^l_q)|$ and put this into the eqs.(4.40/4.41). That means, the coefficients are

$$
\begin{aligned}
a^{0r}_q(t') &= {}^{0r}d^{lmax}_q \exp(-i\,(lmax)\,\omega'(t' - t'_0)), \\
a^{1r}_p(t') &= {}^{1r}d^{lmax+L}_p \exp(-i\ (lmax +\ L)\omega'\,(t' - t'_0)).
\end{aligned}
\tag{4.43}
$$

It is a working hypothesis for the numerical calculations and includes a few questions. Some additional products $({}^1d^{l+\Delta\,L}_p)^*\,({}^0d^l_q))$ near lmax can contribute to a synchronously oscillating current, given in the eqs.(4.40/4.41). The continuity equation reads

$$
\frac{\partial({}^{1\,0}_{p\,q}\tilde{J}_\mu)}{\partial x_\mu} = i\acute{e}\frac{\partial}{\partial x_\mu}(\,D^{1\,0}_{p\,q}\,(\bar{H}^1_p \gamma_\mu H^0_q)\, exp(i\,\varpi\,(t - t_0))) = 0 \tag{4.44}
$$

or

$$
i\acute{e}\, D^{1\,0}_{p\,q}\,(\,\kappa\,(\bar{H}^1_p \gamma_4 H^0_q)\ +\ \tfrac{\partial}{\partial\, x_{\bar{\mu}}}\,(\bar{H}^1_p \gamma_{\bar{\mu}} H^0_q)\,)\exp(i\,\varpi\,(t - t_0)) = 0. \tag{4.45}
$$

Here is

$$\kappa = \varpi/\acute{c} = L\omega'/(\acute{c}\Delta t) \tag{4.46}$$

and due to the eqs.(3.26/3.27), one obtains

$$\frac{\partial}{\partial\, x_{\bar\mu}}\,(\bar H^1_p \gamma_{\bar\mu} H^0_q)\; = \frac{1}{\acute{c}\hbar}(\bar E^1_p - \bar E^0_q). \tag{4.47}$$

Therefore, the eq.(4.47) is fulfilled for constant values of $D^{1\,0}_{p\,q}$ and for

$$\varpi'\Delta t = (\bar E^1_p - \bar E^0_q)/\hbar. \tag{4.48}$$

This can be converted to

$$\varpi' = (E^1_p)' - (E^0_q)', \tag{4.49}$$

because of $(\hbar\Delta t)\; = \acute{e}^2/\acute{a}$ and $\bar E^1_p - \bar E^0_q \cong \frac{\acute{e}^2}{\acute{a}}((E^1_p)' - (E^0_q)')$. $\bar E^1_p,\ \bar E^0_q$ are eigenvalues of the Dirac equation and $(E^1_p)',\ (E^0_q)'$ of the linked Schrödinger equation. The value of

$$\omega' L = \varpi' \tag{4.50}$$

can be obtained in the numerical procedure by selecting of T'. Then, the eqs.(4.44/4.45) describe a "freely" oscillation of a current with a magnitude $|D^{10}_{pq}|$ in a wide range around the centre of the time interval. The Gibbs phenomenon does not allow such an interpretation near the time borders.

An absolutely precise frequency of a "freely" oscillation cannot be determined with a numerical procedure. However, it is possible to show that the force balance of the complete energy-momentum tensor requires the exact fulfilment of eq.(4.48). In preparation for this proof, we consider more details of the radiation process. The retarded potentials are given by

$$-i\acute{e}\,{}^{10}_{pq}\tilde{A}_{\mu}(\mathbf{R},t) = \acute{e}^2 \int \frac{1}{i\acute{e}r_{01}}\,{}^{10}_{pq}\tilde{J}_{\mu}(\mathbf{R}_1,\, t\,-\,r_{01}/\acute{c})dV_1, \tag{4.51}$$

what can be converted in near-field into

$$-i\acute{e}\,{}^{10}_{pq}\tilde{A}_{\bar{\mu}} \cong -i\acute{e}\tilde{A}'^{10}_{\bar{\mu}} \,-\, i(\acute{e}^2/\acute{a})\acute{a}\kappa\,\tilde{B}^{10}_{\bar{\mu}} \tag{4.52}$$

and

$$-i\acute{e}\,{}^{10}_{pq}\tilde{A}_4 = -i\acute{e}\,\tilde{A}'^{10}_4 + (\acute{e}^2/\acute{a})\,(\,-\,\acute{a}\kappa^2\tilde{G}^{10}_4 + i\acute{a}\kappa^3\,\tilde{Z}^{10}_4). \tag{4.53}$$

Besides is

$$-i\acute{e}\,{}^{01}_{qp}\tilde{A}_{\bar{\mu}} \cong -i\acute{e}\tilde{A}'^{01}_{\bar{\mu}} \,+\, i(\acute{e}^2/\acute{a})\acute{a}\kappa\,\tilde{B}^{01}_{\bar{\mu}} \tag{4.54}$$

and

$$-i\acute{e}\,{}^{01}_{qp}\tilde{A}_4 = -i\acute{e}\tilde{A}'^{01}_4 + (\acute{e}^2/\acute{a})\,(\,-\,\acute{a}\kappa^2\tilde{G}^{01}_4 - i\acute{a}\kappa^3\,\tilde{Z}^{01}_4). \tag{4.55}$$

The first term $\tilde{A}'^{10}_{\mu}$ describes the quasi-static share of the oscillating potentials that contributes to ${}^{0\,1}_{q}M^{1\,0}_{\mu}$. For $\mu = 4$ it has the form

$$\begin{array}{c} -i\acute{e}\,\tilde{A}'^{\,1\,0}_{4}(\mathcal{R},t) = (\acute{e}^2/\acute{a})D^{1\,0}_{p\,q}\exp(i\,\varpi\,(t-t_0))\frac{Y_{1\,0}}{Y_{0\,0}}\acute{a}\,r(I^{1\,0}_{p\,q}(\infty) \\ -\int\limits_0^r 1/y^4\int\limits_0^s x^3\varrho^{1\,0}_{p\,q}(x))dxdy). \end{array} \tag{4.56}$$

The dipole approximation of a long time stable oscillation with the normalised frequency κ leads to the dimensionless parameter

$$\tilde{B}^{1\,0}_{\bar{\mu}}(t) = \frac{1}{i\acute{e}}\int {}^{1\,0}_{p\,q}\tilde{J}_{\bar{\mu}}(\mathbf{R}_1,t)dV_1 \tag{4.57}$$

or

$$\tilde{B}^{0\,1}_{\bar{\mu}}(t) = \frac{1}{i\acute{e}}\int {}^{0\,1}_{q\,p}\tilde{J}_{\bar{\mu}}(\mathbf{R}_1,t)dV_1. \tag{4.58}$$

That means

$$\begin{array}{rcl} \tilde{B}^{1\,0}_{\bar{\mu}}(t) &=& exp(i\varpi(t-t_0))B^{1\,0}_{\bar{\mu}}, \\ B^{1\,0}_{\bar{\mu}} &=& D^{1\,0}_{p\,q}\int(\bar{H}^1_p\gamma_{\bar{\mu}}H^0_q)dV_1 \end{array} \tag{4.59}$$

and

$$\begin{array}{rcl} \tilde{B}^{0\,1}_{\bar{\mu}}(t) &=& exp(-i\varpi(t-t_0))B^{0\,1}_{\bar{\mu}}, \\ B^{0\,1}_{\bar{\mu}} &=& D^{0\,1}_{q\,p}\int(\bar{H}^0_q\gamma_{\bar{\mu}}H^1_p)dV_1. \end{array} \tag{4.60}$$

In addition, we find

$$\tilde{G}_4^{1\,0}(\mathbf{R}) = 1/2) D_{p\,q}^{1\,0} \exp(i\,\varpi\,(t - t_0)) \int r_{0\,1}(\bar{H}_p^1 \gamma_4 H_q^0) dV_1 \tag{4.61}$$

and

$$\tilde{Z}_4^{1\,0}(\mathbf{R}) = (1/6) D_{p\,q}^{1\,0} \exp(i\,\varpi\,(t - t_0)) \int (r_{0\,1})^2\,(\bar{H}_p^1 \gamma_4 H_q^0) dV_1. \tag{4.62}$$

Because of ($(r_{0\,1})^2\ = r^2 - 2x_{\bar{\mu}} x_{\bar{\mu}}^1 + (r_1)^2$) becomes

$$\tilde{Z}_4^{1\,0}(\mathbf{R}) = -\frac{x_3}{3} \exp(i\,\varpi\,(t - t_0)) \int x_3^1\, Y_{1\,0}(\vartheta_1) Y_{0\,0} \varrho_{p\,q}^{1\,0}(r_1) dV_1 \tag{4.63}$$

and due to eq.(4.45) is

$$\int\, x_{\bar{\mu}} \bar{H}_p^1 \gamma_4 H_q^0 dV = \frac{1}{\kappa} \int\, \bar{H}_p^1 \gamma_{\bar{\mu}} H_q^0 dV. \tag{4.64}$$

Hence we get

$$D_{p\,q}^{1\,0} \int\, x_3 \bar{H}_p^1 \gamma_4 H_q^0 dV = \frac{1}{\kappa} B_3^{1\,0} \tag{4.65}$$

and

$$\tilde{Z}_4^{1\,0}(\mathbf{R}) = -\frac{x_3}{3\kappa} \tilde{B}_3^{1\,0}. \tag{4.66}$$

The analogous relations, as consequence of $\partial/\partial x_\mu(\,{}_{q\,p}^{0\,1}\tilde{J}_{\bar{\mu}}(\mathbf{R}_1, t)) = 0$, are

$$D_{q\,p}^{0\,1} \int\, x_3 \bar{H}_q^0 \gamma_4 H_p^1 dV = -\frac{1}{\kappa} B_3^{0\,1} \tag{4.67}$$

and

$$\tilde{Z}_4^{0\,1}(\mathbf{R}) = \frac{x_3}{3\kappa}\tilde{B}_3^{0\,1}. \tag{4.68}$$

Such terms cause radiation. That means, the terms $({}^{0\,1}_{q}sm^{1\,0}_{l})$ and $({}^{1\,0}_{q}s\bar{m}^{0\,1}_{l})$ in the eqs.(4.29/4.30) cannot be neglected when the current oscillates synchronously over a long period of time. According to the eqs.(4.1-4.4) and the eqs.(4.32/4.33), the components

$$\begin{gathered} -i\acute{e}(\int(\bar{U}^0\gamma_{\bar{\mu}}U^1\,({}^{1\,0}_{q\,p}\tilde{A}_{\bar{\mu}}))^f dV - i\acute{e}(\int(\bar{U}^0\gamma_4 U^1({}^{1\,0}_{q\,p}\tilde{A}_4))^f dV \\ := \acute{e}^2/\acute{a}\;({}^0d^l_q)^*\,\acute{a}({}^{0\,1}_{q}sm^{1\,0}_{l}) \end{gathered} \tag{4.69}$$

must be taken into account because of which relevance to the far-field. Therefore we replace these terms in the eqs.(4.69) by the following the expressions:

$$\begin{gathered} \bar{U}^0\gamma_\mu U^1 := \exp(-i\,L\omega\,(t-t_0))({}^0d^l_q)^*\;{}^1d^{l+L}_p\bar{H}^0_q\gamma_\mu H^1_p, \\ -i\acute{e}({}^{1\,0}_{q\,p}\tilde{A}_{\bar{\mu}}) := -i(\acute{e}^2/\acute{a})(\acute{a}\kappa)\tilde{B}^{1\,0}_{\bar{\mu}}, \\ -i\acute{e}\,{}^{1\,0}_{q\,p}\tilde{A}_4 := i(\acute{e}^2/\acute{a})\acute{a}\kappa^3\,\tilde{Z}^{1\,0}_4 = -i(\acute{e}^2/\acute{a})\tfrac{\acute{a}\kappa^2\,x_3}{3}\tilde{B}^{1\,0}_3. \end{gathered} \tag{4.70}$$

Here $({}^0d^l_q)^*\;{}^1d^{l+L}_p$ stands for $D^{0\,1}_{q\,p}$ and $({}^1d^{l+L}_p)^*\;{}^0d^l_q$ for $D^{1\,0}_{p\,q}$. Then the eq.(4.69) is given by

$$\begin{gathered} (\acute{e}^2/\acute{a})(({}^0d^l_q)^*\;{}^1d^{l+L}_p(i\acute{a}\kappa\int\bar{H}^0_q\gamma_3 H^1_q dV(-B^{1\,0}_3) - i\tfrac{\acute{a}\kappa^2}{3}\int x_3\bar{H}^0_q\gamma_4 H^1 dV B^{1\,0}_3) \\ = -\tfrac{2}{3}i(\acute{e}^2/\acute{a})(({}^0d^l_q)^*\;{}^1d^{l+L}_p\acute{a}\kappa\int\bar{H}^0_q\gamma_3 H^1_q dV B^{1\,0}_3. \end{gathered} \tag{4.71}$$

That means, we have replaced the eqs.(4.69) by

$$\begin{aligned}&(\acute{e}^2/\acute{a})\acute{a}\ \ ({}^0d^l_q)^*\,({}^{0\,1}_{q}sm^{1\,0})\\ =\ & -i\tfrac{2}{3}(\acute{e}^2/\acute{a})(({}^0d^l_q)^*\ {}^1d^{l+L}_p\acute{a}\kappa\int\bar{H}^0_q\gamma_3H^1_q dV\,B^{1\,0}_3.\end{aligned}\tag{4.72}$$

Due to the eqs.(4.64/4.65) is $({}^0d^l_q)^*\ {}^1d^{l+L}_p\int\bar{H}^0_q\gamma_3H^1_p dV = B^{0\,1}_3$; Therefore becomes

$$(\acute{e}^2/\acute{a})\acute{a}\ \ ({}^0d^l_q)^*\,({}^{0\,1}_{q}sm^{1\,0}_l) = -i\tfrac{2}{3}(\acute{e}^2/\acute{a})\acute{a}\kappa B^{0\,1}_3B^{1\,0}_3.\tag{4.73}$$

With the eqs.(4.32) one obtains

$$i(\acute{e}^2/\acute{a})({}^0d^l_q)^*\,\tfrac{d}{dt'}(\,{}^0d^l_q)\ = -i(\acute{e}^2/\acute{a})\tfrac{2}{3}\acute{a}\kappa B^{1\,0}_3B^{0\,1}_3.\tag{4.74}$$

In addition, we need for the determination of $({}^{1\,0}_{q}s\bar{m}^{0\,1}_l)$ the shares of

$$\begin{aligned}-i\acute{e}(\textstyle\int(\bar{U}^1\gamma_{\bar{\mu}}U^0\,({}^{0\,1}_{q\,p}\tilde{A}_{\bar{\mu}}))^f dV - i\acute{e}(\int(\bar{U}^1\gamma_4U^0({}^{0\,1}_{q\,p}\tilde{A}_4))^f dV\\ := \acute{e}^2/\acute{a}\,({}^0d^0_q)\,\acute{a}({}^{1\,0}_{q}s\bar{m}^{0\,1}_l)\end{aligned}\tag{4.75}$$

which are relevant in far-field. That means, we substitute in the eqs.(4.75) the following expressions:

$\bar{U}^1\gamma_\mu U^0 := \sum_l\exp(i\,L\omega\,(t-t_0))({}^1d^{1+L}_p)^*\ {}^0d^l_q\bar{H}^1_p\gamma_\mu H^0_p$,
$-i\acute{e}({}^{0\,1}_{q\,p}\tilde{A}_{\bar{\mu}}) :=\ i(\acute{e}^2/\acute{a})(\acute{a}\kappa)\tilde{B}^{0\,1}_{\bar{\mu}}$,

$-i\acute{e}\,{}^{0\,1}_{q\,p}\tilde{A}_4 := -i(\acute{e}^2/\acute{a})(\acute{a}\kappa^3)\,\tilde{Z}_4^{0\,1} = i(\acute{e}^2/\acute{a})\frac{\acute{a}\kappa^2\,x_3}{3}\tilde{B}_3^{0\,1}.$

With eqs.(4.64/4.65) we get

$$\begin{array}{c}(\acute{e}^2/\acute{a})\sum_l({}^1d_p^{1+L})^*\ {}^0d_q^l\acute{a}\kappa\int\bar{H}_p^1\gamma_3H_q^0dV\,(B_3^{0\,1}) + i\frac{\acute{a}\kappa^2}{3}\int x_3\bar{H}_q^1\gamma_4H_q^0dV\,B_3^{0\,1}\\ = \frac{2}{3}(\acute{e}^2/\acute{a})\acute{a}\kappa({}^1d_p^{1+L})^*\ {}^0d_q^l\int\bar{H}_p^1\gamma_3H_q^0dV\,B_3^{0\,1}\end{array} \tag{4.76}$$

and one can eq.(4.75) substitute by

$$\begin{array}{rl} & (\acute{e}^2/\acute{a})\acute{a}\quad({}^{1\,0}_{q}s\bar{m}_l^{1\,0})({}^0d_q^l)\\ = & \frac{2}{3}(\acute{e}^2/\acute{a})\acute{a}\kappa(({}^1d_p^{l+L})^*\ {}^0d_q^l\int\bar{H}_p^1\gamma_3H_q^0dV\,B_3^{0\,1}.\end{array} \tag{4.77}$$

According to eq.(4.32) becomes

$$i(\acute{e}^2/\acute{a})\frac{d}{dt'}[({}^0d_q^l)^*)]\,(\,{}^0d_q^l)\ = -i\frac{2\acute{a}\kappa}{3}(\acute{e}^2/\acute{a})B_3^{1\,0}B_3^{0\,1}. \tag{4.78}$$

The sum of the eq.(4.74) and eq.(4.78) yields ($\acute{a}\kappa = \varpi\acute{\alpha}\Delta t$)

$$\frac{d}{dt'}(({}^0d_q^l)^*\,({}^0d_q^l))\ = -\frac{4\acute{a}\kappa}{3}B_3^{1\,0}B_3^{0\,1}. \tag{4.79}$$

An analogous consideration leads to

$$\frac{d}{dt'}(({}^1d_p^{l1})^*\,({}^1d_p^{l1}))\ = \frac{4\acute{a}\kappa}{3}B_3^{1\,0}B_3^{0\,1}, \tag{4.80}$$

due to the different sign of κ. These relations describe the charge exchange between the first and the zero excitation level during the time interval $t'_0 \le t' \le t'_o + T'$. It shows, how the well-known dipole equations, embedded in a transition field, describe radiation. Numerical calculations exhibit that in our cases the values of $|B_3^{1\,0}|$ are small and therefore, the related radiation loss too. An emission of one "photon" it takes about $10^{-6}s$ for the presented p-s transition. However, this process can run much faster if more atoms contribute to the radiation field. This will be shown in the next sections. Finally we mention that the far-field approximation of eq.(4.51) has the form $\tilde{A}_\mu = (c(\omega)/r)\exp(-i(\omega t - r/\acute{c}))$ where only for the radiation frequencies the coefficients $c(\omega)$ are not zero. This is, due to the different numbers of wavelengths up to the sources in the atom. Therefore the other frequencies of the charge dynamics can be excluded [27].

4.1.3 *A general approach to radiation effects*

Now we try to describe the influence of radiation in a more general way. The combination of eq.(4.1) with eq.(4.2) leads to

$$\hbar\tfrac{\partial}{\partial x_4}\bar{U}^0\gamma_4 U^0 = -\hbar\tfrac{\partial}{\partial x_{\bar{\mu}}}\bar{U}^0\gamma_{\bar{\mu}}U^0 + i\tfrac{\acute{e}}{c}(\tilde{A}_\mu^{1\,0}\bar{U}^0\gamma_\mu U^1 - \tilde{A}_\mu^{0\,1}\bar{U}^1\gamma_\mu U^0). \tag{4.81}$$

It results in

$$\int\hbar\tfrac{\partial}{\partial x_\mu}\{\bar{U}^0\gamma_\mu\, U^0\}dV = i\tfrac{\acute{e}}{c}\int(\tilde{A}_\mu^{1\,0}\bar{U}^0\gamma_\mu U^1 - \tilde{A}_\mu^{0\,1}\bar{U}^1\gamma_\mu U^0)dV. \tag{4.82}$$

This is not zero for a retarded potential $\tilde{A}$ and therefore the relationship

$$i\frac{\hbar}{\acute{c}}\frac{d}{dt}((a_q^0)^* a_q^0) = -i\frac{\acute{e}}{c}\int(\tilde{A}_\mu^{1\,0}\bar{U}^0\gamma_\mu U^1 - \tilde{A}_\mu^{0\,1}\bar{U}^1\gamma_\mu U^0)dV \neq 0 \tag{4.83}$$

must be taken into account. We insert the functions, presented in section 4.1.2, into the eqs.(4.82/4.83). Only the currents ${}^{1\,0}_{p\,q}\tilde{J}_\mu = \tilde{J}_\mu^{1\,0}$ and ${}^{0\,1}_{p\,q}\tilde{J}_\mu = \tilde{J}_\mu^{0\,1}$ form radiation moments. With these additional terms, given in the eqs.(4.69/4.72), and the eqs.(4.74/4.78), the right side of eq.(4.83) reads in our case

$$\begin{array}{c} -i\acute{e}\int(\ \tilde{A}_\mu^{1\,0}\bar{U}^0\gamma_\mu U^1 -\ \tilde{A}_\mu^{0\,1}\bar{U}^1\gamma_\mu U^0)dV \\ = \frac{\acute{e}^2}{\acute{a}}\int\frac{\acute{a}}{r_{0\,1}}\bar{U}^0(\mathbf{R},t')\gamma_\mu U^1(\mathbf{R},t')\,\bar{U}^1(\mathbf{R}_1,t')\gamma_\mu U^0(\mathbf{R}_1,t')dV_1 dV \\ -\frac{\acute{e}^2}{\acute{a}}\int\frac{\acute{a}}{r_{0\,1}}\bar{U}^1(\mathbf{R},t')\gamma_\mu U^0)(\mathbf{R},t')\,\bar{U}^0(\mathbf{R}_1,t')\gamma_\mu U^1(\mathbf{R}_1))dV_1 dV \\ -i\frac{4\acute{a}\kappa}{3}(\acute{e}^2/\acute{a})B_3^{1\,0}B_3^{0\,1}. \end{array} \tag{4.84}$$

Then, we can assume that the value of

$$i\frac{\hbar}{\acute{c}}\frac{d}{dt}((a_q^0)^* a_q^0) = -i\frac{4\acute{a}\kappa}{3}(\acute{e}^2/\acute{a})B_3^{1\,0}(t'_n)B_3^{0\,1}(t'_n) \tag{4.85}$$

is approximately constant in the interval $t'_n < t' < t'_n + T'$. A similar consideration is necessary to estimate the change of $\bar{U}^1\gamma_4 U^1$. The eqs.(4.3/4.4) lead to

$$\hbar\frac{\partial}{\partial x_4}\bar{U}^1\gamma_4 U^1 = -\hbar\frac{\partial}{\partial x_{\bar{\mu}}}\bar{U}^1\gamma_{\bar{\mu}}U^1 - i\frac{\acute{e}}{c}(\tilde{A}_\mu^{1\,0}\bar{U}^0\gamma_\mu U^1 - \tilde{A}_\mu^{0\,1}\bar{U}^1\gamma_\mu U^0) \tag{4.86}$$

and to

$$i\frac{\hbar}{\acute{c}}\frac{d}{dt}((a^1_p)^* a^1_p) = i\frac{4\acute{a}\kappa}{3}(\acute{e}^2/\acute{a})B_3^{1\,0}(t'_n)B_3^{0\,1}(t'_n). \tag{4.87}$$

The Eqs.(4.85/4.87)) describe the correction of the conservation laws of excitation levels by radiation. Using this result we calculate the influence of the radiation on the energy component of the associated tensor. One obtains for the time average of the volume integral over the components of such tensors the expressions

$$\frac{1}{T'}\int_{t'_n}^{T'+t_n}({}^{0\,0}_{q\,q}T^d_{4\,4}(t'))'\,dV\,dt' \;=\; -\frac{\acute{e}^2}{\acute{a}}\,{}^0d^0_q(t'_n)^*\,({}^0d^l_q(t'_n))\,(1/\acute{\alpha}^2 +\,l\;\omega')\,\bar{H}^0_q\gamma_4\,H^0_q \tag{4.88}$$

and

$$\frac{1}{T'}\int_{t'_n}^{T'+t'_n}({}^{1\,1}_{p\,p}T^d_{4\,4}(t'))\,dV\,dt' \;=\; -\,\frac{\acute{e}^2}{\acute{a}}\,{}^1d^{l1}_p(t'_n)^*\,({}^1d^{l1}_p(t'_n))(\,1/\acute{\alpha}^2 +\,l1\,\omega'). \tag{4.89}$$

Using these equations the linked change in the mean energy of the Dirac field becomes

$$\begin{aligned} -\frac{1}{T'}\frac{d}{dt'_n}\int_{t'_n}^{T'+t'_n}({}^{1\,1}T^d_{4\,4}(t') +\,{}^{0\,0}T^d_{4\,4}(t'))\,dV\,dt' \;=\\ -\,\frac{\acute{e}^2}{\acute{a}}\,\frac{d}{dt'_n}\,{}^1d^{l+L}_p(t'_n)^*\,({}^1d^{l+L}_p(t'_n))(\,1/\acute{\alpha}^2 +\,l\,\omega' + L\,\omega')\\ -\,\frac{\acute{e}^2}{\acute{a}}\,\frac{d}{dt'_n}\,{}^0d^l_q(t'_n)^*\,({}^0d^l_q(t'_n))(\,1/\acute{\alpha}^2 +\,l\,\omega'). \end{aligned} \tag{4.90}$$

That means,

$$-\tfrac{1}{T'}\tfrac{d}{dt'_n}\int_{t'_n}^{T'+t'_n}({}^{1\,1}T^d_{4\,4}(t') + {}^{0\,0}T^d_{4\,4}(t'))\,dV\,dt' \;=\; -\tfrac{\acute{e}^2}{\acute{a}}\;L\omega'\,\tfrac{d}{dt'_n}({}^1d_p^{l+L}(t'_n)^*\,({}^1d_p^{l+L}(t'_n))) \tag{4.91}$$

or with the eqs.(4.85/4.87)

$$-\tfrac{1}{T'}\tfrac{d}{dt'_n}\int_{t'_n}^{T'+t'_n}({}^{1\,1}T^d_{4\,4}(t') + {}^{0\,0}T^d_{4\,4}(t'))\,dV\,dt' \;=\; -\tfrac{\acute{e}^2}{\acute{a}}\;\tfrac{4\acute{a}\acute{c}\kappa^2}{3}B_3^{1\,0}(t'_n)B_3^{0\,1}(t'_n). \tag{4.92}$$

This can be replaced by

$$-\tfrac{1}{T'}\tfrac{d}{dt'_n}\int_{t'_n}^{T'+t'_n}({}^{1\,1}T^d_{4\,4}(t') + {}^{0\,0}T^d_{4\,4}(t'))\,dV\,dt' \;=\; \hbar\omega\tfrac{4\acute{a}\kappa}{3}\,(B_{\bar{\mu}}^{1\,0})^*(t'_n)B_{\bar{\mu}}^{1\,0}(t'_n) \tag{4.93}$$

because of $(B_{\bar{\mu}}^{1\,0})^* = -B_{\bar{\mu}}^{0\,1}$ and $\tfrac{\acute{e}^2}{\acute{a}}\;L\omega/\Delta t = \hbar\varpi$. Eq.(4.92) describes the radiation losses per Δt in $\acute{e}^2/\acute{a}$ and eq.(4.93) in $\hbar\omega$. Here is assumed that a share of the field oscillates synchronously in interval $t'_n \leq t' \leq t'_n + T'$ with the frequency $\varpi/(2\pi)$ and the fixed magnitude $D_{p\,q}^{1\,0}(t'_n) := {}^1d_p^{l+L}(t'_n)^*\,({}^1d_p^{l+L}(t'_n)$. The magnitude is slightly different in adjacent time intervals. In addition, one can also use the eqs.(4.79/4.80) to determine change in the integral $LS_{\bar{\mu}}$ over the torque. For example, because of

$$LS_3 = -\tfrac{i}{\acute{c}}\int(x_1(T^d_{2\,4} + \Delta T^m_{2\,4}) - x_2(T^d_{1\,4} + \Delta T^m_{1\,4}))dV, \tag{4.94}$$

one obtains

$$-\frac{1}{T'}\frac{d}{dt'_n}\int_{t'_n}^{T'+t'_n} LS_3 dt' \;=\; \hbar\Delta m\,\frac{d}{dt'_n}(d^l_p(t'_n)^*\,(d^l_p(t'_n))) \qquad (4.95)$$

where Δm describes the difference of magnet quantum numbers. In the discussed case is $\Delta m = 0$. It is also possible to show that the surface integrals over the far-field yield the same results as the near-field expressions [27].

4.1.4 *The field of multiple radiation moments*

If the wavelength $\lambda = 2\pi\acute{c}/\varpi$ is significantly larger than the distance between the atoms, we can assume that further atoms, located on $\mathbf{R}_k$ around a central atom with the coordinates $\mathbf{R}_0 = \mathbf{R}$, also have a transition state. This is a consequence of an overlap frequency $\varpi/(2\pi)$ of the electromagnetic field. Hence the complete radiation current is given by

$$ {}^{1\,0}_{p\,q}\tilde{J}^g_\mu \cong \sum_k {}^{1\,0}_{p\,q}J^k_\mu(\mathbf{R}'_k)\exp(i\,\varpi'\,(t'-t'_0)), \qquad (4.96)$$

with

$$\begin{aligned} {}^{1\,0}_{p\,q}J^n_\mu(\mathbf{R}'_k) &= i\acute{e}\;{}^kD^{1\,0}_{p\,q}(k)(j^k_{p\,q})_\mu(\mathbf{R}'_k),\\ (j^k_{p\,q})_\mu(\mathbf{R}'_k) &= \bar{H}_p(\mathbf{R}'_k)\gamma_\mu H_q(\mathbf{R}'_k),\\ D^{1\,0}_{p\,q}(k) &= \textstyle\sum_l({}^{1k}d^{l+L}_p)^*\,({}^{0k}d^l_q)\\ \mathbf{R}'_k &= \mathbf{R}-\mathbf{R}_k. \end{aligned} \qquad (4.97)$$

$\mathbf{R}_k$ describes the zero coordinates of the atom k. The radiation relevant vector potentials of the central atom are presented in the eqs.(4.52/4.55).

As long as the condition $\varpi \, r_{0\,k}/\acute{c} \;<<\; 1$ is fulfilled, the near-field approximation can be applied to determine all expressions ${}^{1\,0}_{p\,q}\tilde{A}^{k}_{\mu}(\mathbf{R}, t')$ of a transition field. We characterise these atoms with k=1, 2, ..., N. In our special case is $\varpi/\acute{c} = 0.002737\acute{a}^{-1}$, because of $\lambda = 2\pi\acute{c}/\varpi = 2296\acute{a} = 121.5nm$. This should cover a number of additional radiation sources. Of which vector potentials have around $\mathbf{R}'_k = (0, 0, 0)$ the form

$$-i\acute{e}\,{}^{1\,0}_{p\,q}\tilde{A}^{k}_{\bar{\mu}}(\mathbf{R}'_k, t') \cong -i\acute{e}\tilde{A}'^{1\,0}_{\bar{\mu}}(k)(\mathbf{R}'_k, t') - \;\acute{e}^2\kappa\,\tilde{B}^{1\,0}_{\bar{\mu}}(k), \tag{4.98}$$

and

$$-i\acute{e}\,{}^{1\,0}_{p\,q}\tilde{A}^{k}_{4}(\mathbf{R}'_k, t') = -i\acute{e}\,\tilde{A}'^{1\,0}_{4}(k)(\mathbf{R}'_k, t') + \acute{e}^2\,(\,-\,\kappa^2\tilde{G}^{1\,0}_{4}(k) + i\kappa^3\,\tilde{Z}^{1\,0}_{4}(k)). \tag{4.99}$$

The sums over radiation relevant components of the vector potentials on $\mathbf{R} = (0, 0, 0)$ are given by

$$-i\acute{e}\,({}^{1\,0}_{p\,q}\tilde{A}_{\bar{\mu}})^f \cong -\,\acute{e}^2\sum_k \kappa\,\tilde{B}^{1\,0}_{\bar{\mu}}(k), \tag{4.100}$$

and

$$-i\acute{e}\,({}^{1\,0}_{p\,q}\tilde{A}_{4})^f \cong \,\acute{e}^2\sum_k (i\kappa^3\,\tilde{Z}^{1\,0}_{4}(\mathbf{R}'_k). \tag{4.101}$$

According to eq.(4.59) or eq.(4.66), we get

$$\begin{aligned} \tilde{B}^{1\,0}_{\bar{\mu}}(k) &= \;exp(i\varpi'(t' - t'_0))B^{1\,0}_{\bar{\mu}}(k), \\ B^{1\,0}_{\bar{\mu}}(k) &= \;D^{1\,0}_{p\,q}(k)\,\int(\bar{H}^1_p\gamma_{\bar{\mu}}H^0_q)dV'_k \end{aligned} \tag{4.102}$$

or

$$\tilde{Z}^{1\,0}_{4}(\mathbf{R}'_k) = -\frac{1}{3}x_{\bar{\mu}}\tilde{B}^{1\,0}_{\bar{\mu}}(k) \tag{4.103}$$

because of $x^k_{\bar\mu}\int\bar H^1_p\gamma_4 H^0_q dV = 0$. The values of $B^{1\,0}_{\bar\mu}(k)$ are approximately constant in the time interval, as long as the near field conditions are met. If the transition states of these atoms are synchronised, one must add all linked radiation terms. Then, because of the eqs.(4.100/4.102), one obtains at the location of the central atom ($\mathbf{R}_0 = (0,0,0)$) instead of the eqs.(4.72) the following values

$$\begin{array}{c}({}^0\bar U^0)\gamma_\mu({}^0U^1) := \exp(-i\,\bar{\omega}'(t'-t'_0)({}^0d^l_q)^*\;{}^1d^{l+L}_p\bar H^0_q\gamma_\mu H^1_p,\\ -i\acute{e}\,({}^{1\,0}_{p\,q}\tilde A_{\bar\mu}(\mathbf{R}))^f \cong -(\acute{e}^2/\acute{a})\exp(i(\bar{\omega}'(t'-t'_0))\sum_k(\acute{a}\kappa\, B^{1\,0}_{\bar\mu}(k))\\ \times C^k_{p\,q}(\mathbf{R}-\mathbf{R}_k)\exp(i\phi^k_{p\,q}(\mathbf{R}-\mathbf{R}_k)),\\ -i\acute{e}\,({}^{1\,0}_{p\,q}\tilde A_4(\mathbf{R}))^f \cong -(\acute{e}^2/\acute{a})\exp(i(\bar{\omega}'(t'-t'_0))\sum_k(\frac{\acute{a}\kappa^2 x_{\bar\mu}}{3}B^{1\,0}_{\bar\mu}(k))\\ \times C^k_{p\,q}(\mathbf{R}-\mathbf{R}_k)\exp(i\phi^k_{p\,q}(\mathbf{R}-\mathbf{R}_k)).\end{array} \tag{4.104}$$

The parameters $B^{1\,0}_{\bar\mu}(k)$, $C^k_{p\,q}$, and $\phi^k_{p\,q}$, depend on the phase and the orientation of the currents $\tilde J^k_\mu(\mathbf{R}'_k,t')$ and of which distance from the origin. These expressions can be put in eq.(4.71). Then one obtains instead of eq.(4.74) the equation

$$\sum_{l,\,k}({}^0d^l_q)^*\frac{d}{dt'}({}^0d^l_q) = -\frac{2\acute{a}\kappa}{3}B^{0\,1}_{\bar\mu}(0)\sum_k({}^kO^{1\,0}_{p\,q})B^{1\,0}_{\bar\mu}(k) \tag{4.105}$$

and for eq.(4.80)

$$\sum_{l,\,k}\frac{d}{dt'}[({}^1d^l_q)^*)]({}^1d^l_q) = -\frac{2\acute{a}\kappa}{3}B^{1\,0}_{\bar\mu}(0)\sum_k({}^kO^{0\,1}_{q\,p})B^{0\,1}_{\bar\mu}(k), \tag{4.106}$$

The determination of the amount and the phase of $({}^kO^{1\,0}_{p\,q})$ and $({}^kO^{0\,1}_{q\,p})$ requires studying a more detailed model. However, one can assume that a dominating transition current of a central atom determines the phase angles of the transition currents of all atoms involved, as in the antennas of technical dipole fields. The reason is that a multiplication of $\exp(i\varphi_k)$ on the transition functions ${}^kU^0(\mathbf{R}'_k, t')$ and ${}^kU^1(\mathbf{R}'_k, t')$ has no influence on the solutions of the linked system as long as the radiation field is neglected. However, the phase angle can be influenced by the radiation field. Therefore a state with $|\sum_l \frac{d}{dt'}(({}^0d^l_q)^*)\,({}^0d^l_q))| \;>>\; \frac{2\acute{a}\kappa}{3}(B^{1\,0}_{\acute{\mu}})^* B^{1\,0}_{\acute{\mu}}$ is possible, which causes stronger charge exchange. Consequently, the model allows an emission of one "photon" faster than the mentioned 10^{-6}s. However, a precise description of radiation processes requires an intensive investigation of the system.

4.2 *The Dirac field of a transition state*

Now we look at the force and energy balance for a general solution of the Dirac equation which is only influenced by the quasi-static potentials. This is necessary to prove the causal behaviour of the system. The solution functions are

$$\begin{gathered} U = \sum_{s=0,\pm1,..} U^{z-s} \\ U^{z-s} = \sum_q {}^{z-s}a_q(t)\, exp(-i(\,(\acute{M}\,\acute{c}^2/\hbar)\,(t\,-\,t_0))H^{z-s}_q \end{gathered} \tag{4.107}$$

and

$$\begin{gathered} \bar{U} = \sum_z \bar{U}^z \\ \bar{U}^z = \sum_p ({}^z a_p(t))^*\, exp(i(\,(\acute{M}\,\acute{c}^2/\hbar)\,(t\,-\,t_0))\bar{H}^z_p, \end{gathered} \tag{4.108}$$

which lead to

$$\begin{gathered} J_\mu = \sum_{z,s} {}^{z\,z-s}_{p\,q}J_\mu(\mathbf{R},t) = \sum_{z',s'} {}^{z'+s'\,z'}_{p\,q}J_\mu(\mathbf{R},t), \\ {}^{z+s'\,z'}_{p\,q}J_\mu(\mathbf{R},t) = i\acute{e}\bar{U}^{z'+s'}\gamma_\mu U^{z'} \end{gathered} \tag{4.109}$$

and the quasi-static potentials

$$A_\mu^{s'} = \sum_z {}^{z+s'\,z}_{p\,q}A_\mu(\mathbf{R},t) = \sum_{z'} {}^{z'\,z'-s'}_{p\,q}A_\mu(\mathbf{R},t)$$
$${}^{z+s'\,z}_{p\,q}A_\mu(\mathbf{R},t) = i\acute{e}\sum_{p,q}(a_p^{z+s'}(t))^*\,a_q^z(t)\int\frac{1}{r_{01}}\,\bar{H}_p^{z+s'}(\mathbf{R}_1)\gamma_\mu H_q^z(\mathbf{R}_1)dV_1. \qquad (4.110)$$

We also need the following forms of the eigenvalue eqs.(3.26/3.27):

$$\gamma_{\bar{\mu}}(\hbar\,\frac{\partial\,H_q^{z-s}}{\partial\,x_{\bar{\mu}}}\;) = \frac{1}{\acute{c}}(\bar{E}_q^{z-s} + \frac{\acute{e}^2}{r}\,)\,\gamma_4 H_q^{z-s} - \acute{M}\,\acute{c}H_q^{z-s}, \qquad (4.111)$$

$$(\hbar\,\frac{\partial\,\bar{H}_p^z}{\partial\,x_{\bar{\mu}}}\;)\gamma_{\bar{\mu}} = \frac{1}{\acute{c}}\;\bar{H}_p^z\gamma_4(-\,\bar{E}_p^z\; - \;\frac{\acute{e}^2}{r}\,) + \acute{M}\,\acute{c}\,\bar{H}_p^z\;. \qquad (4.112)$$

The divergences of the lines of the energy-momentum tensor, formed with the functions of the eqs.(4.107/4.108) read

$$\frac{\partial\,{}^{z\,z-s}T^d_{\nu\,\mu}}{\partial\,x_\mu} = \frac{\acute{c}\,\hbar}{2}\,\frac{\partial}{\partial\,x_\mu}\,[\bar{U}^z\gamma_\mu\,\frac{\partial\,U^{z-s}}{\partial\,x_\nu} - \frac{\partial\,\bar{U}^z}{\partial\,x_\nu}\,\gamma_\mu\,U^{z-s}]. \qquad (4.113)$$

Using the eqs.(4.107/4.108) and the eqs.(4.111/4.112), we get for these divergences

$$\begin{aligned}\frac{\partial\,{}^{z\,z-s}T^d_{\nu\,\bar{\mu}}}{\partial\,x_{\bar{\mu}}} &= \tfrac{1}{2}(\,\textstyle\sum_p\,(a_p^z)^*\bar{H}_p^z\,(-\bar{E}_p^z - \tfrac{\acute{e}^2}{r}\,)\,\gamma_4\tfrac{\partial\,U^{z-s}}{\partial\,x_\nu}\\ &-\tfrac{\partial\,\bar{U}^z}{\partial\,x_\nu}\,\textstyle\sum_q\,a_q^{z-s}(\bar{E}_q^{z-s} + \tfrac{\acute{e}^2}{r}\,)\,\gamma_4\,H_q^{z-s}\\ &+\bar{U}^z\,\tfrac{\partial}{\partial\,x_\nu}\,[(\textstyle\sum_q\,a_q^{z-s}(\bar{E}_q^{z-s} + \tfrac{\acute{e}^2}{r}\,)\,\gamma_4\,H_q^{z-s}],\\ &-\tfrac{\partial}{\partial\,x_\nu}\,[\textstyle\sum_p(\,(a_p^z)^*\,\bar{H}_p^z\,(-\bar{E}_p^z - \tfrac{\acute{e}^2}{r}\,)\,]\,\gamma_4 U^{z-s}).\end{aligned} \qquad (4.114)$$

After adding the missing term for $\mu = 4$, it becomes for $\nu = \bar{\nu}$

$$\begin{array}{c} \frac{\partial\, ^{z\, z-s}T^d_{\bar{\nu}\,\mu}}{\partial\, x_\mu} \;=\; \frac{1}{2}\big(\,\sum_p \bar{H}^z_p\,(\acute{c}\hbar\frac{\partial\,(a^z_p)^*}{\partial\, x_4}\;-(a^z_p)^*\,(\bar{E}^z_p+\frac{\acute{e}^2}{r}\,))\,\gamma_4\frac{\partial\, U^{z-s}}{\partial\, x_{\bar{\nu}}} \\ -\frac{\partial\,\bar{U}^z}{\partial\, x_{\bar{\nu}}}\,\sum_q(\acute{c}\hbar\,\frac{\partial\, a^{z-s}_q}{\partial\, x_4}\,+\,c^{z-s}_q(\bar{E}^{z-s}_q+\,\frac{\acute{e}^2}{r}\,))\,\gamma_4\,H^{z-s}_q \\ +\bar{U}^z\,\frac{\partial}{\partial\, x_{\bar{\nu}}}\,\big[\sum_q((\acute{c}\hbar\,\frac{\partial\, a^{z-s}_q}{\partial\, x_4}\,+\,a^{z-s}_q(\bar{E}^{z-s}_q+\frac{\acute{e}^2}{r}\,))\;\gamma_4\,H^{z-s}_q\big] \\ -\frac{\partial}{\partial\, x_{\bar{\nu}}}\,\big[\sum_p(\,\bar{H}^z_p\,(\acute{c}\hbar\frac{\partial\,(a^z_p)^*}{\partial\, x_4}\;-(a^z_p)^*\,(\bar{E}^z_p+\frac{\acute{e}^2}{r}\,))\,\big]\,\gamma_4 U^{z-s}). \end{array} \tag{4.115}$$

Using the Fourier transformations of the functions

$$\bar{U}^z = \sum_{p,l1}(^z d^{l1}_p)^*\, exp(i(\,(\,\acute{M}\,\acute{c}^2/\hbar\,+\,l1\omega\,)(\,t\,-\,t_0))\bar{H}^z_p \tag{4.116}$$

and

$$U^{z-s} = \sum_{q,l}\,^{z-s}d^l_q\, exp(-i(\,(\,\acute{M}\,\acute{c}^2/\hbar\,+\,l\omega)\,(\,t\,-\,t_0))H^{z-s}_q, \tag{4.117}$$

one obtains

$$\begin{array}{c} \frac{\partial\, ^{z\, z-s}T^d_{\bar{\nu}\,\mu}}{\partial\, x_\mu} \;=\; \frac{1}{2}\sum_{p,q,l,l1}\,((^z d^{l1}_p)^*\,(^{z-s}d^l_q)\;\exp(i\;\;(l1\,-l)\omega\,(t\,-\,t_0)) \\ \times((\bar{H}^z_p\,\gamma_4\frac{\partial\, H^{z-s}_q}{\partial\, x_{\bar{\nu}}}\;-\;\frac{\partial\,\bar{H}^z_p}{\partial\, x_{\bar{\nu}}}\,\gamma_4\,H^{z-s}_q)\,(\,(\,l1\,-l\,)\omega\,\hbar\,-\bar{E}^z_p\;+\,\bar{E}^{z-s}_q\,) \\ +\,2\,\bar{H}^z_p\gamma_4\,H^{z-s}_q\,(\frac{\partial\,(\,\acute{e}^2/r)}{\partial\, x_{\bar{\nu}}}\,)). \end{array} \tag{4.118}$$

This can be described for s=1 and z=1 by

$$\frac{\partial\, {}^{1\,0}T^{d}_{\bar\nu\,\mu}}{\partial\, x_\mu} \;\cong\; \frac{1}{2}\sum_{p,q,l,l1}\left(({}^{1}d^{l1}_{p})^{*}\,({}^{0}d^{l}_{q})\;\exp(i\;\;(l1\,-l)\omega\,(t\,-\,t_0)\,)\right.$$
$$\times((\bar H^{1}_{p}\,\gamma_4\frac{\partial\, H^{0}_{q}}{\partial\, x_{\bar\nu}}\;-\frac{\partial\,\bar H^{1}_{p}}{\partial\, x_{\bar\nu}}\;\gamma_4\, H^{0}_{q})\,(\,(\,l1\,-l\,)\omega\,\hbar\,-\bar E^{1}_{p}\;+\;\bar E^{0}_{q}\,)$$
$$+\,2\,\bar H^{1}_{p}\gamma_4\, H^{0}_{q}\,(\frac{\partial\,(\,\acute e^{2}/r)}{\partial\, x_{\bar\nu}}\,)). \tag{4.119}$$

For $\nu = 4$ the components are

$$ {}^{z\,z-s}T^{d}_{4\,\bar\mu} \;=\; -\frac{1}{2}\sum_{p,q,l,l1}\left(({}^{z}d^{l1}_{p})^{*}\,({}^{z-s}d^{l}_{q})\;\exp(i\;\;(l1\,-l)\omega\,(t\,-\,t_0))\right.$$
$$\times(2\,\acute M\,\acute c^{2}+\,\hbar(l1\,+\,l)\,\omega)\,\bar H^{z}_{p}\gamma_{\bar\mu}\,H^{z-s}_{q} \tag{4.120}$$

and

$$ {}^{z\,z-s}T^{d}_{4\,4} \;=\; -\frac{1}{2}\sum_{p,q,l,l1}\left(({}^{z}d^{l1}_{p})^{*}\,({}^{z-s}d^{l}_{q})\;\exp(i\;\;(l1\,-l)\omega\,(t\,-\,t_0))\right.$$
$$\times(2\,\acute M\,\acute c^{2}+\,\hbar(l1\,+\,l)\,\omega)\,\bar H^{z+s}_{p}\gamma_4\,H^{z}_{q}. \tag{4.121}$$

The divergence of the fourth line is

$$\frac{\partial}{\partial\, x_\mu}\,({}^{z\,z-s}T^{d}_{4\,\mu}) \;=\; -\frac{1}{2\acute c}\sum_{p,q,l,l1}\left(({}^{z}d^{l1}_{p})^{*}\,({}^{z-s}d^{l}_{q})\;\exp(i\;\;(l1\,-l)\omega\,(t\,-\,t_0))\right.$$
$$\times\,(\hbar(l1\,-l)\omega\;-\;\bar E^{z}_{p}\;+\;\bar E^{z-s}_{q})(2\,\acute M\,\acute c^{2}+\,\hbar(l1\,+\,l)\,\omega)\,\bar H^{z}_{p}\gamma_4\,H^{z-s}_{q} \tag{4.122}$$

because of

$$\frac{\partial}{\partial\, x_{\bar\mu}}\bar H^{z}_{p}\gamma_{\bar\mu}\,H^{z-s}_{q} \;=\; \frac{1}{2\acute c\hbar}(-\,\bar E^{z}_{p}\;+\;\bar E^{z-s}_{q})\,\bar H^{z}_{p}\gamma_4\,H^{z-s}_{q} \tag{4.123}$$

and

$$\frac{\partial}{\partial\, x_4}\,({}^{z\,z-s}T^{d}_{4\,4}) \;=\; -\frac{1}{2\acute c}\sum_{p,q,l,l1}\left(({}^{z}d^{l1}_{p})^{*}\,({}^{z-s}d^{l}_{q})\;\exp(i\,(l1\,-l)\omega\,(t\,-\,t_0))\right.$$
$$\times\,(l1\,-l)\omega\,(2\,\acute M\,\acute c^{2}+\,\hbar(l1\,+\,l)\,\omega)\,\bar H^{z}_{p}\gamma_4\,H^{z-s}_{q}. \tag{4.124}$$

One gets for the space integral over the eq.(4.122), due to the orthogonality of H_q^{z-s} and H_p^z, the expression

$$\begin{array}{c} \int \frac{\partial}{\partial x_\mu} \, ({}^{z\,z-s}T^d_{4\,\mu}) \, dV \;=\; -\,\delta(s) \frac{1}{2\acute{c}} \sum_{q,l,l1} \left(({}^{z}d_q^{l1})^* \, ({}^{z-s}d_q^{l}) \right. \\ \left. \times\, \exp(i\;\;(l1\,-l)\omega\,(t\,-\,t_0))\,(l1\,-l)\omega\,(2\,\acute{M}\,\acute{c}^2 + \hbar(l1\,+\,l)\,\omega\right). \end{array} \tag{4.125}$$

Its mean value in time is given for s=0 by

$$\frac{1}{T} \int_{t_n}^{T+t_n} \int \frac{\partial}{\partial x_\mu} \, ({}^{z\,z}T^d_{4\,\mu}(t)) \, dV dt \;=\; 0. \tag{4.126}$$

This is a consequence of the continuity eq.(3.122), which yields for $2\,\acute{M}\,\acute{c}^2 \;>>\; \hbar(l1\,+\,l)\,\omega$ and z=1, s=0 the constant value

$$\frac{1}{T} \int_{t_n}^{T+t_n} ({}^{1\,1}T^d_{4\,4}(t)) \, dV dt \;=\; -\,\sum_{p,l1} \left(({}^{1}d_p^{l1})^* \, ({}^{1}d_p^{l1}\right) \acute{M}\,\acute{c}^2 \tag{4.127}$$

and for z=0, s=0

$$\frac{1}{T} \int_{t_n}^{T+t_n} ({}^{0\,0}T^d_{4\,4}(t))) \, dV dt \;=\; -\,\sum_{q,l} \left(({}^{0}d_q^{l})^* \, ({}^{0}d_q^{l}\right) \acute{M}\,\acute{c}^2, \tag{4.128}$$

if the radiation can be neglected.

To describe the influence of purely quasi-static potentials on the force and energy balance, we use the following Dirac equations

$$
\begin{array}{c}
\bar{U}\sum_z \gamma_\mu\,\hbar\frac{\partial U^z}{\partial\,x_\mu} \;=\; \frac{1}{\acute{c}}(\acute{e}^2/r\,)\bar{U}\gamma_4\,U \\
+\,i\frac{\acute{e}}{\acute{c}}\,\sum_{z,s,s'=0,\pm1,..}(A_\mu^{-s'}\,\bar{U}^z\gamma_\mu\,U^{z-s}\,)\;-\;\acute{M}\,\acute{c}\bar{U}\,U
\end{array}
\tag{4.129}
$$

and

$$
\begin{array}{c}
\sum_z(\hbar\,\frac{\partial\,\bar{U}^z}{\partial\,x_\mu}\,)\gamma_\mu U = -\,\frac{1}{c}(\acute{e}^2/r)\bar{U}\gamma_4 U \\
-i\frac{\acute{e}}{c}\,\sum_{z,s,s'=0,\pm1,..}(A^{-s'}\bar{U}^z\,\gamma_\mu\,U^{z-s}\,)+\;\acute{M}\,\acute{c}\,\bar{U}U.
\end{array}
\tag{4.130}
$$

We don't use of the relation

$$
\begin{array}{c}
i\frac{\acute{e}}{\acute{c}}\,\sum_{s'=0,\pm1,..}\int(A_\mu^{-s'}\,\bar{U}^z\gamma_\mu\,U^{z-s}dV \\
=\,i\frac{\acute{e}^2}{c}\,\sum_{z',s'=0,\pm1,..}\int(\int\,\bar{U}^z\gamma_\mu\,U^{z-s}\,\int\frac{1}{r_{01}}\bar{U}^{z'-s'}\gamma_\mu U^{z'}dV_1 dV \\
=\,i\frac{\acute{e}^2}{c}\,\sum_{z'}\int(\int\,\bar{U}^z\gamma_\mu\,U^{z-s}\,\int\frac{1}{r_{01}}\bar{U}^{z'-s}\gamma_\mu U^{z'}dV_1 dV
\end{array}
\tag{4.131}
$$

because the terms with $s\neq s'$ are necessary for the local description of the force balance. The eqs.(4.129/4.130) enables the formulation of the divergence eq.(4.119) in an alternate way. The eqs.(4.129/4.130) lead to

$$
\begin{array}{c}
\sum_{z,s}\frac{\partial\,{}^{z\,z-s}T^d_{\nu\mu}}{\partial\,x_\mu}\;=\sum_{z,\,s}\,\bar{U}^z\gamma_4\,U^{z-s}\frac{\partial}{\partial\,x_\nu}\,(\acute{e}^2/r) \\
+i\acute{e}\sum_{z,s,s'}(\frac{\partial\,A_\mu^{-s'}}{\partial\,x_\nu}\,)\bar{U}^z\gamma_\mu U^{z-s}\,).
\end{array}
\tag{4.132}
$$

It agrees with eq.(2.35) if we neglect the radiation. With the help of the Fourier transformation, the eq.(4.129) can be replaced by

$$\sum_{z,s} \bar{U}^z \gamma_\mu (\hbar \frac{\partial U^{z-s}}{\partial x_\mu}) =$$
$$\frac{1}{c} \sum_{p,q,z,z',s,s',l,l1,\mu} ({}^z d_p^{l1})^* \, {}^{z-s}d_q^l \, \exp(-i\,(\acute{M}\,\acute{c}^2/\hbar \, + \, (l-l1)\,\omega\,)(t \, - \, t_0))$$
$$\times(- \, \frac{\acute{e}^2}{\acute{a}}((_q^{z\;z-s} m g_l^{z'-s'\,z'})_\mu / ({}^{z-s}d_q^l\,) + \frac{\acute{e}^2}{r}\,) \, \bar{H}_p^z \gamma_4 H_q^{z-s} \, - \, \acute{M}\,\acute{c}^2 \, \bar{H}_p^z H_q^{z-s}) \tag{4.133}$$

and eq.(4.130) by

$$\sum_{z,s} (\hbar \frac{\partial \bar{U}^z}{\partial x_\mu} \gamma_\mu U^{z-s}) \, =$$
$$\frac{1}{c} \sum_{p,q,z,z',s,s',l,l1,\mu} ({}^z d_p^{l1})^* \, {}^{z-s}d_q^l \, \exp(-i\,(\acute{M}\,\acute{c}^2/\hbar \, + \, (l-l1)\,\omega\,)(t \, - \, t_0))$$
$$\times(- \, \frac{\acute{e}^2}{\acute{a}}((_p^{z\;z-s} \bar{m} g_{l1}^{z'-s'\,z'})_\mu / ({}^z d_p^{l1})^*) \, + \frac{\acute{e}^2}{r}\,) \bar{H}_p^z \gamma_4 H_q^{z-s} \, + \, \acute{M}\,\acute{c}^2 \, \bar{H}^z H_q^{z-s}) \tag{4.134}$$

Here is

$$(_q^{z\;z-s} m g_l^{z'-s'\,z'})_\mu = \, \frac{1}{T} \, \int_{t_0}^{t_0+T} \exp(il\omega\,(t-t_0)) \, {}_q^{z\;z-s'} M_\mu^{z'-s'\,z'}(t) dt \tag{4.135}$$

with

$$\sum_\mu \, {}_q^{z\,z-s} M_\mu^{z'-s'\,z'}(t) = -i\acute{a}\, exp(i(t-t_0)/\acute{\alpha}^2)$$
$$\times \int \bar{H}_q^z \gamma_\mu U^{z-s}(t) \;\; \int \frac{1}{r_{01}} \bar{U}^{z'\,-s'} \gamma_\mu U^{z'} dV_1 dV \tag{4.136}$$

or

$$(_p^{z\;z-s} \bar{m} g_l^{z'-s'\,z'})_\mu = \, \frac{1}{T} \, \int_{t_0}^{t_0+T} \exp(-il1\omega\,(t-t_0)) \, {}_p^{z\;z-s} \bar{M}_\mu^{z'-s'\,z'}(t) dt \tag{4.137}$$

with

$$\begin{array}{c} \sum_\mu {}^{z\,z-s}_{\;p}\bar{M}^{z'-s'\,z'}_\mu(t) = -i\acute{a}\,exp(-i(t-t_0)/\acute{\alpha}^2) \\ \times\int \bar{U}^z\gamma_\mu H^{z-s'}_p(t)\ \int \frac{1}{r_{01}}\bar{U}^{z'-s'}\gamma_\mu U^{z'}dV_1 dV\ . \end{array} \tag{4.138}$$

According to eq.(4.111) becomes

$$\gamma_\mu\,\hbar\frac{\partial U^{z-s}}{\partial\,x_\mu}\ =\ \frac{1}{\acute{c}}\sum_q\,(\,-\,i\hbar\frac{d}{dt}a^{z-s}_q + a^{z-s}_q(\,E^{z-s}_q + (\acute{e}^2/r\,)))\gamma_4 H^{z-s}_q. \tag{4.139}$$

In addition, one can write in purely quasi-static cases (see eqs.(4.12/4.13))

$$i\hbar\frac{d}{dt}a^{z-s}_q =\ E^{z-s}_q a^{z-s}_q +\ \frac{\acute{e}^2}{\acute{a}}\sum_{\mu,\,z',s'}\ {}^{z\,z-s}_{\;q}M^{z'-s'\,z'}_\mu(t). \tag{4.140}$$

After changing the dimension is

$$\frac{\acute{e}^2}{\acute{a}}\frac{d}{dt'}a^{z-s}_q = i\frac{\acute{e}^2}{\acute{a}}((E^{z-s}_q)'a^{z-s}_q(t') +\ \sum_{\mu,\,z',s'}\ {}^{z\ z-s}_{\;q}M^{z'-s'\ z'}_\mu(t')). \tag{4.141}$$

Besides, one obtains (see eq.4.30)

$$\begin{array}{c} i\frac{\acute{e}^2}{\acute{a}}\frac{d}{dt'}(a^z_p(t'))^*\ =\ -\frac{\acute{e}^2}{\acute{a}}\,((E^z_p)'(a^z_p(t'))^* \\ +\sum_{\mu,\,z',\,s'}\ {}^{z\,z-s}_{\;p}\bar{M}^{z'-s'\,z'}_\mu(t')). \end{array} \tag{4.142}$$

The Fourier transformation leads to

$$\begin{aligned} i\frac{\acute{e}^2}{\acute{a}}\frac{d}{dt'}a_q^{z-s}(t') = (\acute{e}^2/\acute{a})\textstyle\sum_l (l\,\omega'\ {}^{z-s}d_q^l \\ +\, i\frac{d\,({}^{z-s}d_q^l)}{d\,t'})\exp(-i(\,l\,\omega'\,)(t'-t_0')). \end{aligned} \tag{4.143}$$

Therefore in quasi-static case ($\frac{d\,({}^{z-s}d_q^l)}{d\,t'} = 0$), the eq.(4.142) results in

$$(\acute{e}^2/\acute{a})(l\,\omega'\,)\,{}^{z-s}d_q^l = (\acute{e}^2/\acute{a})(\,(E_q^{z-s})'\ {}^{z-s}d_q^l + \textstyle\sum_{\mu,z',s'},({}_q^{z}\,{}^{z-s}mg_l^{z'-s'\,z'})_\mu)\,. \tag{4.144}$$

and the eq.(4.142) in

$$\begin{aligned} (-\acute{e}^2/\acute{a})(l1\,\omega\,)\,({}^z d_p^{l1})^* = (\acute{e}^2/\acute{a})(\,-\,(E_p^z)'\,({}^z d_p^{l1})^* \\ -\,\textstyle\sum_{\mu,\,z',\,s'}({}_p^{z}\,{}^{z-s}\bar{m}g_{l1}^{z'-s'\,z'})_\mu)\,. \end{aligned} \tag{4.145}$$

These expressions yield

$$\begin{aligned} &(\acute{e}^2/\acute{a})\tfrac{1}{2}\textstyle\sum_{l,l1}\,(({}^z d_p^{l1})^*\,({}^{z-s}d_q^l)\,\exp(i\ \ (l1\,-l)\omega'\,(t'\,-\,t_0'))\,(\,l1\,-l\,)\omega' \\ &= \tfrac{1}{2}\textstyle\sum_{l,l1}\,(({}^z d_p^{l1})^*\,({}^{z-s}d_q^l)\,\exp(i\ \ (l1\,-l)\omega'\,(t'\,-\,t_0'))(\,(E_p^z)'\,-\,(E_q^{z-s})'\,) \\ &\quad +\textstyle\sum_{z',s',\mu}(({}_q^{z}\,{}^{z-s}\bar{m}g_{l1}^{z'-s'\,z'})_\mu/({}^z d_p^{l1})^*\,-\,({}_q^{z}\,{}^{z-s}mg_l^{z'-s'\,z'})_\mu/({}^{z-s}d_q^l)) \end{aligned} \tag{4.146}$$

and thus

$$
\begin{aligned}
&(\acute{e}^2/\acute{a})\tfrac{1}{2}\sum_{l,l1}\left(({}^{z}d_p^{l1})^*\,({}^{z-s}d_q^{l})\,\exp(i\;(l1-l)\omega'\,(t'-t_0'))\,(\,l1-l\,)\omega'\right.\\
&\qquad\times(\bar{H}_p^z\,\gamma_4\frac{\partial\,H_q^{z-s}}{\partial\,x_{\bar\nu}}-\frac{\partial\,\bar{H}_p^z}{\partial\,x_{\bar\nu}}\,\gamma_4\,H_q^{z-s})\\
&=\tfrac{1}{2}\sum_{l,l1}\left(({}^{z}d_p^{l1})^*\,({}^{z-s}d_q^{l})\,\exp(i\;(l1-l)\omega'\,(t'-t_0'))(\,(E_p^z)'-(E_q^{z-s})'\,)\right.\\
&\quad+\sum_{z',s',\mu}\left(({}_q^{z\,z-s}\bar{m}g_{l1}^{z'-s'\,z'})_\mu/({}^{z}d_p^{l1})^*\right)-({}_q^{z\,z-s}mg_l^{z'-s'\,z'})_\mu/({}^{z-s}d_q^{l}))\\
&\qquad\times(\bar{H}_p^z\,\gamma_4\frac{\partial\,H_q^{z-s}}{\partial\,x_{\bar\nu}}-\frac{\partial\,\bar{H}_p^z}{\partial\,x_{\bar\nu}}\,\gamma_4\,H_q^{z-s})\,.
\end{aligned}
\tag{4.147}
$$

If one puts this expression in eq.(4.118), one obtains

$$
\begin{aligned}
\frac{\partial\,{}^{z\,z-s}T_{\bar\nu\,\mu}^{d}}{\partial\,x_\mu}&=\frac{\acute{e}^2}{2\acute{a}}\sum_{p,q,l,l1}({}^{z}d_p^{l1})^*\,{}^{z-s}d_q^{l}\,\exp(i\;(l1-l)\omega'\,(t'-t_0'))\\
&\times((\,\sum_{z',s',\mu}(-({}_p^{z\,z-s}\bar{m}g_{l1}^{z'-s'\,z'})_\mu/({}^{z}d_p^{l1})^*+({}_q^{z\,z-s}mg_l^{z'-s'\,z'})_\mu/({}^{z-s}d_q^{l})\\
&\quad-(E_p^z)'+(E_{q-s}^z)'\,)\,(\bar{H}_p^z\,\gamma_4\frac{\partial\,H_q^{z-s}}{\partial\,x_{\bar\nu}}-\frac{\partial\,\bar{H}_p^z}{\partial\,x_{\bar\nu}}\,\gamma_4\,H_q^{z-s})\\
&\quad+\bar{H}_p^z\gamma_4\,H_q^{z-s}\,\frac{\partial\,(\acute{e}^2/r)}{\partial\,x_{\bar\nu}}\,))
\end{aligned}
\tag{4.148}
$$

and for z=1, s=1 it is

$$
\begin{aligned}
\frac{\partial\,{}^{1\,0}T_{\bar\nu\,\mu}^{d}}{\partial\,x_\mu}&=\frac{\acute{e}^2}{2\acute{a}}\sum_{p,q,l,l1}({}^{1}d_p^{l1})^*\,{}^{0}d_q^{l}\,\exp(i\;(l1-l)\omega'\,(t'-t_0'))\\
&\times((\,\sum_{z',s',\mu}(-({}_p^{1\,0}\bar{m}g_{l1}^{z'-s'\,z'})_\mu/({}^{1}d_p^{l1})^*+({}_q^{1\,0}mg_l^{z'-s'\,z'})_\mu/({}^{0}d_q^{l}))\\
&\quad-(E_p^1)'+(E_q^0)'\,)\,(\bar{H}_p^1\,\gamma_4\frac{\partial\,H_q^{0}}{\partial\,x_{\bar\nu}}-\frac{\partial\,\bar{H}_p^1}{\partial\,x_{\bar\nu}}\,\gamma_4\,H_q^{0})\\
&\quad+\bar{H}_p^1\gamma_4\,H_q^{0}\,\frac{\partial\,(\acute{e}^2/r)}{\partial\,x_{\bar\nu}}\,)).
\end{aligned}
\tag{4.149}
$$

Using eq.(4.147), the eq.(4.122) reads

$$
\begin{aligned}
\frac{\partial}{\partial\,x_\mu}\,({}^{z\,z-s}T_{4\,\mu}^{d})&=\frac{\acute{e}^2}{2\acute{a}}\sum_{p,q,l,l1}\left(({}^{z}d_p^{l1})^*\,({}^{z-s}d_q^{l})\,\exp(i\;(l1-l)\omega'\,(t'-t_0'))\right.\\
&\times((\sum_{z',s'}(-({}_p^{z\,z-s}\bar{m}g_{l1}^{z'-s'\,z'})_\mu/({}^{z}d_p^{l1})^*+({}_q^{z\,z-s'}mg_l^{z'-s'\,z'})_\mu/({}^{z}d_q^{l}))\\
&\quad-(E_p^z)'+(E_q^{z-s})')(2/\acute{\alpha}^2+(l1+l)\,\omega')\,\bar{H}_p^z\gamma_4\,H_q^{z-s}
\end{aligned}
\tag{4.150}
$$

and for z=1, s=1 becomes

$$\begin{array}{c} \frac{\partial}{\partial x_\mu}\,({}^{1\,0}T^{d}_{4\,\mu}) \;=\; \frac{\acute{e}^2}{2\acute{a}}\sum_{p,q,l,l1}(({}^{1}d^{l1}_{p})^*\,({}^{0}d^{l}_{q})\,\exp(i\;\,(l1\,-l)\omega'\,(t'\,-\,t'_0)) \\ \times\,(\sum_{z',s',\mu}(-({}^{1\,0}_{p}\bar{m}g^{z'-s'\,z'}_{l1})_\mu/({}^{1}d^{l1}_{p})^*\;+\;({}^{1\,0}_{q}mg^{z'-s'\,z'}_{l})_\mu/({}^{0}d^{l}_{q})) \\ -\,(E^1_p)'\,+\,(E^0_q)')(2/\acute{\alpha}^2\,+\,(l1\,+\,l)\,\omega')\,\bar{H}^1_p\gamma_4\,H^0_q. \end{array} \tag{4.151}$$

The comparison of eq.(4.118) and eq.(4.149) leads to

$$\begin{array}{c} \sum_{p,q,l,l1}(({}^{z}d^{l1}_{p})^*\,({}^{z-s}d^{l}_{q})\,\exp(i\;\,(l1\,-l)\omega'\,(t'\,-\,t'_0))(\,(l1-l)\,\omega' \\ -(E^z_p)'\,+\,(E^{z-s}_q)'\,-\,\sum_{z',s',\mu}(({}^{z\,z-s}_{p}\bar{m}g^{z'-s'\,z'}_{l1})_\mu/({}^{z}d^{l1}_{p})^* \\ +\,({}^{z\,z-s}_{q}mg^{z'-s'\,z'}_{l})_\mu/({}^{z-s}d^{l}_{q}))) \\ \times(\;\bar{H}^z_p\,\gamma_4\frac{\partial\,H^{z-s}_q}{\partial\,x_\nu}\;-\frac{\partial\,\bar{H}^z_p}{\partial\,x_\nu}\,\gamma_4\,H^{z-s}_q\;+2\,\bar{H}^z_p\gamma_4\,H^{z-s}_q\,\frac{\partial\,(\acute{e}^2/r)}{\partial\,x_\nu}\,)\;=\;0 \end{array} \tag{4.152}$$

or eq.(4.119) and (4.150) to

$$\begin{array}{c} \sum_{p,q,l,l1}\,(({}^{z}d^{l1}_{p})^*\,({}^{z-s}d^{l}_{q})\,\exp(i\;\,(l1\,-l)\omega'\,(t'\,-\,t'_0))((l1-l)\,\omega' \\ -(E^z_p)'\,+\,(E^{z-s}_q)'\,-\,\sum_{z',s',\mu}(({}^{z\,z-s}\bar{m}g^{z'-s'\,z'}_{l1})_\mu/({}^{z}d^{l1}_{p})^* \\ +\,({}^{z\,z-s}_{q}mg^{z'-s'\,z'}_{l})_\mu/({}^{z-s}d^{l}_{q})))(\,2/\acute{\alpha}^2 \\ +\,(l1\,+\,l)\,\omega')\,\bar{H}^z_p\gamma_4\,H^{z-s}_q\;\;=\;0. \end{array} \tag{4.153}$$

Both results are obtained directly by multiplying of eq.(4.147) with different functions and adding up of all terms. These equations become for z=1, z'=1 and s=1, s'= 1 the form

$$
\begin{aligned}
\Sigma_{p,q,l,l1}\,(({}^1d_p^{l1})^*\,({}^0d_q^l)\exp(i\ (l1\ -l)\omega'\,(t'\ -\ t_0'))(\,(l1-l)\,\omega' \\
-(E_p^1)'\ +\ (E_q^0)'\ -\ \Sigma_\mu((^{1\,0}_{\,p}\bar{m g}^{0\,1}_{l1})_\mu/({}^1d_p^{l1})^* \\
+\ (^{1\,0}_{\,q}mg^{0\,1}_{l})_\mu/({}^0d_q^l)))(\ \bar{H}_p^1\,\gamma_4\frac{\partial\,H_q^0}{\partial\,x_\nu} \\
-\frac{\partial\,\bar{H}_p^1}{\partial\,x_\nu}\ \gamma_4\,H_q^0\ +2\,\bar{H}_p^1\gamma_4\,H_q^0\,\frac{\partial\,(\acute{e}^2/r)}{\partial\,x_\nu}\,)\ =\ 0
\end{aligned}
\qquad (4.154)
$$

and

$$
\begin{aligned}
\Sigma_{p,q,l,l1}\ (({}^1d_p^{l1})^*\,({}^0d_q^l)\exp(i\ (l1\ -l)\omega'\,(t'\ -\ t_0'))((l1-l)\,\omega' \\
-(E_p^1)'\ +\ (E_q^0)'\ -\ \ \Sigma_\mu((^{1\,0}\bar{m g}^{0\,1}_{l1})_\mu/({}^1d_p^{l1})^* \\
+(^{1\,0}_{\,q}mg^{0\,1}_{l})_\mu/({}^0d_q^l)\,))(\,2/\acute{\alpha}^2\ +\ (l1\ +\ l)\,\omega')\,\bar{H}_p^1\gamma_4\,H_q^0\ \ =\ 0.
\end{aligned}
\qquad (4.155)
$$

The Eqs.(4.148/4.150) must be valid for all frequencies $\Delta\, l\,\omega/\pi\ =\ (l-l1)\omega'/\pi$ in the interval $t_0'\ \leq\ t'\ \leq\ T'+t_0'$ for every combinations of p and q. The basis equations must thus be fulfilled exactly. This is only possible when all excitation levels and all eigenfunction are used. That means, the solutions are formed by the following sums

$$
\begin{aligned}
\Sigma_z\,\bar{U}^z(\mathbf{R},t) \\
=\ \Sigma_{z,p,l1}\,({}^zd_p^{l1})^*\,exp(i(\,(\,1/\acute{\alpha}^2\ \ +\ l1\omega'\,)(\,t'\ -\ t_0'))\bar{H}_p^z(\mathbf{R})
\end{aligned}
\qquad (4.156)
$$

and

$$
\sum_{z'}U^{z'}(\mathbf{R},t) =\ \sum_{z',q,l}\ {}^{z'}d_q^l\,exp(-i(\,(\,1/\acute{\alpha}^2\ \ +\ l\omega')\,(\,t'\ -\ t_0)')H_q^{z'}(\mathbf{R}). \qquad (4.157)
$$

If these functions fulfils the basic equations exactly in the mentioned time interval, then the eqs.(4.152/4.153) are also met. Such accuracy is not to be expected in the presented approximation.

4.3 *The complete energy-momentum tensors of a transition state*

In this section we compare the energy-momentum tensor of the electromagnetic field and the Dirac field of a transition state. Since the radiation field does not contribute to the stability of such system, we take into account only the quasi-static potentials. These fields are caused by the following currents

$$J_\mu = \hat{J}_\mu + \delta(\mu - 4) J_4^K \tag{4.158}$$

with

$$\begin{array}{c} \hat{J}_\mu = ie\bar{U}\gamma_\mu U, \quad U = \sum_z U^z, \\ J_4^K = -ie\varrho^K. \end{array} \tag{4.159}$$

One needs for description of the dynamic behaviour of such fields the tensor components $\hat{T}^d_{\nu\mu}$, $\hat{T}^w_{\nu\mu}$, *and* $\hat{T}^e_{\nu\mu}$, which are given by

$$\begin{array}{c} \hat{T}^d_{\nu\mu} = \frac{c\hbar}{2}(\bar{U}\gamma_\mu \frac{\partial U}{\partial x_\nu} - \frac{\partial \bar{U}}{\partial x_\nu}\gamma_\mu U), \\ \hat{T}^w_{\nu\mu} = T^w_{\nu\mu} + \grave{T}^w_{\nu\mu}, \\ \hat{T}^e_{\nu\mu} = T^e_{\nu\mu} + \grave{T}^e_{\nu\mu} \end{array} \tag{4.160}$$

with

$$\begin{array}{c} T^w_{\nu\mu} = -\hat{A}_\nu \hat{J}_\mu, \\ \grave{T}^w_{\nu\mu} = -\hat{A}_\nu J^K_\mu - A^K_\nu \hat{J}_\mu, \\ \breve{T}^w_{\nu\mu} = -A^K_\nu J^K_\mu. \end{array} \tag{4.161}$$

We use for $\grave{T}^e_{\nu\mu}$ the tensor

$$\grave{T}^e_{\nu\mu} = \frac{1}{4\pi}(F^K_{\nu\sigma}F^e_{\mu\sigma} + \hat{F}^e_{\nu\sigma}F^K_{\mu\sigma} - \frac{1}{4}\delta(\nu-\mu)(F^e_{\sigma\lambda}F^K_{\sigma\lambda} + F^K_{\sigma\lambda}F^e_{\sigma\lambda})). \tag{4.162}$$

Generally the divergence of $\hat{T}^d_{\nu\,\mu} = T^d_{\nu\,\mu}$, given in eqs.(4.149/4.150), must be compensated by the divergence of $\hat{T}^w_{\nu\,\mu} + \hat{T}^e_{\nu\,\mu}$. These have the following forms

$$\begin{aligned}
\frac{\partial}{\partial x_\mu}(T^w_{\nu\mu} + T^e_{\nu\mu}) &= -\hat{J}_\mu \frac{\partial \hat{A}_\mu}{\partial x_\nu}\,, \\
\frac{\partial}{\partial x_\mu}(\grave{T}^w_{\nu\mu} + \grave{T}^e_{\nu\mu}) &= (-\hat{J}_\mu \frac{\partial A^K_\mu}{\partial x_\nu} - J^K_\mu \frac{\partial \hat{A}_\mu}{\partial x_\nu})\,, \\
\frac{\partial}{\partial x_\mu}(\breve{T}^w_{\nu\mu} + \breve{T}^e_{\nu\mu}) &= -J^K_\mu \frac{\partial A^K_\mu}{\partial x_\nu}\,.
\end{aligned} \tag{4.163}$$

We regard $\varrho^K(r)$ to be a small expanded charge distribution of the nucleus with spherical symmetry and not as a singularity. Then all transformations are allowed. $\frac{\partial}{\partial x_\mu}(\breve{T}^w_{\nu\mu} + \breve{T}^e_{\nu\mu})$ describes the divergence of an independent field and can be separated. The remaining divergence equations are

$$\frac{\partial}{\partial x_\mu}(T^w_{\nu\mu} + T^e_{\nu\mu}) = \textstyle\sum_{z,\,s} \frac{\partial}{\partial x_\mu}({}^{z\,z-s}T^w_{\nu\mu} + {}^{z\,z-s}T^e_{\nu\mu}) \tag{4.164}$$

respectively

$$\begin{aligned}
\frac{\partial}{\partial x_\mu}(T^w_{\nu\mu} + T^e_{\nu\mu}) &= -\textstyle\sum_{z,\;s,\,s'} \frac{\partial A^{-s'}_\mu}{\partial x_\nu} J^{z\,z-s}_\mu, \\
\frac{\partial}{\partial x_\mu}(\grave{T}^w_{\nu\mu} + \grave{T}^e_{\nu\mu}) &= -\textstyle\sum_{z,\,s}(J^{z\,z-s}_\mu \frac{\partial A^K_\mu}{\partial x_\nu} + J^K_\mu \frac{\partial A^{-s'}_\mu}{\partial x_\nu})
\end{aligned} \tag{4.165}$$

and the sum is (see eq.(4.132))

$$\frac{\partial}{\partial x_\mu}(T^w_{\nu\mu} + T^e_{\nu\mu} + \grave{T}^w_{\nu\mu} + \grave{T}^e_{\nu\mu}) = -\sum_{z,\,s} \bar{U}^z\gamma_4 U^{z-s}\frac{\partial}{\partial x_\nu}(\acute{e}^2/r)$$
$$-i\acute{e}\sum_{z,\,s,\,s'}((\frac{\partial A^{-s'}_\mu}{\partial x_\nu})(\bar{U}^z\gamma_\mu U^{z-s} - \sum_{z,\,s} J^K_\mu \frac{\partial A^s_\mu}{\partial x_\nu}) \tag{4.166}$$

or

$$\frac{\partial}{\partial x_\mu}(T^w_{\nu\mu} + T^e_{\nu\mu} + \grave{T}^w_{\nu\mu} + \grave{T}^e_{\nu\mu}) = -\sum_{z,s}\frac{\partial\, {}^{z\,z-s}T^d_{\nu\mu}}{\partial x_\mu} - \sum_{z,\,s} J^K_\mu \frac{\partial A^s_\mu}{\partial x_\nu}. \tag{4.167}$$

These relations yield the general force balance equations

$$\sum_{z,\,s}\frac{\partial}{\partial x_\mu}({}^{z\,z-s}T^d_{\nu\,\mu} + {}^{z\,z-s}T^w_{\nu\,\mu} + {}^{z\,z-s}T^e_{\nu\mu} + {}^{z\,z-s}\grave{T}^w_{\nu\mu}$$
$$+ {}^{z\,z-s}\grave{T}^e_{\nu\mu}) = -\sum_{z,\,s} J^K_\mu \frac{\partial A^s_4}{\partial x_\nu} \tag{4.168}$$

which in our special case can be approximated by

$$\frac{\partial}{\partial x_\mu}({}^{10}T^d_{\nu\,\mu} + {}^{10}T^w_{\mu} + {}^{10}T^e_{\nu\mu} + {}^{10}\grave{T}^w_{\nu\mu} + {}^{10}\grave{T}^e_{\nu\mu}) = -J^K_4\frac{\partial A^{10}_\mu}{\partial x_\nu}. \tag{4.169}$$

Here the ride side becomes for $v = \bar{\nu}$ (see eq.(4.110))

$$\sum_{z,\,s} J^K_4 \frac{\partial A^s_4}{\partial x_{\bar{\nu}}} = (\acute{e}^2/\acute{a})\sum_{z,p,q,l,l1}(({}^z d^{l1}_p)^* ({}^{z-s}d^l_q)$$
$$\times \exp(i\ (l1 - l)\omega'(t' - t'_0))({}^K_{p\,q}mk^{z\,z-s})_{\bar{\nu}} \tag{4.170}$$

with

$$({}^{K}_{p\,q}mk^{z\,z-s})_{\bar\nu}(\mathbf{R}) \;=\; J_4^K(\mathbf{R})\frac{\partial}{\partial x_{\bar\nu}}(\int \frac{1}{r_{0\,1}}\,\bar H_p^z(\mathbf{R}_1)\gamma_\mu H_q^{z-s}(\mathbf{R}_1)dV_1). \qquad (4.171)$$

We insert the generalised eq.(4.148) into (4.168) and obtain after changing the dimension

$$\begin{array}{c}\frac{\partial}{\partial x_\mu}\,({}^{z\,z-s}T^w_{\bar\nu\mu} + {}^{z\,z-s}T^e_{\bar\nu\mu} + {}^{z\,z-s}\grave T^w_{\bar\nu\mu} + {}^{z\,z-s}\grave T^e_{\bar\nu\mu}) \\ = -(\acute e^2/\acute a)\frac{1}{2}\sum_{s,p,q,l,l1}({}^{z}d_p^{l1})^*\,({}^{z-s}d_q^l)\exp(i\,(l1\,-l)\omega'\,(t'\,-\,t_0')) \\ \times(\sum_{z',s'}(({}^{z\,z-s}_{p}\bar m g_{l1}^{z'-s'\,z'})_\mu/({}^{z}d_p^{l1})^* \,-\, ({}^{z\,z-s'}_{q}mg_l^{z'-s'\,z'})_\mu)/({}^{z-s}d_q^l)) \\ \times(\,\bar H_p^z\,\gamma_4\frac{\partial\,H_q^{z-s}}{\partial\,x_{\bar\nu}} \,-\frac{\partial\,\bar H_p^z}{\partial\,x_{\bar\nu}}\,\gamma_4\,H_q^{z-s} \,+\, \bar H_p^z\gamma_4\,H_q^{z-s}\,\frac{\partial\,(\,2/r)}{\partial\,x_{\bar\nu}}\,) \\ -\,2({}^{K}_{p\,q}mk^{z\,z-s})_{\bar\nu}).\end{array} \qquad (4.172)$$

That means the eq.(4.169) yields for $\nu = \bar\nu$

$$\begin{array}{c}\frac{\partial}{\partial x_\mu}\,({}^{z\,z-s}T^d_{\bar\nu\,\mu} + {}^{z\,z-s}T^w_{\bar\nu\,\mu} + {}^{z\,z-s}T^e_{\bar\nu\mu} + {}^{z\,z-s}\grave T^w_{\bar\nu\mu} + {}^{z\,z-s}\grave T^e_{\bar\nu\mu}) \;= \\ (\acute e^2/\acute a)\frac{1}{2}\sum_{p,q,l,l1}(({}^{z}d_p^{l1})^*\,({}^{z-s}d_q^l)\exp(i\;(l1\,-l)\omega'\,(t\,-\,t_0))(\,(l1-l)\,\omega' \\ -(E_p^z)' \,+\, (E_q^{z-s})' \,-\, \sum_{z',s'}(({}^{z\,z-s}_{p}\bar m g_{l1}^{z'\,z'-s'})_\mu/({}^{z}d_p^{l1})^* \\ -\,({}^{z\,z-s}_{q}mg_l^{z'\,z'-s'})_\mu/({}^{z-s}d_q^l)))(\,\bar H_p^z\,\gamma_4\frac{\partial\,H_q^{z-s}}{\partial\,x_\nu} \,-\frac{\partial\,\bar H_p^z}{\partial\,x_{\bar\nu}}\,\gamma_4\,H_q^{z-s} \\ +2\,\bar H_p^z\gamma_4\,H_q^{z-s}\,\frac{\partial\,(\,\acute e^2/r)}{\partial\,x_{\bar\nu}} \,-\, 2({}^{K}_{p\,q}mk_{\bar\nu}^{z\,z-s)}\,)\end{array} \qquad (4.173)$$

and for $\nu = 4$

$$\begin{array}{c}\frac{\partial}{\partial x_\mu}\,({}^{z\,z-s}T^d_{4\,\mu} + {}^{z\,z-s}T^w_{4\,\mu} + {}^{z\,z-s}T^e_{4\mu} + {}^{z\,z-s}\grave T^w_{4\mu} + {}^{z\,z-s}\grave T^e_{4\mu}) \;= \\ (\acute e^2/\acute a)\frac{1}{2}\sum_{p,q,l,l1}\;(({}^{z}d_p^{l1})^*\,({}^{z-s}d_q^l)\exp(i\;(l1\,-l)\omega'\,(t'\,-\,t_0'))((l1-l)\,\omega' \\ -(E_p^z)' \,+\, (E_q^{z-s})' \,-\, \sum_{z',s'}(({}^{z\,z-s}\bar m g_{l1}^{z'-s'\,z'})_\mu/({}^{z}d_p^{l1})^* \\ -\,({}^{z\,z-s}_{q}mg_l^{z'-s'\,z'})_\mu/({}^{z-s}d_q^l)\,))((2/\acute\alpha^2 +\,(l1\,+\,l)\,\omega')\,\bar H_p^z\gamma_4\,H_q^{z-s} \\ -\,2({}^{K}_{p\,q}mk^{z\,z-s})_4)\,.\end{array} \qquad (4.174)$$

Here is

$$({}^{K}_{p\,q} mk^{z\,z-s})_4 \;=\; -(l1-l)\omega'\, J_4^K(\mathbf{R})(\textstyle\int \frac{1}{r_{01}}\, \bar{H}_p^z(\mathbf{R}_1)\gamma_\mu H_q^{z-s}(\mathbf{R}_1) dV_1). \qquad (4.175)$$

Due to eq.(4.168), it becomes for all ν

$$\begin{array}{c} \frac{\partial}{\partial x_\mu}\,({}^{z\,z-s}T^d_{\nu\,\mu} + {}^{z\,z-s}T^w_{\nu\,\mu} + {}^{z\,z-s}T^e_{\nu\mu} + {}^{z\,z-s}\grave{T}^w_{\nu\mu} + {}^{z\,z-s}\grave{T}^e_{\nu\mu}) \;= \\ -(\acute{e}^2/\acute{a})\sum_{p,q,l,l1}({}^{z}d_p^{l1})^*\,({}^{z-s}d_q^{l})\exp(i\;(l1\;-l)\omega'\,(t'\;-\;t_0'))\,({}^{K}_{p\,q}mk^{z\,z-s})_\nu, \end{array} \qquad (4.176)$$

Last equation contains the divergence of following oscillating field

$$\begin{array}{c} {}^{1\,0}\tilde{T}^{dr}_{\nu\,\mu} = (\acute{e}^2/\acute{a})\frac{1}{2}\sum_l\,(({}^{1r}d_k^{l+L})^*\,exp(i(\,(1/\acute{\alpha}^2\;+\;(l+L)\omega'\,)(\,t'\;-\;t_0')) \\ \times\bar{H}_k^1(\mathbf{R})\gamma_\mu\frac{\partial}{\partial x_\nu}[{}^{0r}d_m^l\,exp(-i(\,(1/\acute{\alpha}^2\;+\;(l+L)\omega'\,)(\,t'\;-\;t_0'))H_m^0(\mathbf{R})] \\ -\frac{\partial}{\partial x_\nu}[({}^{1r}d_k^{l+L})^*\,exp(i(\,(1/\acute{\alpha}^2\;+\;l\omega'\,)(\,t'\;-\;t_0')))\bar{H}_k^1(\mathbf{R})] \\ \times\gamma_\mu\,{}^{0r}d_m^l\,exp(-i(\,(1/\acute{\alpha}^2\;+\;l\omega'\,)(\,t'\;-\;t_0'))H_m^0(\mathbf{R})), \end{array} \qquad (4.177)$$

which can be replaced by

$$\begin{array}{c} {}^{1\,0}\tilde{T}^{dr}_{\bar{\nu}\,\mu} = \;(\acute{e}^2/\acute{a})\frac{1}{2}D^{1\,0}_{k\,m}\exp(i\varpi'(t'-t_0')(\bar{H}_k\gamma_{\bar{\nu}}\frac{\partial H_m}{\partial x_{\bar{\nu}}} - \frac{\partial\bar{H}_k}{\partial x_{\bar{\nu}}}\gamma_\mu H_m) \\ and \\ {}^{1\,0}\tilde{T}^{dr}_{4\,\mu} \;=\; -\,(\acute{e}^2/\acute{a})\frac{1}{\acute{\alpha}^2}D^{1\,0}_{k\,m}\;\;\exp(i\;\;\varpi'\,(t'\;-\;t_0'))\,\bar{H}_k^1\gamma_\mu\,H_m^0. \end{array} \qquad (4.178)$$

Due to the eqs.(4.119), the divergences of this tensor read

$$\begin{array}{c} \frac{\partial\, {}^{10}\tilde{T}^{dr}_{\bar{\nu}\,\mu}}{\partial\, x_{\mu}} \;\cong\; (\acute{e}^2/\acute{a})\frac{1}{2}D^{1\,0}_{k\,m}\;\exp(i\,\varpi'\,(t' \,-\, t'_0)\,) \\ \times((\bar{H}^1_k\,\gamma_4\frac{\partial\, H^0_m}{\partial\, x_{\bar{\nu}}} \;-\frac{\partial\,\bar{H}^1_k}{\partial\, x_{\bar{\nu}}}\;\gamma_4\,H^0_m)\,(\,\varpi' \,-(E^1_k)' \,+\,(E^0_m)' \\ +\,\bar{H}^1_k\gamma_4\,H^0_m\,(\frac{\partial\,(2/r)}{\partial\, x_{\bar{\nu}}}\,)) \end{array} \tag{4.179}$$

or

$$\begin{array}{c} \frac{\partial}{\partial\, x_{\mu}}\,({}^{1\,0}\tilde{T}^{dr}_{4\,\mu}) \;=\; (\acute{e}^2/\acute{a})\frac{1}{\acute{\alpha}^2}D^{1\,0}_{k\,m}\;\exp(i\;\;\varpi'\,(t' \,-\, t'_0)) \\ \times\,(\varpi' \,-\, (E^1_k)' \,+\, (E^0_m)')\,\bar{H}^1_k\gamma_4\,H^0_m \end{array} \tag{4.180}$$

if the radiation loss is neglected. The compensating terms of these fields are

$$\begin{array}{c} {}^{1\,0}\tilde{T}^{w}_{\bar{\nu}\,\mu} = \,-D^{1\,0}_{k\,m}\;\exp(i\,\varpi\,(t\,-\,t_0)\,(\,{}^{k\,m}A^{1\,0}_{\bar{\nu}}\,J^K_{\mu} + A^K_{\bar{\nu}}\;{}^{k\,m}J^{1\,0}_{\mu}), \\ {}^{1\,0}\tilde{T}^{e}_{\bar{\nu}\,\mu} = \,\frac{1}{4\pi}D^{1\,0}_{k\,m}\exp(i\,\varpi\,(t\,-\,t_0)\,(F^K_{\bar{\nu}\,\sigma}\,({}^{k\,m}F^{e,\,1\,0}_{\mu\sigma}) + \;({}^{k\,m}F^{e,\,1\,0}_{\bar{\nu}\sigma})F^K_{\mu\,\sigma} \\ -\frac{1}{2}\delta(\bar{\nu}\,-\mu)F^K_{\sigma\lambda}\,({}^{k\,m}F^{e,\,1\,0}_{\sigma\lambda})) \end{array} \tag{4.181}$$

with

$$\begin{array}{c} {}^{k\,m}A^{1\,0}_4(\mathbf{R}) = i\acute{e}\int\frac{1}{r_{0\,1}}\,\bar{H}^1_k\gamma_4\,H^0_m\,(\mathbf{R}_1,)dV_1, \\ {}^{k\,m}F^{e,\,1\,0}_{\bar{\nu}\,\mu} \cong \delta(4-\mu)\frac{\partial^{k\,m}A^{1\,0}_4}{\partial x_{\bar{\nu}}}, \\ {}^{k\,m}F^{e,\,1\,0}_{\nu\bar{\mu}} \cong -\delta(\nu-4)\frac{\partial^{k\,m}A^{1\,0}_4}{\partial x_{\bar{\mu}}}, \\ {}^{k\,m}\tilde{J}^{1\,0}_{\mu} = i\acute{e}\exp(i\varpi(t-t_0)\bar{H}^1_k\gamma_{\mu}H^0_m. \end{array} \tag{4.182}$$

The tensor components have the form

$$
\begin{aligned}
{}^{10}\tilde{T}^{w}_{\bar\nu\,\mu} &= -\delta(4-\mu) D^{1\,0}_{k\,m}\ \exp(i\,\varpi\,(t\,-\,t_0))\,(\,{}^{k\,m}A^{1\,0}_{\bar\nu}J^{K}_{4} + A^{K}_{4}\,{}^{k\,m}J^{1\,0}_{\bar\nu}), \\
{}^{10}\tilde{T}^{e}_{\bar\nu\,\mu} &= \tfrac{1}{4\pi} D^{1\,0}_{k\,m}\ \exp(i\,\varpi\,(t\,-\,t_0))\,(F^{K}_{\bar\nu\,\sigma}\,{}^{k\,m}F^{e,\,1\,0}_{\mu\sigma} + {}^{k\,m}F^{e,\,1\,0}_{\nu\sigma}F^{K}_{\bar\nu\,\sigma} \\
&\qquad -\delta(\bar\nu-\bar\mu)\,\tfrac{\partial\,{}^{k\,m}A^{1\,0}_{4}}{\partial\,x_{\bar\sigma}}\tfrac{\partial\,A^{K}_{4}}{\partial\,x_{\bar\sigma}}).
\end{aligned}
\tag{4.183}
$$

Due to $\Delta\,{}^{k\,m}A^{1\,0}_{\bar\nu} = -\,4\pi\,\,{}^{k\,m}J^{1\,0}_{\bar\nu} \cong 0$ and ${}^{k\,m}A^{1\,0}_{\bar\nu} \cong 0$, we get ${}^{1\,0}\tilde{T}^{w}_{\bar\nu\mu} \cong 0$ and for the sum of the divergences of these tensors

$$
\begin{aligned}
\tfrac{\partial}{\partial\,x_\mu}\,({}^{1\,0}\tilde{T}^{w}_{\bar\nu\mu} + {}^{1\,0}\tilde{T}^{e}_{\bar\nu\mu}) \ &= \ -(\acute{e}^2/\acute{a})D^{1\,0}_{k\,m}\ \exp(i\,\varpi'\,(t'\,-\,t'_0)\,(\bar{H}^{1}_{k}\gamma_4\,H^{0}_{m}\,(\tfrac{\partial\,(1/r)}{\partial\,x_{\bar\nu}}\,) \\
&\qquad + J^{K}_{4}\tfrac{\partial\,{}^{k\,m}A^{1\,0}_{4}}{\partial\,x_{\bar\nu}}).
\end{aligned}
\tag{4.184}
$$

Using the eqs.(4.119/4.180), it becomes

$$
\begin{aligned}
\tfrac{\partial}{\partial\,x_\mu}(\,{}^{1\,0}\tilde{T}^{dr}_{\bar\nu\,\mu} + \,{}^{1\,0}\tilde{T}^{w}_{\bar\nu\,\mu} + \,{}^{1\,0}\tilde{T}^{e}_{\bar\nu\,\mu}) \ &= \ (\acute{e}^2/\acute{a})\tfrac{1}{2}D^{1\,0}_{k\,m}\,\exp(i\,\varpi'\,(t'\,-\,t'_0)\,) \\
&\times((\bar{H}^{1}_{k}\,\gamma_4\tfrac{\partial\,H^{0}_{m}}{\partial\,x_{\bar\nu}}\ -\tfrac{\partial\,\bar{H}^{1}_{k}}{\partial\,x_{\bar\nu}}\,\gamma_4\,H^{0}_{m})\,(\,\varpi' - (E^{1}_{k})' \,+ (E^{0}_{m})'\,) \\
&\qquad -\,D^{1\,0}_{k\,m}\,J^{K}_{4}\tfrac{\partial\,{}^{k\,m}A^{1\,0}_{4}(\mathbf{R})}{\partial\,x_{\bar\nu}}.
\end{aligned}
\tag{4.185}
$$

In addition, it is $\frac{\partial\,{}^{k\,m}A^{1\,0}_{4}(\mathbf{R})}{\partial\,x_{\bar\nu}} \cong 0$ for a spherically, only small expanded charge distribution of the nucleus, if ${}^{k\,m}A^{1\,0}_{4}$ has an extreme value in this area. Then the last term in eq.(4.185) can be approximated by

$$
i\acute{e}\frac{\partial\,{}^{k\,m}A^{1\,0}_{4}(\mathbf{R})}{\partial\,x_{\bar\nu}}\,\varrho^{K}(r) \,=\, 0. \tag{4.186}
$$

With eqs.(4.49/4.186) the eq.(4.185) reads

$$\frac{\partial}{\partial x_\mu}({}^{10}\tilde{T}^d_{\bar{\nu}\,\mu} + {}^{10}\tilde{T}^w_{\bar{\nu}\,\mu} + {}^{10}\tilde{T}^e_{\bar{\nu}\,\mu}) = 0. \qquad (4.187)$$

In addition, because of: $\acute{M}\,\acute{c}^2 \gg {}^{10}\tilde{T}^w_{4\,\mu}, {}^{10}\tilde{T}^e_{4\,\mu}$, it becomes $\frac{\partial}{\partial x_\mu}({}^{10}\tilde{T}^d_{4\,\mu} + {}^{10}\tilde{T}^w_{4\,\mu} + {}^{10}\tilde{T}^e_{4\,\mu}) \cong \frac{\partial}{\partial x_\mu}({}^{10}\tilde{T}^d_{4\,\mu})$. That means, the divergence of the fourth line, given in eq.(4.174), describes the continuity equation in the applied approximation.

The consideration exhibits that the dynamics of a transition field fulfils the causality condition because the divergences of the energy-momentum tensor of the system are zero. Such tensors include normally radiation moments. These can interact with a radiation field and cause radiation effects if a frequency condition is fulfilled. Prerequisites are a noticeable size of such spectral line and the fulfilment of an independent continuity equation such as eq.(4.44). Also the value of the emitted energy must amount to $E = h\nu$ when the transition process is completed. Details on the radiation fields are described in section 4.1.2. The combination of the results allows to say that this model explains the emission of a "photon" through a causal process.

Chapter 5

The two-electron system

5.1 *The basic equations*

In the next step we consider a classical field model in which two electrons are bound to a double charged nucleus. According to our interpretation, the electron 1 and 2 should be described by the following separated field functions $U(\mathcal{R}_1, t)$ and $V(\mathcal{R}_2, t)$ in $\mathcal{R}_1$-space and $\mathcal{R}_2$-space, respectively. Nevertheless, the current and tensor components can be interpreted finally as field functions in the real space $\mathcal{R}$. It is also useful sometimes to set $\mathcal{R} = \mathcal{R}'$ to characterise the source of real electromagnetic fields. The functions satisfy the conditions $\int \bar{U}(\mathcal{R}_1, t)\gamma_4^1 U(\mathcal{R}_1, t) dV_1 = 1$ and $\int \bar{V}(\mathcal{R}_2, t)\gamma_4^2 V(\mathcal{R}_2, t) dV_2 = 1$. These are influenced by the potentials

$$
\begin{aligned}
A_4^1(\mathcal{R}_1, t) &= i\acute{e} \int \tfrac{1}{r_{1\,1'}} \bar{U}(\mathcal{R}_1', t)\gamma'^1_4 U(\mathcal{R}_1', t) dV_1', \\
A_4^2(\mathcal{R}_1, t) &= i\acute{e} \int \tfrac{1}{r_{1\,2'}} \bar{V}(\mathcal{R}_2', t)\gamma'^2_4 V(\mathcal{R}_2', t) dV_2', \\
\\
A_4^1(\mathcal{R}_2, t) &= i\acute{e} \int \tfrac{1}{r_{2\,1'}} \bar{U}(\mathcal{R}_1', t)\gamma'^1_4 U(\mathcal{R}_1', t) dV_1', \\
A_4^2(\mathcal{R}_2, t) &= i\acute{e} \int \tfrac{1}{r_{2\,2'}} \bar{V}(\mathcal{R}_2', t)\gamma'^2_4 V(\mathcal{R}_2', t) dV_2'.
\end{aligned}
\tag{5.1}
$$

That leads to the Dirac equations

$$(\gamma^1_\mu(\hbar\frac{\partial}{\partial x^1_\mu} - i\frac{\acute{e}}{\acute{c}}(A^K_4 + A^1_4(\mathcal{R}_1,t) + A^2_4(\mathcal{R}_1,t)) + \acute{M}\acute{c})U(\mathcal{R}_1,t) = 0 \qquad (5.2)$$

and

$$(\gamma^2_\mu(\hbar\frac{\partial}{\partial x^2_\mu} - i\frac{\acute{e}}{\acute{c}}(A^K_4 + A^1_4(\mathcal{R}_2,t) + A^2_4(\mathcal{R}_2,t)) + \acute{M}\acute{c})V(\mathcal{R}_2,t) = 0, \qquad (5.3)$$

where the potential of the nucleus is given by

$A^K_4 = -2i\acute{e}/r_1$ $(or = -2i\acute{e}/r_2)$.

The potentials A^1_4 and A^2_4 are defined in the eqs.(5.1). The completions by A^j_μ in the eqs.(5.2/5.3) shall no be discussed here. These have only a small influence. We try to solve the eqs.(5.2/5.3) applying the following series

$$\begin{array}{c} U(\mathcal{R}_1,t) = \sum_n a_n(t)H_n(\mathcal{R}_1) = \sum_{n,l} c^l_n \exp(-il\omega t)H_n(\mathcal{R}_1), \\ \bar{U}(\mathcal{R}_1,t) = \sum_{n'} a_{n'}(t)^*\bar{H}_{n'}(\mathcal{R}_1) = \sum_{n',l'} (c^{l'}_{n'})^* \exp(il'\omega t)\bar{H}_{n'}(\mathcal{R}_1), \\ V(\mathcal{R}_2,t) = \sum_m b_m(t)K_m(\mathcal{R}_2) = \sum_{m,l1} d^{l1}_m \exp(-il\omega t)K_m(\mathcal{R}_2), \\ \bar{V}(\mathcal{R}_2,t) = \sum_{m'} b_{m'}(t)^*\bar{K}_{m'}(\mathcal{R}_2) = \sum_{m',l1'} (d^{l1'}_{m'})^* \exp(il1'\omega t)\bar{K}_{m'}(\mathcal{R}_2), \end{array} \qquad (5.4)$$

where $H_n(\mathcal{R}_1)$ and $K_m(\mathcal{R}_2)$ satisfy the eigenvalue equations

$$\acute{c}(\gamma^1_{\bar{\mu}}\hbar\frac{\partial}{\partial x^1_{\bar{\mu}}} - \frac{2\acute{e}^2}{\acute{c}r_1}\gamma^1_4 + \acute{M}\acute{c})H_n(\mathcal{R}_1) = \gamma^1_4\bar{E}^H_n H_n(\mathcal{R}_1) \tag{5.5}$$

and

$$\acute{c}(\gamma^2_{\bar{\mu}}\hbar\frac{\partial}{\partial x^2_{\bar{\mu}}} - \frac{2\acute{e}^2}{\acute{c}r_2}\gamma^2_4 + \acute{M}\acute{c})K_m(\mathcal{R}_2) = \gamma^2_4\bar{E}^K_m K_m(\mathcal{R}_2). \tag{5.6}$$

$H_n(\mathcal{R})$ and $K_m(\mathcal{R})$ fulfil the orthogonal conditions in $\mathcal{R}$. One can put $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}$ in all final results, denoting properties in the real space, such as $J^2_\mu(\mathcal{R}_2, t) = J^2_\mu(\mathcal{R}, t)$.

First we describe the solutions of the eqs.(5.2/5.3) by the functions

$$\begin{array}{c} U(\mathcal{R}_1, t) = \exp(-i\bar{E}\,t/\hbar)\breve{U}(\mathcal{R}_1), \\ \breve{U}(\mathcal{R}_1) = \sum_n \bar{a}_n H_n(\mathcal{R}_1) \end{array} \tag{5.7}$$

and

$$\begin{array}{c} V(\mathcal{R}_2, t) = \exp(-i\bar{E}\,t/\hbar)\breve{V}(\mathcal{R}_2), \\ \breve{V}(\mathcal{R}_2) = \sum_m \bar{b}_m K_m(\mathcal{R}_2). \end{array} \tag{5.8}$$

These lead to the Dirac equations

$$(\gamma^1_4 i\tfrac{\hbar\partial}{\partial t} - \acute{c}\overrightarrow{P}_1 U(\mathcal{R}_1, t) + i\acute{e}(A^1_4(\mathcal{R}_1) + A^2_4(\mathcal{R}_1))\gamma^1_4)U(\mathcal{R}_1 t) = 0 \tag{5.9}$$

and

$$(\gamma^2_4 i\tfrac{\hbar\partial}{\partial t} - \acute{c}\overrightarrow{P}_2)V(\mathcal{R}_2, t) + i\acute{e}(A^1_4(\mathcal{R}_2) + A^2_4(\mathcal{R}_2))\gamma^2_4)V(\mathcal{R}_2, t) = 0, \tag{5.10}$$

where the operators are given by

$$\begin{aligned}\overrightarrow{P}_1 = \hbar\gamma^1_{\bar{\mu}}(\overrightarrow{\partial}/\partial x^1_{\bar{\mu}}) - 2\acute{e}^2\gamma^1_4/(\acute{c}r_1) + \acute{M}\acute{c},\\ \overrightarrow{P}_2 = \hbar\gamma^2_{\bar{\mu}}(\overrightarrow{\partial}/\partial x^2_{\bar{\mu}}) - 2\acute{e}^2\gamma^2_4/(\acute{c}r_2) + \acute{M}\acute{c}.\end{aligned} \tag{5.11}$$

The high indices hint at the related space. We also use the following notations

$$\begin{aligned}\overleftarrow{P}_1 = \hbar\gamma^1_{\bar{\mu}}(\overleftarrow{\partial}/\partial x^1_{\bar{\mu}}) + 2\acute{e}^2\gamma^1_4/(\acute{c}r_1) - \acute{M}\acute{c},\\ \overleftarrow{P}_2 = \hbar\gamma^2_{\bar{\mu}}(\overleftarrow{\partial}/\partial x^2_{\bar{\mu}}) + 2\acute{e}^2\gamma^2_4/(\acute{c}r_2) - \acute{M}\acute{c}.\end{aligned} \tag{5.12}$$

If one inserts the eqs.(5.7/5.8) in the eqs.(5.9/5.10), one obtains

$$\begin{gathered}(\bar{E} - \bar{E}^H_n)\bar{a}_n\\ +i\acute{e}\int(A^1_4(\mathcal{R}_1) + A^2_4(\mathcal{R}_1))\bar{H}_n(\mathcal{R}_1)\gamma^1_4\breve{U}(\mathcal{R}_1)dV_1 = 0,\\ (\bar{E} - \bar{E}^K_m)\bar{b}_m\\ +i\acute{e}\int(A^1_4(\mathcal{R}_2) + A^2_4(\mathcal{R}_2))K_m(\mathcal{R}_2)\gamma^2_4\breve{V}(\mathcal{R}_2)dV_2 = 0.\end{gathered} \tag{5.13}$$

Such static solutions should satisfy the minimum condition

$$-\int T_{44}dV = min. \tag{5.14}$$

Here is

$$T_{\nu\mu} = T^{d1}_{\nu\mu} + T^{d2}_{\nu\mu} + T^{w1}_{\nu\mu} + T^{w2}_{\nu\mu} + T^{e}_{\nu\mu}. \tag{5.15}$$

These tensors have in dynamic cases the form

$$\begin{array}{c} T^{d1}_{\nu\,\mu} = \frac{c\hbar}{2}(\bar{U}(\mathcal{R}_1,t)\gamma^1_\mu \frac{\partial U(\mathcal{R}_1,t)}{\partial x^1_\nu} - \frac{\partial \bar{U}(\mathcal{R}_1,t)}{\partial x^1_\nu}\gamma^1_\mu U(\mathcal{R}_1,t)), \\ T^{d2}_{\nu\,\mu} = \frac{c\hbar}{2}(\bar{V}(\mathcal{R}_2,t)\gamma^2_\mu \frac{\partial V(\mathcal{R}_2,t)}{\partial x^2_\nu} - \frac{\partial \bar{V}(\mathcal{R}_2,t)}{\partial x^2_\nu}\gamma^2_\mu V_(\mathcal{R}_2,t)), \\ T^{w1}_{\nu\,\mu} = -(A^1_\nu(\mathcal{R}_1,t) + A^2_\nu(\mathcal{R}_1,t))J^1_\mu(\mathcal{R}_1,t)), \\ T^{w2}_{\nu\,\mu} = -(A^1_\nu(\mathcal{R}_2,t) + A^2_\nu(\mathcal{R}_2,t))J^2_\mu(\mathcal{R}_2,t), \\ T^{e}_{\nu\,\mu} = \frac{1}{4\pi}((F^1_{\nu\,\sigma}(\mathcal{R},t) + F^2_{\nu\,\sigma}(\mathcal{R},t))(F^1_{\nu\,\sigma}(\mathcal{R},t) + F^2_{\nu\,\sigma}(\mathcal{R},t)) \\ -\frac{1}{4}\delta_{\nu\,\mu}(F^1_{\sigma\,\lambda}(\mathcal{R},t) + F^2_{\sigma\,\lambda}(\mathcal{R},t))^2). \end{array} \tag{5.16}$$

We write again in the final results $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}$. The missing term $-\int A^K_\mu(J^1_\mu + J^2_\mu)dV$ is compensated by the nucleus field and has no meaning in the minimum condition eq.(5.14). For static solutions the integrals over $T^{d1}_{4\,4} + T^{d2}_{4\,4}$ are given, according to the eqs.(5.13), by

$$\begin{array}{c} -\int(T^{d1}_{4\,4} + T^{d2}_{4\,4})dV = \sum_n \bar{E}^H_n(a_n)^* a_n \\ + \sum_m \bar{E}^K_m(b_m)^* b_m - \int(A^1_4 + A^2_4)(J^1_4 + J^2_4)dV. \end{array} \tag{5.17}$$

Therefore is

$$-\int(T^{d1}_{4\,4} + T^{d2}_{4\,4} + T^{w1}_{4\,4} + T^{w2}_{4\,4})dV = \sum_n \bar{E}^H_n(\bar{a}_n)^*\bar{a}_n + \sum_m \bar{E}^K_m(\bar{b}_m)^*\bar{b}_m \tag{5.18}$$

and with

$$-\int T^e_{4\,4}\,dV \cong -\frac{1}{2}\int(A^1_\mu + A^2_\mu)(J^1_\mu + J^2_\mu)dV \tag{5.19}$$

becomes

$$
\begin{gathered}
-\int T_{4\,4} dV = \sum_n \bar{E}_n^H (\bar{a}_n)^* \bar{a}_n \\
+\sum_m \bar{E}_m^K (\bar{b}_m)^* \bar{b}_m - \tfrac{1}{2} \int (A_4^1 + A_4^2)(J_4^1 + J_4^2) dV.
\end{gathered} \tag{5.20}
$$

In addition, it is

$$
\begin{gathered}
-\int T_{4\,4} dV = \bar{E}(\sum_n ((\bar{a}_n)^* \bar{a}_n \\
+\sum_m (\bar{b}_m)^* \bar{b}_m) + \tfrac{1}{2} \int (A_4^1 + A_4^2)(J_4^1 + J_4^2) dV.
\end{gathered} \tag{5.21}
$$

Comparing of eq.(5.20) with eq.(5.21) leads to the relation

$$
\begin{gathered}
\sum_n (\bar{E} - \bar{E}_n^H)(\bar{a}_n)^* \bar{a}_n + \sum_m (\bar{E} - \bar{E}_m^K)(\bar{b}_m)^* \bar{b}_m \\
+\int (A_4^1 + A_4^2)(J_4^1 + J_4^2) dV = 0.
\end{gathered} \tag{5.22}
$$

This also follows from the eqs.(5.13) by multiplication and addition.

5.2 *The energy-momentum tensor*

In the next point we consider the tensor of dynamic solutions. For this aim we apply the eqs.(5.2-5.4). The divergences of the related tensors

$$
\hat{T}^{dj}_{\nu\,\mu}(\mathcal{R}_j, t) = \tfrac{1}{2} \bar{U}(\mathcal{R}_j, t) c\hbar \gamma^j_\mu \frac{\overrightarrow{\partial}}{\partial x^j_\nu} - \tfrac{1}{2} \bar{U}(\mathcal{R}_j, t) c\hbar \gamma^j_\mu \frac{\overleftarrow{\partial}}{\partial x^j_\nu} U(\mathcal{R}_j, t) \tag{5.23}
$$

are

$$
\begin{array}{c}
\frac{\partial}{\partial x^1_\mu}\hat{T}^{d1}_{\bar{\nu}\,\mu}(\mathcal{R}_1,t) = \frac{1}{2}(\bar{U}(\mathcal{R}_1,t)c\hbar\gamma^1_{\bar{\mu}}(\frac{\overleftarrow{\partial}}{\partial x^1_{\bar{\mu}}}\frac{\overrightarrow{\partial}}{\partial x^1_{\bar{\nu}}} + \frac{\overrightarrow{\partial}}{\partial x^1_{\bar{\nu}}}\frac{\overrightarrow{\partial}}{\partial x^1_{\bar{\mu}}})\,U(\mathcal{R}_1,t) \\
-\frac{1}{2}(\bar{U}(\mathcal{R}_1,t)c\hbar\gamma^1_{\bar{\mu}}(\frac{\overleftarrow{\partial}}{\partial x^1_{\bar{\mu}}}\frac{\overleftarrow{\partial}}{\partial x^1_{\bar{\nu}}} + \frac{\overleftarrow{\partial}}{\partial x^1_{\bar{\nu}}}\frac{\overrightarrow{\partial}}{\partial x^1_{\bar{\mu}}})\,U(\mathcal{R}_1,t) + \frac{\partial}{\acute{i}c\partial t}\hat{T}^{d1}_{\bar{\nu}\,\mu}(\mathcal{R}_1,t) \\
and \\
\frac{\partial}{\partial x^1_\mu}\hat{T}^{d2}_{\bar{\nu}\,\mu}(\mathcal{R}_2,t) = \frac{1}{2}(\bar{V}(\mathcal{R}_2,t)c\hbar\gamma^2_{\bar{\mu}}(\frac{\overleftarrow{\partial}}{\partial x^2_{\bar{\mu}}}\frac{\overrightarrow{\partial}}{\partial x^2_{\bar{\nu}}} + \frac{\overrightarrow{\partial}}{\partial x^2_{\bar{\nu}}}\frac{\overrightarrow{\partial}}{\partial x^2_{\bar{\mu}}})\;V(\mathcal{R}_2,t) \\
-\bar{V}(\mathcal{R}_2,t)c\hbar\gamma^2_{\bar{\mu}}(\frac{\overleftarrow{\partial}}{\partial x^2_{\bar{\mu}}}\frac{\overleftarrow{\partial}}{\partial x^2_{\bar{\nu}}} + \frac{\overleftarrow{\partial}}{\partial x^2_{\bar{\nu}}}\frac{\overrightarrow{\partial}}{\partial x^2_{\bar{\mu}}})\,V(\mathcal{R}_2,t)) + \frac{\partial}{\acute{i}c\partial t}\hat{T}^{d2}_{\bar{\nu}\,\mu}(\mathcal{R}_2,t).
\end{array}
\qquad (5.24)
$$

We can replace $c\hbar\gamma^1_{\bar{\mu}}\frac{\overrightarrow{\partial}}{\partial x^1_{\bar{\mu}}}U(\mathcal{R}_1,t)$, $\bar{U}(\mathcal{R}_1,t)\frac{\overleftarrow{\partial}}{\partial x^1_{\bar{\nu}}}c\hbar\gamma^1_{\bar{\mu}}$ through

$$
\begin{array}{rcl}
c\hbar\gamma^1_{\bar{\mu}}\frac{\overrightarrow{\partial}}{\partial x^1_{\bar{\mu}}}U(\mathcal{R}_1,t) & = & (\acute{c}\overrightarrow{P}_1 + \acute{e}^2\gamma^1_4\frac{2}{r_1} - \acute{M}\acute{c}^2)\,U(\mathcal{R}_1,t) \\
 & or & \\
\bar{U}(\mathcal{R}_1,t)\frac{\overleftarrow{\partial}}{\partial x^1_{\bar{\nu}}}c\hbar\gamma^1_{\bar{\mu}} & = & \bar{U}(\mathcal{R}_1,t)(\acute{c}\overleftarrow{P}_1 - \acute{e}^2\gamma^1_4\frac{2}{r_1} + \acute{M}\acute{c}^2)
\end{array}
\qquad (5.25)
$$

and $c\hbar\gamma^2_{\bar{\mu}}\frac{\overrightarrow{\partial}}{\partial x^1_{\bar{\mu}}}V(\mathcal{R}_2,t)$, $\bar{V}(\mathcal{R}_2,t)\frac{\overleftarrow{\partial}}{\partial x^1_{\bar{\nu}}}c\hbar\gamma^2_{\bar{\mu}}$ through analogous equations. Then one obtains for the eqs.(5.24) the expressions

$$
\begin{array}{c}
\frac{\partial}{\partial x^1_\mu}\hat{T}^{d1}_{\bar{\nu}\,\mu}(\mathcal{R}_1,t) = \frac{1}{2}\bar{U}(\mathcal{R}_1,t)((\acute{c}\overleftarrow{P}_1 - \acute{e}^2\gamma^1_4\frac{2}{r_1} + \acute{M}\acute{c}^2)\frac{\overrightarrow{\partial}}{\partial x^1_{\bar{\nu}}} \\
+\frac{\overrightarrow{\partial}}{\partial x^1_{\bar{\nu}}}(\acute{c}\overrightarrow{P}_1 + \acute{e}^2\gamma^1_4\frac{2}{r_1} - \acute{M}\acute{c}^2) - (\acute{c}\overleftarrow{P}_1 - \acute{e}^2\gamma^1_4\frac{2}{r_1} + \acute{M}\acute{c}^2)\frac{\overleftarrow{\partial}}{\partial x^1_{\bar{\nu}}} \\
-\frac{\overleftarrow{\partial}}{\partial x^1_{\bar{\nu}}}(\acute{c}\overrightarrow{P}_1 + \acute{e}^2\gamma^1_4\frac{2}{r_1} - \acute{M}\acute{c}^2))\,U(\mathcal{R}_1,t) + \frac{\partial}{\acute{i}c\partial t}\hat{T}^{d1}_{\bar{\nu}\,\mu}(\mathcal{R}_1,t)
\end{array}
\qquad (5.26)
$$

or

$$
\begin{array}{c}
\frac{\partial}{\partial x^2_\mu}\hat{T}^{d2}_{\bar{\nu}\,\mu}(\mathcal{R}_2,t) = \frac{1}{2}\bar{V}(\mathcal{R}_2,t)\gamma^1_4((\acute{c}\overleftarrow{P}_2 - \acute{e}^2\gamma^2_4\frac{2}{r_2} + \acute{M}\acute{c}^2)\frac{\overrightarrow{\partial}}{\partial x^2_{\bar{\nu}}} \\
+\frac{\overrightarrow{\partial}}{\partial x^2_{\bar{\nu}}}(\acute{c}\overrightarrow{P}_2 + \acute{e}^2\gamma^2_4\frac{2}{r_2} - \acute{M}\acute{c}^2) - (\acute{c}\overleftarrow{P}_2 - \acute{e}^2\gamma^1_4\frac{2}{r_1} + \acute{M}\acute{c}^2)\frac{\overleftarrow{\partial}}{\partial x^2_{\bar{\nu}}} \\
-\frac{\overleftarrow{\partial}}{\partial x^2_{\bar{\nu}}}(\acute{c}\overrightarrow{P}_2 + \acute{e}^2\gamma^2_4\frac{2}{r_2} - \acute{M}\acute{c}^2))\,V(\mathcal{R}_2,t) + \frac{\partial}{\acute{i}c\partial t}\hat{T}^{d2}_{\bar{\nu}\,\mu}(\mathcal{R}_1,t).
\end{array}
\qquad (5.27)
$$

We insert into these equations the functions described in eqs.(5.4). Then the eqs.(5.26/5.27) read

$$\tfrac{\partial}{\partial x^1_\mu}\hat{T}^{d1}_{\nu\,\mu}(\mathcal{R}_1,t) = \textstyle\sum_{n,n',l,l'} \tfrac{\partial}{\partial x^1_\mu}({}^{l'\,l}_{n'\,n}T^{d1}_{\nu\,\mu}(\mathcal{R}_1,t)) \tag{5.28}$$

with

$$\begin{array}{c} \tfrac{\partial}{\partial x^1_\mu}({}^{l'\,l}_{n'\,n}T^{d1}_{\nu\,\mu}(\mathcal{R}_1,t)) = \tfrac{1}{2}(c^{l'}_{n'})^*(c^l_n) \\ \times\exp(i(l'-l)\omega t)(\bar{H}_{n'}(\mathcal{R}_1)\gamma^1_4(\bar{E}^H_n - \bar{E}^H_{n'})(\tfrac{\overrightarrow{\partial}}{\partial x^1_{\bar\nu}} \\ -\tfrac{\overleftarrow{\partial}}{\partial x^1_{\bar\nu}}) + \acute{e}^2\tfrac{\partial}{\partial x^1_{\bar\nu}}[\tfrac{2}{r_1}])\,H_n(\mathcal{R}_1)\, + \tfrac{\partial}{\acute{i}c\partial t}({}^{l'\,l}_{n'\,n}T^{d1}_{\nu\,\mu}(\mathcal{R}_1,t)) \end{array} \tag{5.29}$$

and (from now on: $({}^{l'\,l}_{n'\,n}T^{d1}_{\nu\,\mu}(\mathcal{R}_1,t)) = T^{d1}_{\nu\,\mu}(\mathcal{R}_1,t))$

$$\begin{array}{c} \tfrac{\partial}{\acute{i}c\partial t}T^{d1}_{\bar\nu\,4}(\mathcal{R}_2,t) = \tfrac{1}{2}(c^{l'}_{n'})^*(c^l_n)\exp(i(l'-l)\omega t) \\ \times(\hbar(l'-l)\omega)(\bar{H}_{n'}(\mathcal{R}_1)\gamma^1_4\tfrac{\partial H_n(\mathcal{R}_1)}{\partial x^1_{\bar\nu}} - \tfrac{\partial(\bar{H}_{n'}(\mathcal{R}_1)}{\partial x^2_{\bar\nu}}\gamma^1_4 H_n(\mathcal{R}_1)). \end{array} \tag{5.30}$$

Applying the abbreviations (see eq.5.4)

$$\begin{array}{c} {}^{l'\,l}_{n'\,n}C_1(t) = (c^{l'}_{n'})^*(c^l_n)\exp(i(l'-l)\omega t), \\ {}^{l1'\,l1}_{m'\,m}C_2(t) = (d^{l1'}_{m'})^*(d^{l1}_m)\exp(i(l1'-l1)\omega t), \end{array} \tag{5.31}$$

the product of the complete charges can be described by

$$\begin{array}{c} \varrho(\mathcal{R}_1)\varrho(\mathcal{R}_2) = \acute{e}^2\textstyle\sum_{n',n,l,l',m',m,l1',l1}\, {}^{l'\,l}_{n'\,n}C_1(t)\,{}^{l1'\,l1}_{m'\,m}C_2(t) \\ \times\bar{H}_{n'}(\mathcal{R}_1)\gamma^1_4 H_n(\mathcal{R}_1)\bar{K}_{m'}(\mathcal{R}_2)\gamma^2_4 K_m(\mathcal{R}_2). \end{array} \tag{5.32}$$

With the eigenvalue equations

$$
\begin{aligned}
\acute{c}\hbar\gamma^1_{\bar\mu}\frac{\overrightarrow{\partial}}{\partial x^1_{\bar\mu}}H_n(\mathcal{R}_1) &= \gamma^1_4(E_n + \acute{e}^2\tfrac{2}{r_1})\,H_n(\mathcal{R}_1),\\
\bar H_{n'}(\mathcal{R}_1)\frac{\overleftarrow{\partial}}{\partial x^1_{\bar\mu}}c\hbar\gamma^1_{\bar\mu} &= \bar H_n(\mathcal{R}_1)\gamma^1_4(-E_{n'} - \acute{e}^2\tfrac{2}{r_1}),
\end{aligned}
\tag{5.33}
$$

it becomes

$$
\begin{aligned}
\tfrac{\partial}{\partial x^1_\mu}T^{d1}_{\bar\nu\,\mu}(\mathcal{R}_1,t) = \tfrac{1}{2}\,{}^{l'\,l}_{n'\,n}C_1(t)((\bar E^H_n - \bar E^H_{n'} + \hbar(l'-l)\omega)\\
\times(\bar H_{n'}(\mathcal{R}_1)\gamma^1_4\tfrac{\partial H_n(\mathcal{R}_1)}{\partial x^1_{\bar\nu}} - \tfrac{\partial(\bar H_{n'}(\mathcal{R}_1)}{\partial x^1_{\bar\nu}}\gamma_4 H_n(\mathcal{R}_1)\\
+\acute{e}^2\bar H_{n'}(\mathcal{R}_1)\gamma^1_4\,H_n(\mathcal{R}_1)\,\tfrac{\partial}{\partial x^1_{\bar\nu}}[\tfrac{2}{r_1}])\,.
\end{aligned}
\tag{5.34}
$$

Analogously reads

$$
\begin{aligned}
\tfrac{\partial}{\partial x^2_\mu}T^{d2}_{\bar\nu\,\mu}(\mathcal{R}_2,t) = \tfrac{1}{2}\,{}^{l1'\,l1}_{m'\,m}C_2(t)((\bar E^K_m - \bar E^K_{m'} + \hbar(l1-l1')\omega)\\
\times(\bar K_{m'}(\mathcal{R}_2)\gamma^2_4\tfrac{\partial K_m(\mathcal{R}_2)}{\partial x^2_{\bar\nu}} - \tfrac{\partial(\bar K_{m'}(\mathcal{R}_2)}{\partial x^2_{\bar\nu}}\gamma^2_4 K_m(\mathcal{R}_2))\\
+\acute{e}^2\bar K_{m'}(\mathcal{R}_2)\gamma^2_4\,K_m(\mathcal{R}_2)\,\tfrac{\partial}{\partial x^2_{\bar\nu}}[\tfrac{2}{r_2}]).
\end{aligned}
\tag{5.35}
$$

Due to $\hbar\frac{\partial}{i\partial t}\exp(-i\acute{M}\acute{c}^2/\hbar) = -\acute{M}\acute{c}^2\exp(-i\acute{M}\acute{c}^2/\hbar)$ and

$$
\begin{aligned}
\tfrac{\partial}{\partial x^1_\mu}T^{d1}_{4\,\mu}(\mathcal{R}_1,t) = -\acute{M}\acute{c}^2(\bar H_{n'}(\mathcal{R}_1)({}^{l'\,l}_{n'\,n}C_1(t)c\hbar\gamma^1_{\bar\mu}(\tfrac{\overleftarrow{\partial}}{\partial x^1_{\bar\mu}} + \tfrac{\overrightarrow{\partial}}{\partial x^1_{\bar\mu}})\\
+\tfrac{\hbar\partial}{i\acute{c}\partial t}\,{}^{l'\,l}_{n'\,n}C_1(t))\,H_n(\mathcal{R}_1),
\end{aligned}
\tag{5.36}
$$

we get

$$
\begin{aligned}
\tfrac{\partial}{\partial x^1_\mu}T^{d1}_{4\,\mu}(\mathcal{R}_1,t) = -\,{}^{l'\,l}_{n'\,n}C_1(t)\acute{M}\acute{c}^2(E^H_n + \acute{e}^2(\tfrac{2}{r_1})\\
-(E^H_{n'} + \acute{e}^2(\tfrac{2}{r_1})) + \hbar(l'-l)\omega)\bar H_{n'}(\mathcal{R}_1)\gamma^1_4 H_n(\mathcal{R}_1)
\end{aligned}
\tag{5.37}
$$

and

$$
\begin{aligned}
\tfrac{\partial}{\partial x^2_\mu}T^{d2}_{4\,\mu}(\mathcal{R}_2,t) = -\,{}^{l1'\,l1}_{m'\,m}C_2(t)\acute{M}\acute{c}^2(E^K_m + \acute{e}^2(\tfrac{2}{r_2})\\
-(E^K_{m'} + \acute{e}^2(\tfrac{2}{r_2})) + \hbar(l1'-l1)\omega)\bar K_{m'}(\mathcal{R}_2)\gamma^2_4 K_m(\mathcal{R}_2).
\end{aligned}
\tag{5.38}
$$

These relations describe the energy and force balance if the interaction between the electrons is neglected. One sees that the conditions $\bar{E}^H_n - \bar{E}^H_{n'} + \hbar(l' - l)\omega = 0$ and $\bar{E}^K_m - \bar{E}^K_{m'} + \hbar(l1 - l1')\omega = 0$ are only fulfilled by different frequencies. However, radiation should be caused if the divergences eq.(5.37) and eq.(5.38) contain synchronously oscillating shares of a common special frequency. Hence we look for combinations of H and K where parts of the divergences meet this condition. Besides, the interaction between the electrons must be taken into account. Therefore we introduce the following operators:

$$\begin{aligned} Q^{\bar{\nu}}_1 &= c\hbar\gamma^1_{\bar{\mu}}\gamma^2_4(\frac{\overleftarrow{\partial}}{\partial x^1_{\bar{\mu}}}\frac{\overrightarrow{\partial}}{\partial x^1_{\bar{\nu}}} + \frac{\overrightarrow{\partial}}{\partial x^1_{\bar{\nu}}}\frac{\overrightarrow{\partial}}{\partial x^1_{\bar{\mu}}}) - c\hbar\gamma^1_{\bar{\mu}}\gamma^2_4(\frac{\overleftarrow{\partial}}{\partial x^1_{\bar{\mu}}}\frac{\overleftarrow{\partial}}{\partial x^1_{\bar{\nu}}} + \frac{\overleftarrow{\partial}}{\partial x^1_{\bar{\nu}}}\frac{\overrightarrow{\partial}}{\partial x^1_{\bar{\mu}}}) \\ &\quad + \hbar\gamma^2_4\gamma^1_4\frac{\partial}{i\partial t}(\frac{\overrightarrow{\partial}}{\partial x^1_{\bar{\nu}}} - \frac{\overleftarrow{\partial}}{\partial x^1_{\bar{\nu}}}) + \acute{e}^2\frac{\partial}{\partial x^1_{\bar{\nu}}}[\frac{2}{r_1} - \frac{1}{r_{12}}]\gamma^1_4\gamma^2_4, \\ Q^{\bar{\nu}}_2 &= c\hbar\gamma^1_4\gamma^2_{\bar{\mu}}(\frac{\overleftarrow{\partial}}{\partial x^2_{\bar{\mu}}}\frac{\overrightarrow{\partial}}{\partial x^2_{\bar{\nu}}} + \frac{\overrightarrow{\partial}}{\partial x^2_{\bar{\nu}}}\frac{\overrightarrow{\partial}}{\partial x^2_{\bar{\mu}}}) - c\hbar\gamma^1_4\gamma^2_{\bar{\mu}}(\frac{\overleftarrow{\partial}}{\partial x^2_{\bar{\mu}}}\frac{\overleftarrow{\partial}}{\partial x^2_{\bar{\nu}}} + \frac{\overleftarrow{\partial}}{\partial x^2_{\bar{\nu}}}\frac{\overrightarrow{\partial}}{\partial x^2_{\bar{\mu}}}) \\ &\quad + \hbar\gamma^2_4\gamma^1_4\frac{\partial}{i\partial t}(\frac{\overrightarrow{\partial}}{\partial x^2_{\bar{\nu}}} - \frac{\overleftarrow{\partial}}{\partial x^2_{\bar{\nu}}}) + \acute{e}^2\frac{\partial}{\partial x^2_{\bar{\nu}}}[\frac{2}{r_2} - \frac{1}{r_{21}}]\gamma^1_4\gamma^2_4, \end{aligned} \tag{5.39}$$

$$\begin{aligned} Q^4_1 &= -\frac{\acute{a}}{\acute{\alpha}}\gamma^2_4((\acute{c}\hbar\gamma^1_{\bar{\mu}}\frac{\overleftarrow{\partial}}{\partial x^1_{\bar{\mu}}} - \gamma^1_4\acute{e}^2(\frac{2}{r_1} - \frac{1}{r_{12}}) \\ &\quad + (\acute{c}\hbar\gamma^1_{\bar{\mu}}\frac{\overrightarrow{\partial}}{\partial x^1_{\bar{\mu}}} + \gamma^1_4\acute{e}^2(\frac{2}{r_1} - \frac{1}{r_{12}}) + \hbar\gamma^1_4\frac{\partial}{i\partial t}), \\ Q^4_2 &= -\frac{\acute{a}}{\acute{\alpha}}\gamma^1_4((\acute{c}\hbar\gamma^2_{\bar{\mu}}\frac{\overleftarrow{\partial}}{\partial x^2_{\bar{\mu}}} - \gamma^2_4\acute{e}^2(\frac{2}{r_2} - \frac{1}{r_{21}}) \\ &\quad + (\hbar\gamma^2_{\bar{\mu}}\frac{\overrightarrow{\partial}}{\partial x^2_{\bar{\mu}}} + \gamma^2_4\acute{e}^2(\frac{2}{r_2} - \frac{1}{r_{21}}) + \hbar\gamma^1_4\frac{\partial}{i\partial t}). \end{aligned} \tag{5.40}$$

The term $\frac{\acute{e}^2}{r_{21}}$ describes the interaction between the electrons. $\acute{a}/\acute{\alpha}$ follows from

$\acute{M}\acute{c}/\hbar = \acute{a}/\acute{\alpha}$. The sum $(Q_1^4 + Q_2^4)$ reads

$$Q_1^4 + Q_2^4 = -\frac{\acute{a}}{\acute{\alpha}}(\overleftarrow{Q}^4_{1\,2} + \overrightarrow{Q}^4_{1\,2} + 2\gamma_4^1\gamma_4^2\frac{\hbar\partial}{i\partial t}) \tag{5.41}$$

with

$$\begin{aligned}
\overrightarrow{Q}^4_{1\,2} &= \gamma_4^2(\acute{c}\hbar\gamma^1_{\bar{\mu}}\frac{\overrightarrow{\partial}}{\partial x^1_{\bar{\mu}}} - \gamma_4^1\acute{e}^2(\tfrac{2}{r_1} - \tfrac{1}{r_{1\,2}})) \\
&+\gamma_4^1(\acute{c}\hbar\gamma^2_{\bar{\mu}}\frac{\overrightarrow{\partial}}{\partial x^2_{\bar{\mu}}} - \acute{e}^2\gamma_4^2(\tfrac{2}{r_2} - \tfrac{1}{r_{2\,1}})), \\
\overleftarrow{Q}^4_{1\,2} &= \gamma_4^2(\acute{c}\hbar\gamma^1_{\bar{\mu}}\frac{\overleftarrow{\partial}}{\partial x^1_{\bar{\mu}}} + \gamma_4^1\acute{e}^2(\tfrac{2}{r_1} - \tfrac{1}{r_{1\,2}})) \\
&+\gamma_4^1(\acute{c}\hbar\gamma^2_{\bar{\mu}}\frac{\overleftarrow{\partial}}{\partial x^2_{\bar{\mu}}} + \acute{e}^2\gamma_4^2(\tfrac{2}{r_2} - \tfrac{1}{r_{2\,1}})).
\end{aligned} \tag{5.42}$$

Applying these operators, one can introduce the following the eigenvalue equations

$$\begin{aligned}
\overrightarrow{Q}^4_{1\,2}\Psi^{\pm}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2) &= \bar{E}^{\pm}_{n\,m}\gamma_4^1\gamma_4^2\Psi^{\pm}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2), \\
\bar{\Psi}^{\pm}_{n'\,m'}(\mathcal{R}_1,\mathcal{R}_2)\overleftarrow{Q}^4_{1\,2} &= -\bar{E}^{\pm}_{n'\,m'}\gamma_4^1\gamma_4^2\bar{\Psi}^{\pm}_{n'\,m'}(\mathcal{R}_1,\mathcal{R}_2)
\end{aligned} \tag{5.43}$$

with

$$\begin{aligned}
\Psi^{\pm}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2) &= \tfrac{1}{\sqrt{2}}(H_n(\mathcal{R}_1)K_m(\mathcal{R}_2) \pm K_m(\mathcal{R}_1)H_n(\mathcal{R}_2)), \\
\bar{\Psi}^{\pm}_{n'\,m'}(\mathcal{R}_1,\mathcal{R}_2) &= \tfrac{1}{\sqrt{2}}(\bar{H}_{n'}(\mathcal{R}_1)\bar{K}_{m'}(\mathcal{R}_2) \pm \bar{K}_{m'}(\mathcal{R}_1)\bar{H}_{n'}(\mathcal{R}_2)), \\
H_n(\mathcal{R}_1)K_m(\mathcal{R}_2) &= \tfrac{1}{\sqrt{2}}(\Psi^{+}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2) + \Psi^{-}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2))
\end{aligned} \tag{5.44}$$

and

$$
\begin{aligned}
\bar{E}^{\pm}_{n\,m} &= \bar{E}^{H}_{n} + \bar{E}^{K}_{m} + {}_4({}^{HH}_{nn}M^{KK}_{mm})_4 \pm {}_4({}^{HK}_{nm}M^{KH}_{mn})_4, \\
\bar{E}^{\pm}_{n'\,m'} &= \bar{E}^{H}_{n'} + \bar{E}^{K}_{m'} + {}_4({}^{HH}_{n'n'}M^{KK}_{m'm'})_4 \pm {}_4({}^{HK}_{n'm'}M^{KH}_{m'n'})_4, \\
{}_\mu({}^{HH}_{nn}M^{KK}_{mm})_\mu &= \textstyle\int(1/r_{1\,2})\bar{H}_n(\mathcal{R}_1)\gamma_\mu H_n(\mathcal{R}_1)\,\bar{K}_m(\mathcal{R}_2)\gamma_\mu K_m(\mathcal{R}_2)dV_2dV_1, \\
{}_\mu({}^{HK}_{nm}M^{KH}_{mn})_\mu &= \textstyle\int(1/r_{1\,2})\bar{H}_n(\mathcal{R}_1)\gamma_\mu K_m(\mathcal{R}_1)\,\bar{K}_m(\mathcal{R}_2)\gamma_\mu H_n(\mathcal{R}_2)dV_2dV_1.
\end{aligned}
\tag{5.45}
$$

We investigate the cases where $m' = m$ and $n' \neq n$. That means, $\int \bar{K}_m(\mathcal{R}_2)\gamma_4^2 K_m(\mathcal{R}_2)dV_2 = 1$ but $\int \bar{H}_{n'}(\mathcal{R}_1)\gamma_4^1 H_n(\mathcal{R}_1)dV_1 = 0$. Then the following sum of the divergences of the fourth line of the tensors becomes

$$
\begin{aligned}
&{}^{l1'\,l1}_{m'\,m}C_2(t)\tfrac{\partial}{\partial x^1_\mu}T^{d1}_{4\,\mu}(\mathcal{R}_1,t) + {}^{l'\,l}_{n'\,n}C_1(t)\tfrac{\partial}{\partial x^1_\mu}T^{d2}_{4\,\mu}(\mathcal{R}_2,t) \\
&= {}^{l1'\,l1}_{m\,m}C_2(t)\textstyle\int \bar{H}_{n'}(\mathcal{R}_1)\bar{K}_m(\mathcal{R}_2)Q^4_1\,{}^{l'\,l}_{n'\,n}C_1(t)H_n(\mathcal{R}_1)K_m(\mathcal{R}_2)dV_2 \\
&+{}^{l'\,l}_{n'\,n}C_1(t)\textstyle\int(\bar{H}_{n'}(\mathcal{R}_1)\bar{K}_m(\mathcal{R}_2)Q^4_2\,{}^{l1'\,l1}_{m\,m}C_2(t)H_n(\mathcal{R}_1)K_m(\mathcal{R}_2)dV_1.
\end{aligned}
\tag{5.46}
$$

The sum of the interaction terms is zero due to the eqs.(5.42). Eq.(5.46) can be decomposed (see the eqs.(5.42)) into

$$
\begin{aligned}
&{}^{l1'\,l1}_{m\,m}C_2(t)\tfrac{\partial}{\partial x^1_\mu}T^{d1}_{4\,\mu}(\mathcal{R}_1,t) + {}^{l'\,l}_{n'\,n}C_1(t)\tfrac{\partial}{\partial x^1_\mu}T^{d2}_{4\,\mu}(\mathcal{R}_2,t) = -\tfrac{\acute{a}}{\acute{\alpha}}\,{}^{l1'\,l1}_{m\,m}C_2(t)\textstyle\int(\bar{H}_{n'}(\mathcal{R}_1)\bar{K}_m(\mathcal{R}_2) \\
&\quad\times(\overleftarrow{Q}^4_{1\,2} + \overrightarrow{Q}^4_{1\,2} + 2\gamma^1_4\gamma^2_4\tfrac{\hbar\partial}{i\partial t})\,{}^{l'\,l}_{n'\,n}C_1(t))\,H_n(\mathcal{R}_1)K_m(\mathcal{R}_2)dV_2 \\
&\quad-\tfrac{\acute{a}}{\acute{\alpha}}\,{}^{l'\,l}_{n'\,n}C_1(t)\textstyle\int(\bar{H}_{n'}(\mathcal{R}_1)\bar{K}_m(\mathcal{R}_2)(\overleftarrow{Q}^4_{1\,2} + \overrightarrow{Q}^4_{1\,2} \\
&\quad+2\gamma^1_4\gamma^2_4\tfrac{\hbar\partial}{i\partial t})\,{}^{l1'\,l1}_{m\,m}C_2(t))\,H_n(\mathcal{R}_1)K_m(\mathcal{R}_2)dV_1
\end{aligned}
\tag{5.47}
$$

or

$$
\begin{array}{c}
{}^{l1'l1}_{mm}C_2(t)\frac{\partial}{\partial x^1_\mu}T^{d1}_{4\,\mu}(\mathcal{R}_1,t)+{}^{l'\,l}_{n'\,n}C_1(t)\frac{\partial}{\partial x^2_\mu}T^{d2}_{4\,\mu}(\mathcal{R}_2,t)=\\
-\frac{\acute{a}}{2\acute{\alpha}}({}^{l1'\,l1}_{m\,m}C_2(t)\int(\bar{\Psi}^+_{n'\,m}(\mathcal{R}_1,\mathcal{R}_2)(\overleftarrow{Q}^4_{1\,2}+\overrightarrow{Q}^4_{1\,2}+2\gamma^1_4\gamma^2_4\frac{\hbar\partial}{i\partial t})\,{}^{l'\,l}_{n'\,n}C_1(t)\,\Psi^+_{n\,m}(\mathcal{R}_1,\mathcal{R}_2))dV_2\\
+{}^{l1'\,l1}_{m\,m}C_2(t)\int(\bar{\Psi}^-_{n'\,m}(\mathcal{R}_1,\mathcal{R}_2)(\overleftarrow{Q}^4_{1\,2}+\overrightarrow{Q}^4_{1\,2}+2\gamma^1_4\gamma^2_4\frac{\hbar\partial}{i\partial t})\,{}^{l'\,l}_{n'\,n}C_1(t)\,\Psi^-_{n\,m}(\mathcal{R}_1,\mathcal{R}_2))dV_2\\
+{}^{l'\,l}_{n'\,n}C_1(t)\int(\bar{\Psi}^+_{n'\,m}(\mathcal{R}_1,\mathcal{R}_2)(\overleftarrow{Q}^4_{1\,2}+\overrightarrow{Q}^4_{1\,2}+2\gamma^1_4\gamma^2_4\frac{\hbar\partial}{i\partial t})\,{}^{l1'\,l1}_{m\,m}C_2(t)\,\Psi^+_{n\,m}(\mathcal{R}_1,\mathcal{R}_2))dV_1\\
+{}^{l'\,l}_{n'\,n}C_1(t)\int(\bar{\Psi}^-_{n'\,m}(\mathcal{R}_1,\mathcal{R}_2)(\overleftarrow{Q}^4_{1\,2}+\overrightarrow{Q}^4_{1\,2}+2\gamma^1_4\gamma^2_4\frac{\hbar\partial}{i\partial t})C_2(t)\,\Psi^-_{n\,m}(\mathcal{R}_1,\mathcal{R}_2))dV_1)\\
+\frac{\acute{a}}{2\acute{\alpha}}\Delta O.
\end{array}
\tag{5.48}
$$

That leads to

$$
\begin{array}{c}
{}^{l1'\,l1}_{m\,m}C_2(t)\frac{\partial}{\partial x^1_\mu}T^{d1}_{4\,\mu}(\mathcal{R}_1,t)+{}^{l'\,l}_{n'\,n}C_1(t)\frac{\partial}{\partial x^2_\mu}T^{d2}_{4\,\mu}(\mathcal{R}_2,t)=\\
\frac{\acute{a}}{2\acute{\alpha}}\,{}^{l'\,l}_{n'\,n}C_1(t)\,{}^{l1'\,l1}_{m\,m}C_2(t)\int\bar{\Psi}^+_{n'\,m}(\mathcal{R}_1,\mathcal{R}_2)(\bar{E}^+_{n'\,m}-\bar{E}^+_{n\,m}\\
-2\hbar(l'-l)\omega)\,\Psi^+_{n\,m}(\mathcal{R}_1,\mathcal{R}_2)dV_2\\
+\frac{\acute{a}}{2\acute{\alpha}}\,{}^{l'\,l}_{n'\,n}C_1(t)\,{}^{l1'\,l1}_{m\,m}C_2(t)\int\bar{\Psi}^-_{n'\,m}(\mathcal{R}_1,\mathcal{R}_2)(\bar{E}^-_{n'\,m}-\bar{E}^-_{n\,m}\\
-2\hbar(l'-l)\omega)\,\Psi^-_{n\,m}(\mathcal{R}_1,\mathcal{R}_2)dV_2\\
+\frac{\acute{a}}{2\acute{\alpha}}\,{}^{l'\,l}_{n'\,n}C_1(t)\,{}^{l1'\,l1}_{m\,m}C_2(t)\int\bar{\Psi}^+_{n'\,m}(\mathcal{R}_1,\mathcal{R}_2)(\bar{E}^+_{n'\,m}-\bar{E}^+_{n\,m}\\
-2\hbar(l1'-l1)\omega)\,\Psi^+_{n\,m}(\mathcal{R}_1,\mathcal{R}_2)dV_1\\
+\frac{\acute{a}}{2\acute{\alpha}}\,{}^{l'\,l}_{n'\,n}C_1(t)\,{}^{l1'\,l1}_{m\,m}C_2(t)\int\bar{\Psi}^-_{n'\,m}(\mathcal{R}_1,\mathcal{R}_2)(\bar{E}^-_{n'\,m}-\bar{E}^-_{n\,m}\\
-2\hbar(l1'-l1)\omega)\,\Psi^-_{n\,m}(\mathcal{R}_1,\mathcal{R}_2)dV_1\\
+\frac{\acute{a}}{2\acute{\alpha}}\Delta O
\end{array}
\tag{5.49}
$$

with

$$\Delta O(\mathcal{R}_1, \mathcal{R}_2, t) = \Delta O_a^{4+}(\mathcal{R}_1, t) + \Delta O_b^{4+}(\mathcal{R}_2, t) + \Delta O_a^{4-}(\mathcal{R}_1, t) \\ + \Delta O_b^{4-}(\mathcal{R}_2, t),$$

$$\begin{gathered} \Delta O_a^{4\pm}(\mathcal{R}_1, t) = -({}^{l1'\,l1}_{m\,m}C_2(t) \int (\bar{\Psi}^{\pm}_{n'\,m}(\mathcal{R}_1, \mathcal{R}_2)(\overleftarrow{Q}^4_{1\,2} + \overrightarrow{Q}^4_{1\,2} \\ + 2\gamma_4^1 \gamma_4^2 \tfrac{\hbar \partial}{i \partial t}) \, {}^{l'\,l}_{n'\,n}C_1(t) \, \Psi^{\mp}_{n\,m}(\mathcal{R}_1, \mathcal{R}_2)) dV_2 \\ = {}^{l'\,l}_{n'\,n}C_1(t) \, {}^{l1'\,l1}_{m\,m}C_2(t) \int \bar{\Psi}^{\pm}_{n'\,m}(\mathcal{R}_1, \mathcal{R}_2)(\bar{E}^{\pm}_{n'\,m} - \bar{E}^{\mp}_{n\,m} \\ - 2\hbar(l' - l)\omega)\gamma_4^1 \gamma_4^2 \, \Psi^{\mp}_{n\,m}(\mathcal{R}_1, \mathcal{R}_2) dV_2 = \\ \tfrac{1}{2} \, {}^{l'\,l}_{n'\,n}C_1(t) \, {}^{l1'\,l1}_{m\,m}C_2(t)(E^{\pm}_{n'\,m} - E^{\mp}_{n\,m} - 2\hbar(l' - l)\omega) \bar{H}_{n'}(\mathcal{R}_1) \gamma_4^1 H_n(\mathcal{R}_1) \end{gathered} \tag{5.50}$$

$$\begin{gathered} \Delta O_b^{4\pm}(\mathcal{R}_2, t) = -({}^{l1'\,l1}_{m\,m}C_1(t) \int (\bar{\Psi}^{\pm}_{n'\,m}(\mathcal{R}_1, \mathcal{R}_2)(\overleftarrow{Q}^4_{1\,2} + \overrightarrow{Q}^4_{1\,2} \\ + 2\gamma_4^1 \gamma_4^2 \tfrac{\hbar \partial}{i \partial t}) \, {}^{l'\,l}_{n'\,n}C_2(t) \, \Psi^{\mp}_{n\,m}(\mathcal{R}_1, \mathcal{R}_2)) dV_1 \\ = {}^{l'\,l}_{n'\,n}C_1(t) \, {}^{l1'\,l1}_{m\,m}C_2(t) \int \bar{\Psi}^{\pm}_{n'\,m}(\mathcal{R}_1, \mathcal{R}_2)(\bar{E}^{\pm}_{n'\,m} - \bar{E}^{\mp}_{n\,m} \\ - 2\hbar(l1' - l1)\omega)\gamma_4^1 \gamma_4^2 \, \Psi^{\mp}_{n\,m}(\mathcal{R}_1, \mathcal{R}_2) dV_1 = \\ -\tfrac{1}{2} \, {}^{l'\,l}_{n'\,n}C_1(t) \, {}^{l1'\,l1}_{m\,m}C_2(t)(E^{\pm}_{n'\,m} - E^{\mp}_{n\,m} - 2\hbar(l' - l)\omega) \bar{H}_{n'}(\mathcal{R}_2) \gamma_4^2 H_n(\mathcal{R}_2). \end{gathered}$$

If we consider the coordinates $\mathcal{R}_1 = \mathcal{R}$ and $\mathcal{R}_2 = \mathcal{R}$ in the eqs.(5.49/5.50) as coordinates of the real space, we get for $n' \neq n$ and $l' - l = l1' - l1$ the relations

$$\Delta O_a^{4\pm}(\mathcal{R}, \mathcal{R}, t) + \Delta O_b^{4\pm}(\mathcal{R}, \mathcal{R}, t) = 0. \tag{5.51}$$

Therefore we obtain

$$\begin{gathered} {}^{l1'\,l1}_{m'\,m}C_2(t) \tfrac{\partial}{\partial x^1_\mu} T^{d1}_{4\,\mu}(\mathcal{R}, t) + {}^{l'\,l}_{n'\,n}C_1(t) \tfrac{\partial}{\partial x^2_\mu} T^{d2}_{4\,\mu}(\mathcal{R}, t) = \\ \tfrac{\acute{a}}{2\acute{\alpha}} \, {}^{l'\,l}_{n'\,n}C_1(t) \, {}^{l1'\,l1}_{m\,m}C_2(t)((\bar{E}^{+}_{n'\,m} - \bar{E}^{+}_{n\,m} - 2\hbar(l' - l)\omega) \, \bar{H}_{n'}(\mathcal{R}) \gamma_4^0 H_n(\mathcal{R}) \\ + (\bar{E}^{-}_{n'\,m} - \bar{E}^{-}_{n\,m} - 2\hbar(l' - l)\omega) \, \bar{H}_{n'}(\mathcal{R}) \gamma_4^0 H_n(\mathcal{R})). \end{gathered} \tag{5.52}$$

Eq.(5.52) is describable also by the divergences of currents. To show this, we introduce the following currents

$$J^{\pm d1}_{\bar\mu}(\mathcal{R}_1,t) = \tfrac{i}{2}\acute{e}\,{}^{l'\,l}_{n'\,n}C_1(t)\,{}^{l1'\,l1}_{m\,m}C_2(t)(J^H_{\bar\mu}(\mathcal{R}_1,t)+J^{A\pm}_{\bar\mu}(\mathcal{R}_1)),$$
$$J^{\pm d2}_{\bar\mu}(\mathcal{R}_2) = \tfrac{i}{2}\acute{e}\,{}^{l'\,l}_{n'\,n}C_1(t)\,{}^{l1'\,l1}_{m\,m}C_2(t)(J^H_{\bar\mu}(\mathcal{R}_2)+J^{B\pm}_{\bar\mu}(\mathcal{R}_2)) \quad (5.53)$$

with

$$J^H_{\bar\mu}(\mathcal{R}_1) = \bar H_{n'}(\mathcal{R}_1)\gamma^1_{\bar\mu}H_n(\mathcal{R}_1),$$
$$J^{\pm A}_{\bar\mu}(\mathcal{R}_1)) = -\tfrac{1}{4\pi\hbar\acute{c}}\tfrac{\partial}{\partial x^1_{\bar\mu}}\int\tfrac{1}{r_{1\,1'}}\bar\Psi^{\pm}_{n'\,m}(\mathcal{R}'_1,\mathcal{R}'_2)\gamma^{1'}_4\gamma^{2'}_4\acute{e}^2(\tfrac{1}{r_{1'\,2'}})\Psi^{\pm}_{n\,m}(\mathcal{R}'_1,\mathcal{R}'_2)dV'_2dV'_1$$
$$J^H_{\bar\mu}(\mathcal{R}_2) = \bar H_{n'}(\mathcal{R}_2)\gamma^2_{\bar\mu}H_n(\mathcal{R}_2),$$
$$J^{\pm B}_{\bar\mu}(\mathcal{R}_2)) = -\tfrac{1}{4\pi\acute{c}\hbar}\tfrac{\partial}{\partial x^2_{\bar\mu}}\int\tfrac{1}{r_{2\,2'}}\bar\Psi^{\pm}_{n'\,m}(\mathcal{R}'_1,\mathcal{R}_2)\gamma^{1'}_4\gamma^{2'}_4\acute{e}^2(\tfrac{1}{r_{2'\,1'}})\Psi^{\pm}_{n\,m}(\mathcal{R}'_1,\mathcal{R}'_2)dV'_2dV'_1 \quad (5.54)$$
$$or$$
$$J^{\pm d1}_4(\mathcal{R}_1,t) = i\acute{e}\,{}^{l'\,l}_{n'\,n}C_1(t)\,{}^{l1'\,l1}_{m'\,m}C_2(t)\bar H_{n'}(\mathcal{R}_1\gamma^1_4H(\mathcal{R}_1),\;\; J^{\pm A}_4 = 0$$
$$J^{\pm d2}_4(\mathcal{R}_2,t) = i\acute{e}\,{}^{l'\,l}_{n'\,n}C_1(t)\,{}^{l1'\,l1}_{m\,m}C_2(t)\bar H_{n'}(\mathcal{R}_2\gamma^2_4H(\mathcal{R}_2),\;\; J^{\pm B}_4 = 0.$$

If you use the solutions of the following potential equations

$$-\tfrac{1}{4\pi}\tfrac{\partial^2}{\partial x^1_{\bar\mu}\partial x^1_{\bar\mu}}(\int(\tfrac{1}{r_{1\,1'}}\bar\Psi^{\pm}_{n'\,m}(\mathcal{R}'_1,\mathcal{R}'_2)\,\gamma^{1'}_4\gamma^{2'}_4\acute{e}^2(\tfrac{1}{r_{1'\,2'}})\Psi^{\pm}_{n\,m}(\mathcal{R}'_1,\mathcal{R}'_2)dV'_2dV'_1 =$$
$$= \bar\Psi^{\pm}_{n'\,m}(\mathcal{R}_1,\mathcal{R}'_2)\,\gamma^1_4\gamma^{2'}_4\acute{e}^2(\tfrac{1}{r_{1\,2'}})\Psi^{\pm}_{n\,m}(\mathcal{R}_1,\mathcal{R}'_2)dV'_2)$$
$$and \quad (5.55)$$
$$-\tfrac{1}{4\pi}\tfrac{\partial^2}{\partial x^2_{\bar\mu}\partial x^2_{\bar\mu}}(\int(\tfrac{1}{r_{1'\,2}}\bar\Psi^{\pm}_{n'\,m}(\mathcal{R}'_1,\mathcal{R}'_2)\,\gamma^{1'}_4\gamma^{2'}_4\acute{e}^2(\tfrac{1}{r_{1'\,2'}})\Psi^{\pm}_{n\,m}(\mathcal{R}'_1,\mathcal{R}'_2)dV'_2dV'_1 =$$
$$\bar\Psi^{\pm}_{n'\,m}(\mathcal{R}'_1,\mathcal{R}_2)\,\gamma^{1'}_4\gamma^2_4\acute{e}^2(\tfrac{1}{r_{1'\,2}})\Psi^{\pm}_{n\,m}(\mathcal{R}'_1,\mathcal{R}_2)dV'_1,$$

you obtain for the divergences of the currents:

$$
\begin{array}{c}
\acute{c}\hbar\frac{\partial}{\partial x^1_{\bar\mu}}J^H_{\bar\mu}(\mathcal{R}_1)=(E_{n'}+E_m-E_n-E_m)\bar H_{n'}(\mathcal{R}_1)\gamma^1_4 H_n(\mathcal{R}_1),\\
\acute{c}\hbar\frac{\partial}{\partial x^1_{\bar\mu}}J^{\pm A}_{\bar\mu}(\mathcal{R}_1)=\bar\Psi^{\pm}_{n'\,m}(\mathcal{R}_1,\mathcal{R}'_2)\gamma^1_4\gamma^{2'}_4(\frac{1}{r_{1\,2'}})\Psi^{\pm}_{n\,m}(\mathcal{R}_1,\mathcal{R}'_2)dV'_2,\\
\\
\acute{c}\hbar\frac{\partial}{\partial x^1_{\bar\mu}}J^H_{\bar\mu}(\mathcal{R}_2)=(E_{n'}+E_m-E_n-E_m)\bar H_{n'}(\mathcal{R}_2)\gamma^2_4 H_n(\mathcal{R}_2),\\
\acute{c}\hbar\frac{\partial}{\partial x^1_{\bar\mu}}J^{\pm B}_{\bar\mu}(\mathcal{R}_2)=\bar\Psi^{\pm}_{n'\,m}(\mathcal{R}'_1,\mathcal{R}_2)\gamma^{1'}_4\gamma^2_4(\frac{1}{r_{1'\,2}})\Psi^{\pm}_{n\,m}(\mathcal{R}'_1\mathcal{R}_2)dV'_1,
\end{array}
\tag{5.56}
$$

Here the eq.(4.47) and the relation $\hbar\acute{c}=(\acute{e}^2/\acute{a}\acute{\alpha})$ are applied. Therefore is

$$
\begin{array}{rl}
\acute{c}\hbar\frac{\partial}{\partial x^1_{\mu}}J^{\pm d1}_{\mu}(\mathcal{R}_1)= & \frac{i}{2}\acute{e}\,{}^{l'\,l}_{n'\,n}C_1(t)\,{}^{l1'\,l1}_{m\,m}C_2(t)(E^{\pm}_{n'\,m}-E^{\pm}_{n\,m}\\
 & -2\hbar(l'-l)\omega)\bar H_{n'}(\mathcal{R}_1)\gamma^1_4 H_n(\mathcal{R}_1),\\
\acute{c}\hbar\frac{\partial}{\partial x^2_{\mu}}J^{\pm d2}_{\mu}(\mathcal{R}_2)= & \frac{i}{2}\acute{e}\,{}^{l'\,l}_{n'\,n}C_1(t)\,{}^{l1'\,l1}_{m\,m}C_2(t)(E^{\pm}_{n'\,m}-E^{\pm}_{n\,m}\\
 & -2\hbar(l'-l)\omega)\bar H_{n'}(\mathcal{R}_2)\gamma^2_4 H_n(\mathcal{R}_2).
\end{array}
\tag{5.57}
$$

We consider again the coordinates $\mathcal{R}_1=\mathcal{R}$ and $\mathcal{R}_2=\mathcal{R}$ in these equations as coordinates of the real space. That leads to the relations

$$
\begin{array}{c}
\acute{c}\hbar\frac{\partial}{\partial x_{\mu}}(J^{\pm d1}_{\bar\mu}(\mathcal{R})+J^{\pm d2}_{\bar\mu}(\mathcal{R}))=\frac{i}{2}\,\acute{e}\,{}^{l'\,l}_{n'\,n}C_1(t)\,{}^{l1'\,l1}_{m\,m}C_2(t)(E^{\pm}_{n'\,m}-E^{\pm}_{n\,m}\\
-2\hbar(l'-l)\omega)\bar H_{n'}(\mathcal{R})\gamma^0_4 H_n(\mathcal{R}).
\end{array}
\tag{5.58}
$$

and the comparison with eq.(5.52) to $(\frac{\acute{a}}{\acute{\alpha}}\,\acute{c}\hbar=\acute{M}\acute{c}^2)$

$$
\begin{array}{c}
{}^{l1'\,l1}_{m\,m}C_2(t)\frac{\partial}{\partial x^1_{\mu}}T^{d1}_{4\,\mu}(\mathcal{R},t)+{}^{l'\,l}_{n'\,n}C_1(t)\frac{\partial}{\partial x^2_{\mu}}T^{d2}_{4\,\mu}(\mathcal{R},t)=\\
\frac{\acute{a}}{2\acute{\alpha}}\,{}^{l'\,l}_{n'\,n}C_1(t)\,{}^{l1'\,l1}_{m'\,m}C_2(t)((\bar E^{+}_{n'\,m}-\bar E^{+}_{n\,m}-2\hbar(l'-l)\omega)\,\bar H_{n'}(\mathcal{R})\gamma^0_4 H_n(\mathcal{R})\\
+(\bar E^{-}_{n'\,m}-\bar E^{-}_{n\,m}-2\hbar(l'-l)\omega)\,\bar H_{n'}(\mathcal{R})\gamma^0_4 H_n(\mathcal{R}))\\
=-\acute{M}\acute{c}^2\frac{i}{\acute{e}}\,\frac{\partial}{\partial x_{\mu}}(J^{+d}_{\mu}(\mathcal{R})+J^{-d}_{\mu}(\mathcal{R})).
\end{array}
\tag{5.59}
$$

The factor 1/2 in eq.(5.58) is the consequence of eq.(5.51), where the other half of the associated current is compensated. However, the charge components J_4^{dj} are involved completely. These currents form for $n' \neq n$ radiation moments. That means, it radiates when one of the conditions

$$(E^{\pm}_{n'\,m} - E^{\pm}_{n\,m} - 2\hbar(l' - l)\omega) = 0 \tag{5.60}$$

is fulfilled. Then an interaction with a radiation field is not disturbed by the normal force balance. All terms of the sum over differences $l' - l = l1' - l1$ which meet this condition, contribute to the effect. The expressions $\bar{H}_{n'} = (\hat{H}^*_{n'}\ \check{H}^*_{n'})$ and $H_n = (\hat{H}_n\ over\ \check{H}_n)$ explain the linking with the solutions $\hat{H}^*_{n'} = ()h^*_{n'}$ and $\hat{H}_n = ()h_n$ of the Schrödinger equation. The difference to the eqs.(5.36-5.38) results from the influence of the second electron on the oscillation of the first electron and the opposite reaction.

Finally we consider the divergences of the other tensor lines in a more general form. We use the following expression

$$\begin{array}{l} {}^{l1'\,l1}_{m'\,m}C_2(t)\frac{\partial}{\partial x^1_\mu}T^{d1}_{\bar{\nu}\,\mu}(\mathcal{R}_1,t) + {}^{l'\,l}_{n'\,n}C_1(t)\frac{\partial}{\partial x^1_\mu}T^{d2}_{\bar{\nu}\,\mu}(\mathcal{R}_2,t) \\ = {}^{l1'\,l1}_{m'\,m}C_2(t) \int \bar{H}_{n'}(\mathcal{R}_1)\bar{K}_{m'}(\mathcal{R}_2)Q^{\bar{\nu}}_1\, {}^{l'\,l}_{n'\,n}C_1(t)H_n(\mathcal{R}_1)K_m(\mathcal{R}_2)dV_2 \\ + {}^{l'\,l}_{n'\,n}C_1(t) \int (\bar{H}_{n'}(\mathcal{R}_1)\bar{K}_{m'}(\mathcal{R}_2)Q^{\bar{\nu}}_2\, {}^{l1'\,l1}_{m'\,m}C_2(t)H_n(\mathcal{R}_1)K_m(\mathcal{R}_2)dV_1. \end{array} \tag{5.61}$$

If we add up all terms with n, n', m, m', given eq.(5.61), we obtain the complete force balance. Using the abbreviations

$$
\begin{aligned}
{}^{HH}_{n'\,n}L1^{KK}_{m'\,m}(\mathcal{R}_1,\mathcal{R}_2)_{\bar\nu} &= {}^{l1'\,l1}_{m'\,m}C_2(t)\bar H_{n'}(\mathcal{R}_1)\,\bar K_{m'}(\mathcal{R}_2)Q_1^{\bar\nu}\,{}^{l'\,l}_{n'\,n}C_1(t)\,H_n(\mathcal{R}_1)K_m(\mathcal{R}_2),\\
{}^{HH}_{n'\,n}L2^{KK}_{m'\,m}(\mathcal{R}_1,\mathcal{R}_2)_{\bar\nu} &= {}^{l'\,l}_{n'\,n}C_1(t)\bar H_{n'}(\mathcal{R}_1)\,\bar K_{m'}(\mathcal{R}_2)Q_2^{\bar\nu}\,{}^{l1'\,l1}_{m'\,m}C_2(t)\,H_n(\mathcal{R}_1)K_m(\mathcal{R}_2),
\end{aligned}
\tag{5.62}
$$

this divergence reads

$$
\begin{aligned}
&{}^{l1'\,l1}_{m'\,m}C_2(t)\tfrac{\partial}{\partial x^1_\mu}T^{d1}_{\bar\nu\,\mu}(\mathcal{R}_1,t) + {}^{l'\,l}_{n'\,n}C_1(t)\tfrac{\partial}{\partial x^2_\mu}T^{d2}_{\bar\nu\,\mu}(\mathcal{R}_2,t)\\
&= \int {}^{HH}_{n'\,n}L1^{KK}_{m'\,m}(\mathcal{R}_1,\mathcal{R}_2,t)_{\bar\nu}dV_2 + \int {}^{HH}_{n'\,n}L2^{KK}_{m'\,m}(\mathcal{R}_1,\mathcal{R}_2,t)_{\bar\nu}dV_1.
\end{aligned}
\tag{5.63}
$$

Due to the eqs.(5.34/5.35/5.39) and because of

$$
\begin{aligned}
&{}^{l1'\,l1}_{m'\,m}C_2(t)\bar H_{n'}(\mathcal{R}_1)Q_1^{\bar\nu}\,{}^{l'\,l}_{n'\,n}C_1(t)H_n(\mathcal{R}_1)\bar K_{m'}(\mathcal{R}_2)\gamma_4^2K_n(\mathcal{R}_2)\\
&= {}^{l1'\,l1}_{m'\,m}C_2(t)\bar H_{n'}(\mathcal{R}_1)(-E^H_{n'} - \acute e^2\tfrac{2}{r_1} + E^H_n + \acute e^2\tfrac{2}{r_1})\gamma_4^1\tfrac{\overrightarrow{\partial}}{\partial x^1_{\bar\nu}}\\
&\quad +(E^H_{n'} + \acute e^2\tfrac{2}{r_1} - E^H_n - \acute e^2\tfrac{2}{r_1})\tfrac{\overleftarrow{\partial}}{\partial x^1_{\bar\nu}}\\
&+\hbar\gamma_4^2\gamma_4^1\tfrac{\partial}{i\partial t}(\tfrac{\overrightarrow{\partial}}{\partial x^1_{\bar\nu}} - \tfrac{\overleftarrow{\partial}}{\partial x^1_{\bar\nu}}) + \acute e^2\tfrac{\partial}{\partial x^1_{\bar\nu}}[\tfrac{2}{r_1} - \tfrac{1}{r_{1\,2}}]\gamma_4^1\,{}^{l'\,l}_{n'\,n}C_1(t)H_n(\mathcal{R}_1)\bar K_{m'}\gamma_4^2K_n(\mathcal{R}_2),
\end{aligned}
\tag{5.64}
$$

it becomes

$$
\begin{aligned}
\int {}^{HH}_{n'\,n}L1^{KK}_{m'\,m}(\mathcal{R}_1,\mathcal{R}_2,t)_{\bar\nu}dV_2 &= {}^{l'\,l}_{n'\,n}C_1(t)\,{}^{l1'\,l1}_{m'\,m}C_2(t)\int \bar H_{n'}(\mathcal{R}_1)\bar K_{m'}(\mathcal{R}_2)\gamma_4^2\\
&\times((-E^H_{n'} - \acute e^2\gamma_4^1\tfrac{2}{r_1})\tfrac{\overrightarrow{\partial}}{\partial x^1_{\bar\nu}} + \tfrac{\overleftarrow{\partial}}{\partial x^1_{\bar\nu}}(E^H_n + \acute e^2\gamma_4^1\tfrac{2}{r_1})\\
&(\,H_n(\mathcal{R}_1,t)\,K_m(\mathcal{R}_2)dV_2 + \hbar(l'-l)\omega)\int(\bar H_{n'}(\mathcal{R}_1)\gamma_4^1\tfrac{\partial H_n(\mathcal{R}_1)}{\partial x^1_{\bar\nu}}\\
&-\tfrac{\partial(\bar H_{n'}(\mathcal{R}_1)}{\partial x^1_{\bar\nu}}\gamma_4 H_n(\mathcal{R}_1))\bar K_{m'}(\mathcal{R}_2)\gamma_4^2\,K_m(\mathcal{R}_2)dV_2\\
&+\acute e^2\int \bar H_{n'}(\mathcal{R}_1)\gamma_4^1\,H_n(\mathcal{R}_1)\,\tfrac{\partial}{\partial x^1_{\bar\nu}}[\tfrac{2}{r_1} - \tfrac{1}{r_{1\,2}}])\bar K_{m'}(\mathcal{R}_2)\gamma_4^2K_m(\mathcal{R}_2)dV_2.
\end{aligned}
\tag{5.65}
$$

Hence is

$$\int {}^{HH}_{n'\,n}L1^{KK}_{m'\,m}(\mathcal{R}_1,\mathcal{R}_2,t)_{\bar\nu}dV_2 = {}^{l'\,l}_{n'\,n}C_1(t)\,{}^{l1'\,l1}_{m'\,m}C_2(t)((E^H_n - E^H_{n'} + \hbar(l'-l)\omega) \times(\bar H_{n'}(\mathcal{R}_1)\gamma_4^1\frac{\partial H_n(\mathcal{R}_1)}{\partial x^1_{\bar\nu}} - \frac{\partial \bar H_{n'}(\mathcal{R}_1)}{\partial x^1_{\bar\nu}}\gamma^1_4 H_n(\mathcal{R}_1))\delta(m'-m) + \acute{e}^2\int \bar H_{n'}(\mathcal{R}_1)\gamma_4^1\,H_n(\mathcal{R}_1)\,\frac{\partial}{\partial x^1_{\bar\nu}}[\frac{2}{r_1}-\frac{1}{2r_{1\,2}}])\bar K_{m'}(\mathcal{R}_2)\gamma_4^2 K_m(\mathcal{R}_2)dV_2). \tag{5.66}$$

The analogous formula reads

$$\int {}^{HH}_{n'\,n}L2^{KK}_{m'\,m}(\mathcal{R}_1,\mathcal{R}_2,t)_{\bar\nu}dV_2 = {}^{l'\,l}_{n'\,n}C_1(t)\,{}^{l1'\,l1}_{m'\,m}C_2(t)((E^K_m - E^K_{m'} + \hbar(l1'-l1)\omega) \times(\bar K_{m'}(\mathcal{R}_1)\gamma_4^2\frac{\partial K_m(\mathcal{R}_2)}{\partial x^2_{\bar\nu}} - \frac{\partial \bar K_{m'}(\mathcal{R}_2)}{\partial x^2_{\bar\nu}}\gamma^2_4 K_m(\mathcal{R}_2))\delta(n'-n) + \acute{e}^2\int \bar H_{n'}(\mathcal{R}_1)\gamma_4^1\,H_n(\mathcal{R}_1)\,\frac{\partial}{\partial x^2_{\bar\nu}}[\frac{2}{r_2}-\frac{1}{r_{2\,1}}])\bar K_{m'}(\mathcal{R}_2)\gamma_4^2 K_m(\mathcal{R}_2)dV_1). \tag{5.67}$$

So becomes

$$\begin{aligned} &{}^{l1'\,l1}_{m'\,m}C_2(t)\frac{\partial}{\partial x_\mu}T^{d1}_{\bar\nu\,\mu}(\mathcal{R}_1,t) + {}^{l'\,l}_{n'\,n}C_1(t)\frac{\partial}{\partial x_\mu}T^{d2}_{\bar\nu\,\mu}(\mathcal{R}_2,t) \\ &= {}^{l'\,l}_{n'\,n}C_1(t)\,{}^{l1'\,l1}_{m'\,m}C_2(t)((E^H_n - E^H_{n'} + \hbar(l'-l)\omega) \\ &\times(\bar H_{n'}(\mathcal{R}_1)\gamma_4^1\frac{\partial H_n(\mathcal{R}_1)}{\partial x^1_{\bar\nu}} - \frac{\partial \bar H_{n'}(\mathcal{R}_1)}{\partial x^1_{\bar\nu}}\gamma^1_4 H_n(\mathcal{R}_1))\delta(m'-m) + (E^K_m - E^K_{m'} \\ &+\hbar(l1'-l1)\omega)(\bar K_{m'}(\mathcal{R}_2)\gamma_4^2\frac{\partial K_m(\mathcal{R}_2)}{\partial x^2_{\bar\nu}} - \frac{\partial \bar K_{m'}(\mathcal{R}_2)}{\partial x^2_{\bar\nu}}\gamma^2_4 K_m(\mathcal{R}_2))\delta(n'-n) \\ &+ \acute{e}^2\int (\bar H_{n'}(\mathcal{R}_1)\gamma_4\,H_n(\mathcal{R}_1)\,\frac{\partial}{\partial x^1_{\bar\nu}}[\frac{2}{r_1}-\frac{1}{r_{1\,2}}]\bar K_{m'}(\mathcal{R}_2)\gamma_4^2 K_m(\mathcal{R}_2)dV_2 \\ &+\acute{e}^2\int\ \bar H_{n'}(\mathcal{R}_1)\gamma_4^1 H_n(\mathcal{R}_1)\frac{\partial}{\partial x^2_{\bar\nu}}[\frac{2}{r_2}-\frac{1}{r_{1\,2}}])\bar K_{m'}(\mathcal{R}_2)\gamma_4^2\,K_m(\mathcal{R}_2)\,dV_1)\,. \end{aligned} \tag{5.68}$$

This has to be compensated through the divergence of the linked quasi-static electromagnetic field.

5.3 *The Schrödinger approximation*

All these equations can be replaced by analogous expression based on solutions of the Schrödinger equation. We apply the units of dimensions, specified in section 3.2. The functions linked to the eqs.(5.4) are now

$$u(\mathcal{R}_1, t') = \sum_n a_n(t') h_n(\mathcal{R}_1), \tag{5.69}$$

and

$$v(\mathcal{R}_2, t') = \sum_m b_m(t') \grave{k}_m(\mathcal{R}_2) \tag{5.70}$$

respectively in the static cases

$$\begin{gathered} u(\mathcal{R}_1, t') = \exp(-\, iE'\, t') \breve{u}(\mathcal{R}_1), \\ \breve{u}(\mathcal{R}_1) = \textstyle\sum_n \bar{a}_n h_n(\mathcal{R}_1) \end{gathered} \tag{5.71}$$

and

$$\begin{gathered} v(\mathcal{R}_2, t) = \exp(-\, iE'\, t') \breve{v}(\mathcal{R}_2), \\ \breve{v}(\mathcal{R}_2) = \textstyle\sum_m \bar{b}_m \grave{k}_m(\mathcal{R}_2). \end{gathered} \tag{5.72}$$

These expressions must satisfy the following differential equations

$$\begin{gathered} \tfrac{\acute{e}^2}{\acute{a}}((i\tfrac{\partial}{\partial t'} - \overrightarrow{P}^s_1 + A^1_4(\mathcal{R}_1, t') + A^2_4(\mathcal{R}_1, t'))u(\mathcal{R}_1, t') = 0, \\ \tfrac{\acute{e}^2}{\acute{a}}(i\tfrac{\partial}{\partial t'} - \overrightarrow{P}^s_2 + A^1_4(\mathcal{R}_2, t') + A^2_4(\mathcal{R}_2, t'))v(\mathcal{R}_2, t') = 0, \end{gathered} \tag{5.73}$$

$$\frac{é^2}{á}u(\mathcal{R}_1,t')^*((i\frac{\partial}{\partial t'}-\overleftarrow{P}^s_1-A^1_4(\mathcal{R}_1,t')-A^2_4(\mathcal{R}_1,t'))=0,$$
$$\frac{é^2}{á}v(\mathcal{R}_2,t')^*(i\frac{\partial}{\partial t'}-\overleftarrow{P}^s_2-A^1_4(\mathcal{R}_2,t')-A^2_4(\mathcal{R}_2,t'))=0. \qquad (5.74)$$

The new operators are

$$\overrightarrow{P}^s_1=-\frac{1}{2}\frac{\overrightarrow{\partial}^2}{\partial x^1_{\bar\mu}\partial x^1_{\bar\mu}}-\frac{2}{r_1},\ \overrightarrow{P}^s_2=-\frac{1}{2}\frac{\overrightarrow{\partial}^2}{\partial x^2_{\bar\mu}\partial x^2_{\bar\mu}}-\frac{2}{r_2},$$
$$\overleftarrow{P}^s_1=\frac{1}{2}\frac{\overleftarrow{\partial}^2}{\partial x^1_{\bar\mu}\partial x^1_{\bar\mu}}+\frac{2}{r_1},\ \overleftarrow{P}^s_2=\frac{1}{2}\frac{\overleftarrow{\partial}^2}{\partial x^2_{\bar\mu}\partial x^2_{\bar\mu}}+\frac{2}{r_2} \qquad (5.75)$$

which lead to $\overrightarrow{P}^s_1 h_n(\mathcal{R})=(E^h_n)'h_n(\mathcal{R})$ and $h_n(\mathcal{R})^*\overleftarrow{P}^s_1=-(E^h_n)'h(\mathcal{R})^*$. The eqs.(5.40) read now

$$Q^s_1=-\frac{é^2}{á}((\overleftarrow{P}^s_1-\frac{1}{r_{12}})+(\overrightarrow{P}^s_1+\frac{1}{r_{12}})+\frac{\partial}{i\partial t'}),$$
$$Q^s_2=-\frac{é^2}{á}((\overleftarrow{P}^s_2-\frac{1}{r_{12}})+(\overrightarrow{P}^s_2+\frac{1}{r_{21}})+\frac{\partial}{i\partial t'}) \qquad (5.76)$$

and the eqs.(5.42) obtain the form

$$\overrightarrow{Q}^s_{12}=\frac{é^2}{á}(\overrightarrow{P}^s_1+\overrightarrow{P}^s_2+\frac{1}{r_{12}}),$$
$$\overleftarrow{Q}^s_{12}=\frac{é^2}{á}(\overleftarrow{P}^s_1+\overleftarrow{P}^s_2-\frac{1}{r_{12}}). \qquad (5.77)$$

The eigenfunctions and the eigenvalues are

$$\begin{aligned}\psi^{\pm}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2) &= \tfrac{1}{\sqrt{2}}(h_n(\mathcal{R}_1)\grave{k}_m(\mathcal{R}_2) \pm \grave{k}_m(\mathcal{R}_1)h_n(\mathcal{R}_2)),\\ \psi^{\pm}_{n'\,m}(\mathcal{R}_1,\mathcal{R}_2)^* &= \tfrac{1}{\sqrt{2}}(h_{n'}(\mathcal{R}_1)^*\grave{k}_m(\mathcal{R}_2) \pm \grave{k}_m(\mathcal{R}_1)^*h_{n'}(\mathcal{R}_2))^*\end{aligned} \tag{5.78}$$

and

$$\begin{aligned}(E^{\pm}_{n\,m})' &= \ (E^h_n)' + (E^k_m)' + \ {}_4({}^{hh}_{nn}M^{kk}_{mm})_4 \pm \ {}_4({}^{hk}_{nm}M^{kh}_{mn})_4\\ (E^{\pm}_{n'\,m})' &= \ (E^h_{n'})' + (E^k_m)' + \ {}_4({}^{hh}_{n'n'}M^{kk}_{mm})_4 \pm \ {}_4({}^{hk}_{n'm}M^{kh}_{mn'})_4.\end{aligned} \tag{5.79}$$

With these operators the relations from eq.(5.46) to eq.(5.52) can be approximated as follows:

$$\begin{aligned}&{}^{l1'\,l1}_{m\,m}C_2(t')\tfrac{\partial}{\partial x^1_\mu}T^{d1}_{4\,\mu}(\mathcal{R}_1,t') + \ {}^{l'\,l}_{n'\,n}C_1(t')\tfrac{\partial}{\partial x^1_\mu}T^{d2}_{4\,\mu}(\mathcal{R}_2,t')\\ &= -\tfrac{\acute{a}}{\acute{\alpha}}(\,{}^{l1'\,l1}_{m\,m}C_2(t')\textstyle\int h_{n'}(\mathcal{R}_1)^*\grave{k}_m(\mathcal{R}_2)^*Q^s_1\,{}^{l'\,l}_{n'\,n}C_1(t')h_n(\mathcal{R}_1)\grave{k}_m(\mathcal{R}_2)dV_2\\ &\quad+{}^{l'\,l}_{n'\,n}C_1(t')\textstyle\int h_{n'}(\mathcal{R}_1)^*\grave{k}_m(\mathcal{R}_2)^*Q^s_2\,{}^{l1'\,l1}_{m\,m}C_2(t')h_n(\mathcal{R}_1)\grave{k}_m(\mathcal{R}_2)dV_1)\end{aligned} \tag{5.80}$$

or

$$\begin{aligned}&{}^{l1'\,l1}_{m\,m}C_2(t')\tfrac{\partial}{\partial x^1_\mu}T^{d1}_{4\,\mu}(\mathcal{R}_1,t') + \ {}^{l'\,l}_{n'\,n}C_1(t')\tfrac{\partial}{\partial x^2_\mu}T^{d2}_{4\,\mu}(\mathcal{R}_2,t') =\\ &-\tfrac{\acute{a}}{2\acute{\alpha}}(\,{}^{l1'\,l1}_{m\,m}C_2(t')\textstyle\int(\psi^{+}_{n'\,m}(\mathcal{R}_1,\mathcal{R}_2)^*(\overleftarrow{Q}^s_{1\,2} + \overrightarrow{Q}^s_{1\,2} + 2\tfrac{\partial}{i\partial t'})\,{}^{l'\,l}_{n'\,n}C_1(t')\,\psi^{+}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2))dV_2\\ &+{}^{l1'\,l1}_{m\,m}C_2(t')\textstyle\int(\psi^{-}_{n'\,m}(\mathcal{R}_1,\mathcal{R}_2)(\overleftarrow{Q}^s_{1\,2} + \overrightarrow{Q}^s_{1\,2} + 2\tfrac{\partial}{i\partial t'})\,{}^{l'\,l}_{n'\,n}C_1(t')\,\psi^{-}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2))dV_2\\ &+{}^{l'\,l}_{n'\,n}C_1(t')\textstyle\int(\psi^{+}_{n'\,m}(\mathcal{R}_1,\mathcal{R}_2)^*(\overleftarrow{Q}^s_{1\,2} + \overrightarrow{Q}^s_{1\,2} + 2\tfrac{\partial}{i\partial t'})\,{}^{l1'\,l1}_{m\,m}C_2(t')\,\psi^{+}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2))dV_1\\ &+{}^{l'\,l}_{n'\,n}C_1(t')\textstyle\int(\psi^{-}_{n'\,m}(\mathcal{R}_1,\mathcal{R}_2)^*(\overleftarrow{Q}^s_{1\,2} + \overrightarrow{Q}^s_{1\,2} + 2\tfrac{\partial}{i\partial t'})\,{}^{l1'\,l1}_{m\,m}C_2(t')\,\psi^{-}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2))dV_1)\\ &+\tfrac{\acute{a}}{2\acute{\alpha}}(\Delta O^{s+}_a(\mathcal{R}_1,t') + \Delta O^{s+}_b(\mathcal{R}_2,t') + \Delta O^{s-}_a(\mathcal{R}_1,t') + \Delta O^{s-}_b(\mathcal{R}_2,t'))\end{aligned} \tag{5.81}$$

with

$$\begin{aligned}
\Delta O_a^{s\pm}(\mathcal{R}_1,t) = -({}^{l1'\,l1}_{m\,m}C_2(t) \textstyle\int (\psi^{\pm}_{n'\,m}(\mathcal{R}_1,\mathcal{R}_2)^*(\overleftarrow{Q}^4_{1\,2} + \overrightarrow{Q}^4_{1\,2} \\
+2\tfrac{\hbar\partial}{i\partial t})\, {}^{l'\,l}_{n'\,n}C_1(t)\, \psi^{\mp}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2))dV_2 \\
= {}^{l'\,l}_{n'\,n}C_1(t)\, {}^{l1'\,l1}_{m\,m}C_2(t) \textstyle\int \psi^{\pm}_{n'\,m}(\mathcal{R}_1,\mathcal{R}_2)^*(E^{\pm}_{n'\,m} - E^{\mp}_{n\,m} \\
-\hbar(l'-l)\omega)\, \psi^{\mp}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2)dV_2 \\
\\
\Delta O_b^{s\pm}(\mathcal{R}_2,t) = -({}^{l1'\,l1}_{m\,m}C_1(t) \textstyle\int (\psi^{\pm}_{n'\,m}(\mathcal{R}_1,\mathcal{R}_2)^*(\overleftarrow{Q}^s_{1\,2} + \overrightarrow{Q}^s_{1\,2} \\
+2\tfrac{\hbar\partial}{i\partial t})\, {}^{l'\,l}_{n'\,n}C_2(t)\, \psi^{\mp}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2))dV_1 \\
= {}^{l'\,l}_{n'\,n}C_1(t)\, {}^{l1'\,l1}_{m\,m}C_2(t) \textstyle\int \psi^{\pm}_{n'\,m}(\mathcal{R}_1,\mathcal{R}'_2)^*(E^{\pm}_{n'\,m} - E^{\mp}_{n\,m} \\
-\hbar(l1'-l1)\omega)\, \psi^{\mp}_{n\,m}(\mathcal{R}_1,\mathcal{R}'_2)dV_1.
\end{aligned} \tag{5.82}$$

Therefore is

$$\begin{aligned}
{}^{l1'\,l1}_{m\,m}C_2(t)\tfrac{\partial}{\partial x^1_\mu}T^{d1}_{4\,\mu}(\mathcal{R}_1,t) + {}^{l'\,l}_{n'\,n}C_1(t)\tfrac{\partial}{\partial x^2_\mu}T^{d2}_{4\,\mu}(\mathcal{R}_2,t) = \\
\tfrac{\acute{a}}{4\acute{\alpha}}\, {}^{l'\,l}_{n'\,n}C_1(t)\, {}^{l1'\,l1}_{m\,m}C_2(t)((E^{+}_{n'\,m} - E^{+}_{n\,m} - 2\hbar(l'-l)\omega)\, h_{n'}(\mathcal{R}_1)^*h_n(\mathcal{R}_1) \\
+(E^{-}_{n'\,m} - E^{-}_{n\,m} - 2\hbar(l'-l)\omega)\, h_{n'}(\mathcal{R}_1)^*h_n(\mathcal{R}_1) \\
+(E^{+}_{n'\,m} - E^{+}_{n\,m} - 2\hbar(l1'-l1)\omega)\, h_{n'}(\mathcal{R}_2)^*h_n(\mathcal{R}_2) \\
+(E^{-}_{n'\,m} - E^{-}_{n\,m} - 2\hbar(l1'-l1)\omega)\, h_{n'}(\mathcal{R}_2)^*h_n(\mathcal{R}_2)) \;+\; \tfrac{\acute{a}}{2\acute{\alpha}}\Delta O.
\end{aligned} \tag{5.83}$$

and

$$
\begin{aligned}
\Delta O &= \Delta O_a^{s+}(\mathcal{R}_1,t) + \Delta O_b^{s+}(\mathcal{R}_2,t) + \Delta O_a^{s-}(\mathcal{R}_1,t) + \Delta O_b^{s-}(\mathcal{R}_2,t) \\
&= \tfrac{1}{2} {}^{l'\,l}_{n'\,n}C_1(t)\, {}^{l1'\,l1}_{m\,m}C_2(t)((E^+_{n'\,m} - E^-_{n\,m} - 2\hbar(l'-l)\omega)\, h_{n'}(\mathcal{R}_1)^* h_n(\mathcal{R}_1) \\
&\quad +(E^-_{n'\,m} - E^+_{n\,m} - 2\hbar(l'-l)\omega)\, h_{n'}(\mathcal{R}_1)^* h_n(\mathcal{R}_1) \\
&\quad -(E^+_{n'\,m} - E^+_{n\,m} - 2\hbar(l1'-l1)\omega)\, h_{n'}(\mathcal{R}_2)^* h_n(\mathcal{R}_2) \\
&\quad -(E^-_{n'\,m} - E^-_{n\,m} - 2\hbar(l1'-l1)\omega)\, h_{n'}(\mathcal{R}_2)^* h_n(\mathcal{R}_2)).
\end{aligned} \tag{5.84}
$$

If you consider the coordinates $\mathcal{R}_1 = \mathcal{R}$ and $\mathcal{R}_2 = \mathcal{R}$ in the eqs.(5.83/5.84) as coordinates of the real space, you obtain again for $n' \neq n$ and $l' - l = l1' - l1$ the relation

$$
\begin{aligned}
& {}^{l1'\,l1}_{m\,m}C_2(t)\tfrac{\partial}{\partial x^1_\mu} T^{d1}_{4\,\mu}(\mathcal{R},t) + {}^{l'\,l}_{n'\,n}C_1(t)\tfrac{\partial}{\partial x^2_\mu} T^{d2}_{4\,\mu}(\mathcal{R},t) = \\
& \tfrac{\acute{a}}{2\grave{a}}\, {}^{l'\,l}_{n'\,n}C_1(t)\, {}^{l1'\,l1}_{m\,m}C_2(t)(E^+_{n'\,m} - E^+_{n\,m} - 2\hbar(l'-l)\omega)\, h_{n'}(\mathcal{R})^* h_n(\mathcal{R}) \\
& +2\tfrac{\acute{a}}{\grave{a}}(E^-_{n'\,m} - E^-_{n\,m} - 2\hbar(l'-l)\omega)\, h_{n'}(\mathcal{R})^* h_n(\mathcal{R})
\end{aligned} \tag{5.85}
$$

due to $\Delta O_a^{s\pm}(\mathcal{R},t') + \Delta O_b^{s\pm}(\mathcal{R},t') = 0$. The current components, be formed from the solutions of the Schrödinger equation, are given in eq.(5.60).

5.4 *Properties of the solutions*

In this section we apply the Schrödinger approximation and the dimension units introduced in section 3.2. The considered dynamic solutions are

$$
\begin{aligned}
{}^0u(\mathcal{R}_1,t') &= \textstyle\sum_n {}^0a_n(t')^0h_n(\mathcal{R}_1) \\
{}^1u(\mathcal{R}_1,t') &= \textstyle\sum_n {}^1a_n(t')^1h_n(\mathcal{R}_1) \\
v(\mathcal{R}_2,t') &= \textstyle\sum_m b_m(t')\grave{k}_m(\mathcal{R}_2)
\end{aligned} \tag{5.86}
$$

and

$$
\begin{aligned}
{}^{0}u(\mathcal{R}_1,t')^* &= \textstyle\sum_p\ {}^{0}a_p(t')^*\,{}^{0}h_p(\mathcal{R}_1)^*,\\
{}^{1}u(\mathcal{R}_1,t')^* &= \textstyle\sum_p\ {}^{1}a_p(t')^*\,{}^{1}h_p(\mathcal{R}_1)^*\\
v(\mathcal{R}_2,t')^* &= \textstyle\sum_q\ b_q(t')^*\,\grave{k}_q(\mathcal{R}_2)^*.
\end{aligned}
\tag{5.87}
$$

These form the charges

$$
\begin{aligned}
\rho_1 = \acute{e}({}^{0}u(\mathcal{R}_1,t')^* +^{1} u(\mathcal{R}_1,t')^*)({}^{0}u(\mathcal{R}_1,t') +^{1} u(\mathcal{R}_1,t')),\\
\rho_2 = \acute{e}(v(\mathcal{R}_2,t')^*\,v(\mathcal{R}_2,t')
\end{aligned}
\tag{5.88}
$$

and the associated electrostatic vector-potentials

$$
\begin{aligned}
A_4^1(\mathcal{R},t') = i\acute{e}\textstyle\int \frac{1}{r_{0\,1}}\rho_1(\mathcal{R}_1,t')dV_1,\\
A_4^2(\mathcal{R},t') = i\acute{e}\textstyle\int \frac{1}{r_{0\,2}}\rho_2(\mathcal{R}_2,t')dV_2.
\end{aligned}
\tag{5.89}
$$

In addition, it reads $-i\acute{e}A_4^1(\mathcal{R}_1,t')^0u(\mathcal{R}_1,t') = \sum_n({}^{0h}_{n}M^1(t'))\,{}^{0}h_n(\mathcal{R}_1)$, $-i\acute{e}A_4^2(\mathcal{R}_1,t')^0u(\mathcal{R}_1,t') = \sum_n({}^{0h}_{n}M^2(t'))\,{}^{0}h_n(\mathcal{R}_1)$, etc. Therefore the functions should satisfy the following Schrödinger equations (see eqs.(5.73/5.74)):

$$i\frac{\acute{e}^2}{\acute{a}}\frac{\partial}{\partial t'}\,{}^0u(\mathcal{R}_1,t') = \frac{\acute{e}^2}{\acute{a}}\overrightarrow{P}^s_1\,{}^0u(\mathcal{R}_1,t') + \sum_n ({}^{0h}_{\ n}M^1(t') + {}^{0h}_{\ n}M^2(t'))^0h_n(\mathcal{R}_1),$$
$$i\frac{\acute{e}^2}{\acute{a}}\frac{\partial}{\partial t'}\,{}^1u(\mathcal{R}_1,t') = \frac{\acute{e}^2}{\acute{a}}\overrightarrow{P}^s_1\,{}^1u(\mathcal{R}_1,t') + \sum_n ({}^{1h}_{\ n}M^1(t') + {}^{1h}_{\ n}M^2(t'))^1h_{n'}(\mathcal{R}_1),$$

$$\begin{aligned}
-i\frac{\acute{e}^2}{\acute{a}}\frac{\partial}{\partial t'}\,{}^0u(\mathcal{R}_1,t')^* &= -\frac{\acute{e}^2}{\acute{a}}\,{}^0u(\mathcal{R}_1,t')^*\overleftarrow{P}^s_1 \\
&+ \sum_p ({}^{0h}_{\ p}M^1(t')^* + {}^{0h}_{\ p}M^2(t')^*)\,{}^0h_p(\mathcal{R}_1)^*, \\
-i\frac{\acute{e}^2}{\acute{a}}\frac{\partial}{\partial t'}\,{}^1u(\mathcal{R}_1,t')^* &= -\frac{\acute{e}^2}{\acute{a}}\,{}^1u(\mathcal{R}_1,t')^*\overleftarrow{P}^s_1 \\
&+ \sum_p ({}^{1h}_{\ p}M^1(t')^* + {}^{1h}_{\ p}M^2(t')^*)\,{}^1h_p(\mathcal{R}_1)^*,
\end{aligned} \tag{5.90}$$

$$i\frac{\acute{e}^2}{\acute{a}}\frac{\partial}{\partial t'}\,v(\mathcal{R}_2,t') = \frac{\acute{e}^2}{\acute{a}}\overrightarrow{P}^s_2\,v(\mathcal{R}_2,t') + \sum_m ({}^{k}_{m}M^1(t') + {}^{k}_{m}M^2(t'))\acute{k}_m(\mathcal{R}_2),$$

$$\begin{aligned}
-i\frac{\acute{e}^2}{\acute{a}}\frac{\partial}{\partial t'}\,v(\mathcal{R}_2,t')^* &= -\frac{\acute{e}^2}{\acute{a}}\,v(\mathcal{R}_2,t')^*\overleftarrow{P}^s_2 \\
&+ \sum_q ({}^{k}_{q}M^1(t')^* + {}^{k}_{q}M^2(t')^*)\,\acute{k}_q(\mathcal{R}_2)^*.
\end{aligned}$$

${}^0h_n$, ${}^1h_n$, ${}^0h_p$, ${}^1h_p$, $\grave{k}_m$ and $\grave{k}_q$ are orthogonal eigenfunctions. This system is useful to describe a transition state.

The coefficients ${}^0a_n(t')$, ${}^1a_{n'}$, $b_m(t')$, etc. follow from the eqs.(5.90). It is

$$\begin{aligned}
i\frac{\acute{e}^2}{\acute{a}}\frac{d^0a_n(t')}{dt'} &= \frac{\acute{e}^2}{\acute{a}}\,({}^0E^h_n)'\,{}^0a_n(t') + {}^{0h}_{\ n}M^1(t') + {}^{0h}_{\ n}M^2(t'), \\
i\frac{\acute{e}^2}{\acute{a}}\frac{d^1a_n(t')}{dt'} &= \frac{\acute{e}^2}{\acute{a}}\,({}^1E^h_n)'\,{}^1a_n(t') + {}^{1h}_{\ n}M^1(t')) + {}^{1h}_{\ n}M^2(t'), \\
i\frac{\acute{e}^2}{\acute{a}}\frac{db_m(t')}{dt'} &= \frac{\acute{e}^2}{\acute{a}}\,(E^k_m)'\,b_m(t') + {}^{k}_{m}M^1(t')) + ({}^{k}_{m}M^2(t')
\end{aligned} \tag{5.91}$$

and

$$\begin{aligned}
\frac{i\acute{e}^2}{\acute{a}}\frac{d^0a_p(t')^*}{dt'} &= -\frac{\acute{e}^2}{\acute{a}}\,({}^0E^h_p)'\,{}^0a_n(t')^* - {}^{0h}_{\ p}M^1(t')^* - {}^{0h}_{\ p}M^2(t')^*, \\
i\frac{\acute{e}^2}{\acute{a}}\frac{d^1a_p(t')^*}{dt'} &= -\frac{\acute{e}^2}{\acute{a}}\,({}^1E^h_p)'\,{}^1a_p(t')^* - {}^{1h}_{\ p}M^1(t')^* - {}^{1h}_{\ n'}M^2(t')^*, \\
i\frac{\acute{e}^2}{\acute{a}}\frac{db_q(t')^*}{dt'} &= -\frac{\acute{e}^2}{\acute{a}}\,(E^k_q)'\,b_q(t')^* - {}^{k}_{q}M^1(t')^* - {}^{k}_{q}M^2(t')^*.
\end{aligned} \tag{5.92}$$

We insert into the eqs.(5.90-5.92) the following Fourier transformed expressions

$$\begin{array}{c} {}^0a_n(t') = \sum_l {}^0c_n^l \exp(-il\,\omega'\,(t'-t_0')), \\ {}^1a_n(t') = \sum_l {}^1c_n^l \exp(-il\,\omega'\,(t'-t_0')), \\ b_m(t') = \sum_{l1} d_m^{l1} \exp(-il1\,\omega'\,(t'-t_0')) \\ {}^0a_p(t')^* = \sum_{l'} ({}^0c_p^{l'})^* \exp(il'\,\omega'\,(t'-t_0')), \\ {}^1a_p(t')^* = \sum_{l'} ({}^1c_p^{l'})^* \exp(il'\,\omega'\,(t'-t_0')), \\ b_q(t')^* = \sum_{l1'} (d_q^{l1'})^* \exp(il1'\,\omega'\,(t'-t_0')) \end{array} \tag{5.93}$$

respectively

$$\begin{array}{c} {}^{0h}_{n}M^1(t') = \frac{é^2}{á} \sum_l {}^{0h}ma_n^l \exp(-il\,\omega'\,(t'-t_0')), \\ {}^{1h}_{n}M^1(t') = \frac{é^2}{á} \sum_l {}^{1h}ma_n^l \exp(-il\,\omega'\,(t'-t_0')), \\ {}^{0h}_{n}M^2(t') = \frac{é^2}{á} \sum_{l1} {}^{0h}mb_n^{l1} \exp(-il1\,\omega'\,(t'-t_0')), \\ {}^{1h}_{n}M^2(t') = \frac{é^2}{á} \sum_l {}^{1h}mb_n^l \exp(-il\,\omega'\,(t'-t_0')), \\ {}^{k}_{m}M^1(t') = \frac{é^2}{á} \sum_l {}^{k}ma_m^l \exp(-il\,\omega'\,(t'-t_0')), \\ {}^{k}_{m}M^2(t') = \frac{é^2}{á} \sum_{l1} {}^{k}mb_m^{l1} \exp(-il1\,\omega'\,(t'-t_0')) \end{array} \tag{5.94}$$

and

$$\begin{array}{c} {}^{0h}_{p}M^1(t')^* = \frac{é^2}{á} \sum_{l'} ({}^{0h}ma_p^{l'})^* \exp(il'\,\omega'\,(t'-t_0')), \\ {}^{1h}_{p}M^1(t')^* = \frac{é^2}{á} \sum_{l'} ({}^{1h}ma_p^{l'})^* \exp(il'\,\omega'\,(t'-t_0')), \\ {}^{0h}_{p}M^2(t')^* = \frac{é^2}{á} \sum_{l1'} ({}^{0h}mb_p^{l1'})^* \exp(il1'\,\omega'\,(t'-t_0')), \\ {}^{1h}_{p}M^2(t')^* = \frac{é^2}{á} \sum_{l'} ({}^{1h}mb_p^{l'})^* \exp(il\,\omega'\,(t'-t_0')), \\ {}^{k}_{q}M^1(t')^* = \frac{é^2}{á} \sum_{l'} ({}^{k}ma_q^{l'})^* \exp(il'\,\omega'\,(t'-t_0')), \\ {}^{k}_{q}M^2(t')^* = \frac{é^2}{á} \sum_{l1'} ({}^{k}mb_q^{l1'})^* \exp(il1'\,\omega'\,(t'-t_0')). \end{array} \tag{5.95}$$

Here is

$$
\begin{aligned}
{}^{0h}ma_n^l = -i\frac{\acute{a}}{\acute{e}} \int (A_4^1(\mathcal{R}_1, t') \textstyle\sum_{j,l1} \exp(i(l-l1)\omega'(t'-t_0'))\\
\times {}^0h_n(\mathcal{R}_1)^* ({}^0c_j^{l1}\, {}^0h_j(\mathcal{R}_1) + {}^1c_j^{l1}\, {}^1h_j(\mathcal{R}_1)) dV_1,\\
{}^{1h}ma_n^l = -i\frac{\acute{a}}{\acute{e}} \int (A_4^1(\mathcal{R}_1, t') \textstyle\sum_{j,l1} \exp(i(l-l1)\omega'(t'-t_0'))\\
\times {}^1h_n(\mathcal{R}_1)^* ({}^0c_j^{l1}\, {}^0h_j(\mathcal{R}_1) + {}^1c_j^{l1}\, {}^1h_j(\mathcal{R}_1)) dV_1,\\
{}^{0h}mb_n^l = -i\frac{\acute{a}}{\acute{e}} \int (A_4^2(\mathcal{R}_1, t') \textstyle\sum_{j,l1} \exp(i(l-l1)\omega'(t'-t_0'))\\
\times {}^0h_n(\mathcal{R}_1)^* ({}^0c_j^{l1}\, {}^0h_j(\mathcal{R}_1) + {}^1c_j^{l1}\, {}^1h_j(\mathcal{R}_1)) dV_1,\\
{}^{1h}mb_n^l = -i\frac{\acute{a}}{\acute{e}} \int (A_4^2(\mathcal{R}_1, t') \textstyle\sum_{j,l1} \exp(i(l-l1)\omega'(t'-t_0'))\\
\times {}^1h_n(\mathcal{R}_1)^* ({}^0c_j^{l1}\, {}^0h_j(\mathcal{R}_1) + {}^1c_j^{l1}\, {}^1h_j(\mathcal{R}_1)) dV_1,\\
{}^kma_m^l = -i\frac{\acute{a}}{\acute{e}} \int (A_4^1(\mathcal{R}_1, t') \textstyle\sum_{j,l1}\ {}^0d_j^{l1}\\
\times \exp(i(l-l1)\omega'(t'-t_0'))\, \acute{k}_m(\mathcal{R}_1)^*\, \acute{k}_j(\mathcal{R}_1) dV_1,\\
{}^kmb_m^l(t')) = -i\frac{\acute{a}}{\acute{e}} \int (A_4^2(\mathcal{R}_1, t') \textstyle\sum_{j,l1}\ {}^1d_j^{l1}\\
\times \exp(i(l-l1)\omega'(t'-t_0'))\, \acute{k}_m(\mathcal{R}_1)^*\, \acute{k}_j(\mathcal{R}_1) dV_1
\end{aligned}
\qquad (5.96)
$$

and

$$
\begin{aligned}
({}^{0h}ma_p^{l'})^* &= -i\tfrac{\acute{a}}{\acute{e}} \int (A_4^1(\mathcal{R}_1,t') \textstyle\sum_{j',l1'} \exp(-i(l'-l1')\omega'(t'-t_0')) \\
&\quad \times {}^0h_p(\mathcal{R}_1)({}^0c_{j'}^{l1'}\,{}^0h_{j'}(\mathcal{R}_1) + {}^1c_{j'}^{l1'}\,{}^1h_{j'}(\mathcal{R}_1))^* dV_1, \\
({}^{1h}ma_{j'}^{l'})^* &= -i\tfrac{\acute{a}}{\acute{e}} \int (A_4^1(\mathcal{R}_1,t') \textstyle\sum_{j',l1'} \exp(-i(l'-l1')\omega'(t'-t_0')) \\
&\quad \times {}^1h_p(\mathcal{R}_1)({}^0c_{j'}^{l1'}\,{}^0h_{j'}(\mathcal{R}_1) + {}^1c_{j'}^{l1'}\,{}^1h_{j'}(\mathcal{R}_1))^* dV_1, \\
({}^{0h}mb_p^{l'})^* &= -i\tfrac{\acute{a}}{\acute{e}} \int (A_4^2(\mathcal{R}_1,t') \textstyle\sum_{j',l1'} \exp(-i(l'-l1')\omega'(t'-t_0')) \\
&\quad \times {}^0h_p(\mathcal{R}_1)({}^0c_{j'}^{l1'}\,{}^0h_{j'}(\mathcal{R}_1) + {}^1c_{j'}^{l1'}\,{}^1h_{j'}(\mathcal{R}_1))^* dV_1, \\
({}^{1h}mb_p^{l'})^* &= -i\tfrac{\acute{a}}{\acute{e}} \int (A_4^2(\mathcal{R}_1,t') \textstyle\sum_{j',l1'} \exp(-i(l'-l1')\omega'(t'-t_0')) \\
&\quad \times {}^1h_p(\mathcal{R}_1)({}^0c_{j'}^{l1'}\,{}^0h_{j'}(\mathcal{R}_1) + {}^1c_{j'}^{l1'}\,{}^1h_{j'}(\mathcal{R}_1))^* dV_1, \\
({}^kma_q^{l'})^* &= -i\tfrac{\acute{a}}{\acute{e}} \int (A_4^1(\mathcal{R}_1,t') \textstyle\sum_{j',l1'} \exp(-i(l'-l1')\omega'(t'-t_0')) \\
&\quad \times {}^0d_{j'}^{l1}\,\acute{k}_q(\mathcal{R}_1)^*\,\acute{k}_{j'}(\mathcal{R}_1) dV_1, \\
({}^kmb_q^{l'})^* &= -i\tfrac{\acute{a}}{\acute{e}} \int (A_4^2(\mathcal{R}_1,t') \textstyle\sum_{j',l1'} \exp(-i(l'-l1')\omega'(t'-t_0')) \\
&\quad \times {}^1d_{j'}^{l1'}\,\acute{k}_q(\mathcal{R}_1)^*\,\acute{k}_{j'}(\mathcal{R}_1) dV_1.
\end{aligned}
\tag{5.97}
$$

Now we consider the share $\acute{e}^2\,({}^1u(\mathcal{R}_1)^* v(\mathcal{R}_2)^*\,{}^0u(\mathcal{R}_1)v(\mathcal{R}_2))$ of the total charge product. This and the conjugated complex expression can form radiation moments. The dynamics of this product is determined by

$$
\begin{aligned}
& i\tfrac{\acute{e}^2}{\acute{a}}\tfrac{\partial}{\partial t'}({}^1u(\mathcal{R}_1,t')^* v(\mathcal{R}_2,t')^*\,{}^0u(\mathcal{R}_1,t')v(\mathcal{R}_2,t')) \\
&= i\tfrac{\acute{e}^2}{\acute{a}}(\tfrac{\partial}{\partial t'}[({}^1u(\mathcal{R}_1,t')^*\bar{v}(\mathcal{R}_2,t')^*]\,{}^0u(\mathcal{R}_1,t')v(\mathcal{R}_2,t') \\
&\quad + {}^1u(\mathcal{R}_1,t')^* v(\mathcal{R}_2,t')^* \tfrac{\partial}{\partial t'}\,[{}^0u(\mathcal{R}_1,t')v(\mathcal{R}_2,t')]).
\end{aligned}
\tag{5.98}
$$

The expression

$$
{}^{10}O^* = i\tfrac{\acute{e}^2}{\acute{a}}\tfrac{\partial}{\partial t'}(\,{}^1u(\mathcal{R}_1,t')^*\,v(\mathcal{R}_2,t')^*) \tag{5.99}
$$

leads to (see eq.(5.86/5.87/5.93)

$$ {}^{10}O^{*} = -\frac{\acute{e}^{2}}{\acute{a}} \sum_{p,l',q,l1'} (l'+l1')\,\omega' \,\exp(i(l'+l1')\,\omega'\,(t'-t'_{0})) \\ \times({}^{1}c_{p}^{l'})^{*}\,(d_{q}^{l1'})^{*}({}^{1}h_{p}(\mathcal{R}_{1})^{*})\acute{k}_{q}(\mathcal{R}_{2})^{*}. \tag{5.100} $$

This can be replaced by

$$ {}^{10}O^{*} = (\,-\sum_{p}\,(\frac{\acute{e}^{2}}{\acute{a}}({}^{1}a_{p}(t'))^{*}\,{}^{1}h_{p}(\mathcal{R}_{1})^{*}\overleftarrow{P}_{1}^{s} \\ +\,({}_{p}^{1h}M^{1}(t')^{*} + {}_{p}^{1h}M^{2}(t')^{*})^{1}h_{p}(\mathcal{R}_{1})^{*})v(\mathcal{R}_{2})^{*} \\ +(\,-\sum_{q}\,(\frac{\acute{e}^{2}}{\acute{a}}(b_{q}(t'))^{*}\,\acute{k}_{q}(\mathcal{R}_{2})^{*}\overleftarrow{P}_{2}^{s} \\ +\,({}_{q}^{k}M^{1}(t')^{*} + {}_{q}^{k}M^{2}(t')^{*})\acute{k}_{q}(\mathcal{R}_{2})^{*})u(\mathcal{R}_{1})^{*} \tag{5.101} $$

and, due to the eqs.(5.93-5.95), becomes

$$ {}^{10}O^{*} = -\frac{\acute{e}^{2}}{\acute{a}}(\,\sum_{p,l'}\,\exp(il'\,\omega'\,(t'-t'_{0}))(({}^{1}c_{p}^{l'})^{*}\,({}^{1}E_{p}^{h})' \\ +\,({}^{1h}ma_{p}^{l'})^{*} +\, ({}^{1h}mb_{p}^{l'})^{*})^{1}h_{p}(\mathcal{R}_{1})^{*})v(\mathcal{R}_{2})^{*} \\ +\,\sum_{q,l1'}\,\exp(il1'\,\omega'\,(t'-t'_{0}))(\,(d_{q}^{l1'})^{*}\,(E_{q}^{k})' \\ +\,({}^{k}ma_{q}^{l1'})^{*} +\, ({}^{k}mb_{q}^{l1'})^{*})\acute{k}_{q}(\mathcal{R}_{2})^{*})^{1}u(\mathcal{R}_{1},t')^{*}) \tag{5.102} $$

and with eqs.(5.86/5.90)

$$ {}^{10}O^{*} = -\frac{\acute{e}^{2}}{\acute{a}}(\,\sum_{p,l',q,l1'}\,\exp(i(l'+l1')\,\omega'\,(t'-t'_{0}))({}^{1}c_{p}^{l'})^{*}\,(d_{q}^{l1'})^{*}(({}^{1}E_{p}^{h})' + (E_{q}^{k})' \\ +\,({}^{1h}ma_{p}^{l'})^{*} +\, ({}^{1h}mb_{p}^{l'})^{*} +\, ({}^{k}ma_{q}^{l1'})^{*} \\ +\,({}^{k}mb_{q}^{l1'})^{*})^{1}h_{p}(\mathcal{R}_{1})^{*})\acute{k}_{q}(\mathcal{R}_{2})^{*}))\,. \tag{5.103} $$

After multiplication with ${}^{o}u(\mathcal{R}_1,t')v(\mathcal{R}_2,t')$ we get

$$
\begin{gathered}
{}^{10}O^{*}({}^{o}u(\mathcal{R}_1,t')v(\mathcal{R}_2,t')) = -\tfrac{\acute{e}^2}{\acute{a}}(\textstyle\sum_{p,l',q,l1',n,l,m,l1} \\
\times \exp(i(l'+l1'-l-l1)\,\omega'\,(t'-t'_0))({}^{1}c_p^{l'})^{*}\,(d_q^{l1'})^{*}({}^{0}c_n^{l})\,(d_m^{l1}) \\
\times(({}^{1}E_p^{h})' + (E_q^{k})' + \,({}^{1h}ma_p^{l'})^{*} + \,({}^{1h}mb_p^{l'})^{*} \\
+\,({}^{k}ma_q^{l1'})^{*} + \,({}^{k}mb_q^{l1'})^{*})^{1}h_p(\mathcal{R}_1)^{*})\acute{k}_q(\mathcal{R}_2)^{*})^{0}h_n(\mathcal{R}_1)\acute{k}_m(\mathcal{R}_2))
\end{gathered}
\tag{5.104}
$$

or using eq.(5.100)

$$
\begin{gathered}
{}^{10}O^{*}({}^{o}u(\mathcal{R}_1,t')v(\mathcal{R}_2,t')) = -\tfrac{\acute{e}^2}{\acute{a}}(\textstyle\sum_{p,l',q,l1',n,l,m,l1} \\
\times \exp(i(l'+l1'-l-l1)\,\omega'\,(t'-t'_0))({}^{1}c_p^{l'})^{*}\,(d_q^{l1'})^{*}({}^{0}c_n^{l})\,(d_m^{l1}) \\
\times(\,(l'+l1')\,\omega'\,)^{1}h_p(\mathcal{R}_1)^{*})\acute{k}_q(\mathcal{R}_2)^{*})^{0}h_n(\mathcal{R}_1)\acute{k}_m(\mathcal{R}_2))\,.
\end{gathered}
\tag{5.105}
$$

The comparison of the eq.(5.104) and (5.105) results in

$$
\begin{gathered}
-\tfrac{\acute{e}^2}{\acute{a}}(\textstyle\sum_{p,l',q,l1',n,l,m,l1} \exp(i(l'+l1'-l-l1)\,\omega'\,(t'-t'_0)) \\
\times({}^{1}c_p^{l'})^{*}\,(d_q^{l1'})^{*}({}^{0}c_n^{l})\,(d_m^{l1})(\,(l'+l1')\,\omega'\,)^{1}h_p(\mathcal{R}_1)^{*})\acute{k}_q(\mathcal{R}_2)^{*})^{0}h_n(\mathcal{R}_1)\acute{k}_m(\mathcal{R}_2)) \\
= -\tfrac{\acute{e}^2}{\acute{a}}(\textstyle\sum_{p,l',q,l1',n,l,m,l1} \exp(i(l'+l1'-l-l1)\,\omega'\,(t'-t'_0)) \\
\times({}^{1}c_p^{l'})^{*}\,(d_q^{l1'})^{*}({}^{0}c_n^{l})\,(d_m^{l1})(({}^{1}E_p^{h})' + (E_q^{k})' + \,({}^{1h}ma_p^{l'})^{*} + \,({}^{1h}mb_p^{l'})^{*} \\
+\,({}^{k}ma_q^{l1'})^{*} + \,({}^{k}mb_q^{l1'})^{*})^{1}h_p(\mathcal{R}_1)^{*})\acute{k}_q(\mathcal{R}_2)^{*})^{0}h_n(\mathcal{R}_1)\acute{k}_m(\mathcal{R}_2))\,.
\end{gathered}
\tag{5.106}
$$

and, due to $({}^{1}E_p^{h})' + (E_q^{k})' = 0.5(E_{p\,q}^{+})' + (E_{p\,q)}^{-})') - {}_4({}_{pp}^{hh}M_{qq}^{kk})_4$, becomes

$$\begin{array}{c}\frac{\acute{e}^2}{\acute{a}}(\sum_{p,l',q,l1'}((l'+l1')\,\omega' \;-\frac{1}{2}((E^{+}_{p\,q})'+(E^{-}_{p\,q)})')\exp(i(l'+l1')\,\omega'\,(t'-t'_0))\\ \times({}^1c^{l'}_p)^*\,(d^{l1'}_q)^*)({}^1h_p(\mathcal{R}_1)^*)\acute{k}_q(\mathcal{R}_2)^*\\ =\frac{\acute{e}^2}{\acute{a}}(\sum_{p,l',l1'}\exp(i(l'+l1')\,\omega'\,(t'-t'_0))({}^1c^{l'}_p)^*\,(d^{l1'}_q)^*\\ \times(({}^{1h}ma^{l'}_p)^*+\,({}^{1h}mb^{l'}_p)^*+\,({}^{k}ma^{l1'}_q)^*+\,({}^{k}mb^{l1'}_q)^*)\\ -\,{}_4({}^{hh}_{pp}M^{kk}_{qq})_4({}^0h_p(\mathcal{R}_1)^*)\acute{k}_q(\mathcal{R}_2)^*.\end{array} \qquad (5.107)$$

The next we consider the expression

$$^{0\,1}O = i\frac{\acute{e}^2}{\acute{a}}\frac{\partial}{\partial t'}(\,{}^0u(\mathcal{R}_1,t')v(\mathcal{R}_2,t')). \qquad (5.108)$$

It reads (see eqs.(5.86/5.96)

$$\begin{array}{c}O^{0\,1}=-\frac{\acute{e}^2}{\acute{a}}\sum_{n,l,m,l1}(l+l1)\,\omega'\;\exp(-i(l+l1)\,\omega'\,(t'-t'_0))\\ \times({}^0c^l_n)\,(d^{l1}_m)({}^0h_n(\mathcal{R}_1))\acute{k}_m(\mathcal{R}_2).\end{array} \qquad (5.109)$$

This can be substituted by

$$\begin{array}{c}O^{0\,1}=(\sum_n(\frac{\acute{e}^2}{\acute{a}}\,{}^0a_n(t')\overrightarrow{P}^s_1\,{}^0h_n(\mathcal{R}_1)+\,({}^{0h}_{\;n}M^1(t')+\,{}^{0h}_{\;n}M^2(t'))^0h_n(\mathcal{R}_1))v(\mathcal{R}_2)\\ +(\sum_m(\frac{\acute{e}^2}{\acute{a}}(b_m(t'))\overrightarrow{P}^s_2\,\acute{k}_m(\mathcal{R}_2)+\,{}^k_mM^1(t')\acute{k}_m(\mathcal{R}_2)+\,{}^k_mM^2(t')\acute{k}_m(\mathcal{R}_2))^0u(\mathcal{R}_1)\end{array} \qquad (5.110)$$

and, due to the eqs.(5.90-5.96), becomes

$$O^{0\,1} = \frac{\acute{e}^2}{\acute{a}}\big(\textstyle\sum_{n,l} \exp(-il\,\omega'\,(t'-t_0'))((^0c_n^l)\,(^0E_n^h)' \\ + (^{0h}ma_n^l) + \,(^{0h}mb_n^l))^0h_n(\mathcal{R}_1))v(\mathcal{R}_2) \\ + \textstyle\sum_{m,l1} \exp(-il1\,\omega'\,(t'-t_0'))(\,(d_m^{l1})\,(E_m^k)' \\ + (^kma_m^{l1}) + \,(^kmb_m^{l1}))\acute{k}_m(\mathcal{R}_2)^0u(\mathcal{R}_1,t')) \tag{5.111}$$

or with eqs.(5.86)

$$O^{0\,1} = \frac{\acute{e}^2}{\acute{a}}\big(\textstyle\sum_{n,l,m,l1} \exp(-i(l+l1)\,\omega'\,(t'-t_0'))(^0c_n^l)\,(d_m^{l1})((^0E_n^h)' + (E_m^k)' \\ + (^{0h}ma_n^l) + \,(^{0h}mb_n^l) + \,(^kma_m^{l1}) \\ + (^kmb_m^{l1}))^0h_n(\mathcal{R}_1))\acute{k}_m(\mathcal{R}_2)))\,. \tag{5.112}$$

Therefore is

$$(^1u(\mathcal{R}_1)^*v(\mathcal{R}_2)^*)O^{0\,1} = \frac{\acute{e}^2}{\acute{a}}\big(\textstyle\sum_{p,l',q,l1',n,l,m,l1} \exp(i(l'+l1'-l-l1)\,\omega'\,(t'-t_0')) \\ \times(^1c_p^{l'})^*\,(d_q^{l1'})^*(^0c_n^l)\,(d_m^{l1})((^0E_n^h)' + (E_m^k)' \\ + (^{0h}ma_n^l) + \,(^{0h}mb_n^l) + \,(^kma_m^{l1}) \\ + (^kmb_m^{l1}))^1h_p(\mathcal{R}_1))^*\acute{k}_q(\mathcal{R}_2)^*(^0h_n(\mathcal{R}_1))\acute{k}_m(\mathcal{R}_2))))\,. \tag{5.113}$$

and with eq.(5.109)

$$(^1u(\mathcal{R}_1)^*v(\mathcal{R}_2)^*)O^{0\,1} = \frac{\acute{e}^2}{\acute{a}}\big(\textstyle\sum_{p,l',q,l1',n,l,m,l1} \exp(i(l'+l1'-l-l1)\,\omega'\,(t'-t_0')) \\ \times(^1c_p^{l'})^*\,(d_q^{l1'})^*(^0c_n^l)\,(d_m^{l1})((l+l1)\omega')^1h_p(\mathcal{R}_1))^*\acute{k}_q(\mathcal{R}_2)^*(^0h_n(\mathcal{R}_1))\acute{k}_m(\mathcal{R}_2))))\,. \tag{5.114}$$

That leads to

$$
\begin{aligned}
&\tfrac{\acute{e}^2}{\acute{a}}\big(\textstyle\sum_{p,l',q,l1',n,l,m,l1} \exp(i(l' + l1' - l - l1)\,\omega'\,(t' - t'_0)) \\
&\times({}^1c_p^{l'})^*\,(d_q^{l1'})^*({}^0c_n^l)\,(d_m^{l1})((l+l1)\omega')^1h_p(\mathcal{R}_1))^*\acute{k}_q(\mathcal{R}_2)^*({}^0h_n(\mathcal{R}_1))\acute{k}_m(\mathcal{R}_2)))) \\
&= \tfrac{\acute{e}^2}{\acute{a}}\big(\textstyle\sum_{p,l',q,l1',n,l,m,l1} \exp(i(l' + l1' - l - l1)\,\omega'\,(t' - t'_0)) \\
&\times({}^1c_p^{l'})^*\,(d_q^{l1'})^*({}^0c_n^l)\,(d_m^{l1})(({}^0E_n^h)' + (E_m^k)' \\
&+ ({}^{0h}ma_n^l) + \,({}^{0h}mb_n^l) + \,({}^kma_m^{l1}) \\
&+ ({}^kmb_m^{l1}))^1h_n(\mathcal{R}_1))^*\acute{k}_m(\mathcal{R}_2)^*({}^0h_n(\mathcal{R}_1))\acute{k}_m(\mathcal{R}_2))))\,.
\end{aligned}
\tag{5.115}
$$

Due to the eq,(5.98), one obtains

$$
\begin{aligned}
&i\tfrac{\acute{e}^2}{\acute{a}}\big(\tfrac{\partial}{\partial t'}[({}^1u(\mathcal{R}_1,t')^*v(\mathcal{R}_2,t')^*]\,{}^0u(\mathcal{R}_1,t')v(\mathcal{R}_2,t') \\
&+ {}^1u(\mathcal{R}_1,t')^*v(\mathcal{R}_2,t')^*\tfrac{\partial}{\partial t'}\,({}^0u(\mathcal{R}_1,t')v(\mathcal{R}_2,t'))) \\
&= -\tfrac{\acute{e}^2}{\acute{a}}\big(\textstyle\sum_{p,l',q,l1',n,l,m,l1} \exp(i(l' + l1' - l - l1)\,\omega'\,(t' - t'_0)) \\
&\times({}^1c_p^{l'})^*\,(d_q^{l1'})^*({}^0c_n^l)\,(d_m^{l1})(\,(l' + l1' - l - l1)\,\omega'\,) \\
&\times {}^1h_p(\mathcal{R}_1)^*)\acute{k}_q(\mathcal{R}_2)^*)^0h_n(\mathcal{R}_1)\acute{k}_m(\mathcal{R}_2)).
\end{aligned}
\tag{5.116}
$$

and for the sum of eq.(5.104) and eq.(5.115)

$$
\begin{aligned}
&i\tfrac{\acute{e}^2}{\acute{a}}\big(\tfrac{\partial}{\partial t'}[({}^1u(\mathcal{R}_1,t')^*v(\mathcal{R}_2,t')^*]\,{}^0u(\mathcal{R}_1,t')v(\mathcal{R}_2,t') \\
&+ {}^1u(\mathcal{R}_1,t')^*v(\mathcal{R}_2,t')^*\tfrac{\partial}{\partial t'}\,({}^0u(\mathcal{R}_1,t')v(\mathcal{R}_2,t'))) \\
&= \tfrac{\acute{e}^2}{\acute{a}}\big(\textstyle\sum_{p,l',q,l1',n,l,m,l1} \exp(i(l' + l1' - l - l1)\,\omega'\,(t' - t'_0)) \\
&\times({}^1c_p^{l'})^*\,(d_q^{l1'})^*({}^0c_n^l)\,(d_m^{l1})(\,-\,({}^1E_p^h)' + (E_q^k)') + ({}^0E_n^h)' + (E_m^k)' \\
&- ({}^{1h}ma_p^{l'})^* - \,({}^{1h}mb_p^{l'})^* - \,({}^kma_q^{l1'})^* - \,({}^kmb_q^{l1'})^*) + \,({}^{0h}ma_n^l) + \,({}^{0h}mb_n^l) \\
&+ ({}^kma_m^{l1}) + \,({}^kmb_m^{l1}))^1h_p(\mathcal{R}_1))^*\acute{k}_q(\mathcal{R}_2)^*({}^0h_n(\mathcal{R}_1))\acute{k}_m(\mathcal{R}_2))))\,.
\end{aligned}
\tag{5.117}
$$

Using the relations $({}^1E^h_p)' + (E^k_q)' = 0.5((E^+_{p\,q})' + (E^-_{p\,q)})') - {}_4({}^{hh}_{pp}M^{kk}_{qq})_4$ and $({}^0E^h_n)' + (E^k_m)' = 0.5(E^+_{n\,m})' + (E^-_{n\,m)})') - {}_4({}^{hh}_{nn}M^{kk}_{mm})_4$, it becomes

$$\begin{array}{c} -\frac{\acute{e}^2}{\acute{a}}(\sum_{p,l',q,l1',n,l,m,l1} \exp(i(l'+l1'-l-l1)\,\omega'\,(t'-t'_0)) \\ \times({}^1c^{l'}_p)^*\,(d^{l1'}_q)^*({}^0c^l_n)\,(d^{l1}_m)(\,(l'+l1'-l-l1)\,\omega'\,) \\ \times\,{}^1h_p(\mathcal{R}_1)^*)\acute{k}_q(\mathcal{R}_2)^*)^0h_n(\mathcal{R}_1)\acute{k}_m(\mathcal{R}_2)). \\ = \frac{\acute{e}^2}{\acute{a}}(\sum_{p,l',q,l1',n,l,m,l1} \exp(i(l'+l1'-l-l1)\,\omega'\,(t'-t'_0)) \\ \times({}^1c^{l'}_p)^*\,(d^{l1'}_q)^*({}^0c^l_n)\,(\bar{d}^{l1}_m)(\,-\,0.5({}^1E^+_{p\,q})' + ({}^1E^-_{p\,q)})') - {}_4({}^{hh}_{pp}M^{kk}_{qq})_4 \\ +0.5({}^0E^+_{n\,m})' + ({}^0E^-_{n\,q})') - {}_4({}^{hh}_{nn}M^{kk}_{mm})_4 \\ -\,({}^{1h}ma^{l'}_p)^* - ({}^{1h}mb^{l'}_p)^* - ({}^kma^{l1'}_q)^* - ({}^kmb^{l1'}_q)^*) + ({}^{0h}ma^l_n) + ({}^{0h}mb^l_n) \\ +\,({}^kma^{l1}_m) + ({}^kmb^{l1}_m))^1h_p(\mathcal{R}_1))^*\acute{k}_q(\mathcal{R}_2)^*({}^0h_n(\mathcal{R}_1))\acute{k}_m(\mathcal{R}_2)))). \end{array} \tag{5.118}$$

The integration over V_2 yields, due to the orthogonality of $\acute{k}_q()^*\acute{k}_m()$,

$$\begin{array}{c} -\frac{\acute{e}^2}{\acute{a}}(\sum_{p,l',m,l1',n,l,l1} \exp(i(l'+l1'-l-l1)\,\omega'\,(t'-t'_0)) \\ 0.5({}^1c^{l'}_p)^*\,(d^{l1'}_m)^*({}^0c^l_n)\,(d^{l1}_m)((\,(l'+l1'-l-l1)\,\omega' + ({}^1E^+_{p\,m})' - ({}^0E^+_{n\,m})') \\ +\,(\,(l'+l1'-l-l1)\,\omega' + ({}^1E^-_{p\,m})' - ({}^0E^-_{n\,m})')\,{}^1h_p(\mathcal{R}_1)^*)^0h_n(\mathcal{R}_1)). \\ = \frac{\acute{e}^2}{\acute{a}}(\sum_{p,l',m,l1',n,l,l1} \exp(i(l'+l1'-l-l1)\,\omega'\,(t'-t'_0)) \\ \times({}^1c^{l'}_p)^*\,(d^{l1'}_m)^*({}^0c^l_n)\,(d^{l1}_m)(\,-\,{}_4({}^{hh}_{pp}M^{kk}_{mm})_4 \\ -\,{}_4({}^{hh}_{nn}M^{kk}_{mm})_4 - ({}^{1h}ma^{l'}_p)^* - ({}^{1h}mb^{l'}_p)^* - ({}^kma^{l1'}_m)^* - ({}^kmb^{l1'}_m)^* \\ +\,({}^{0h}ma^l_n) + ({}^{0h}mb^l_n) + ({}^kma^{l1}_m) + ({}^kmb^{l1}_m))^1h_p(\mathcal{R}_1)^*({}^0h_n(\mathcal{R}_1))). \end{array} \tag{5.119}$$

The left side is formed by current divergences analogously to eqs.(5.57). Details of eq.(5.119) should not be considered here. However one must be taken into account, that the solutions of the eqs.((5.90/5.91/5.92) are only influenced by quasi-static potentials. Radiation effects are neglected. But, this assumption must be corrected

if a share of the currents satisfies one of the conditions given in the eqs.(5.60). That share must the divergence eq.(5.58) also fulfils.
The related sub-field satisfies the energy balance as in one-electron case (see section 4.3). However, the eigenfunctions h_n and $\grave{k}_m$ and of which eigenvalues $(E_n)'$ and $(E_m)'$ differ from the expressions of one-electron cases due to the double charge of the nucleus. Such tables have so far only be calculated for s-states and therefore, we restrict our numerical documentation on these cases and set $h_n = Y_{0\,0}h'_n(r)$ and $\grave{k}_m = Y_{0\,0}h'_m(r)$. These combinations do not form radiation moments and the introduction of excitation levels makes not sense. In addition, the border of the calculation range RR and the number of eigenfunctions must be greater than in the one-electron case. This leads to longer calculation times. The Figs. 10-12 and Table 10 show some results. The parameter e represents the total energy of the system, which depends on the starting distribution. However, here we have used a reduced number of eigenfunctions and a short calculation time. That means, we have accepted a lower level of accuracy. Therefore the shapes in Fig. 10 and Fig. 11 are wider and show dents. Nevertheless, it allows a reasonable interpretation of the results. Fig. 11 exhibits a similar frequency behaviour of $c_1^1 = |c_1^l|$ and $d_1^1 = |d_1^1(l1 = l)|$ (see eqs.(5.86/5.87/5.93). However, the accuracy of the examples presented is too low for a more detailed interpretation.

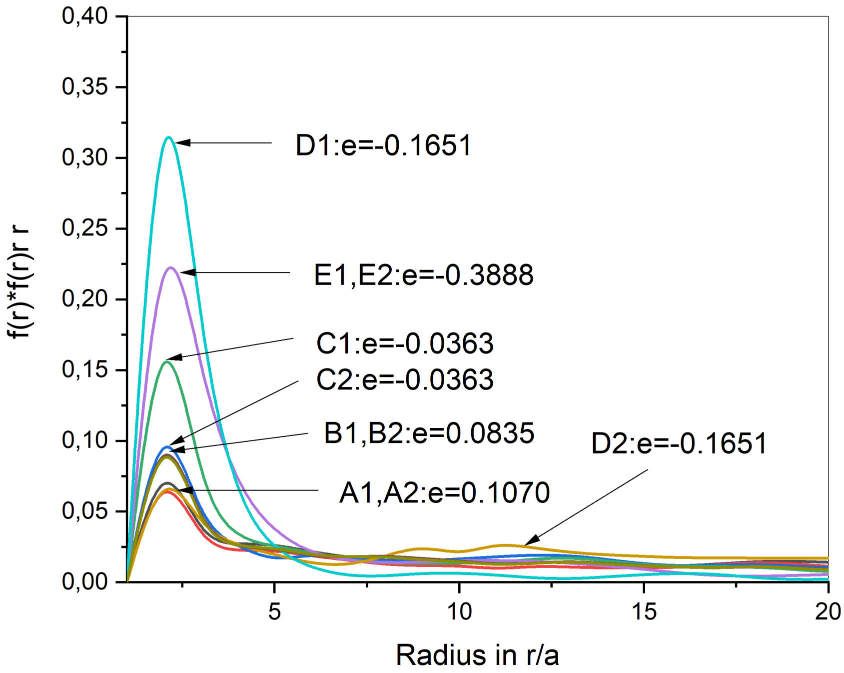

Fig. 10: (see the eqs.(5.91/5.92)

Mean $f^*(r)f(r)r^2 = \frac{1}{4\pi T'}\int_0^{T'} u(r_1,t')^* u(r_1,t')dt'$ of electron 1 (cases A1-E1) and mean $f^*(r)f(r)r^2 = \frac{1}{4\pi T'}\int_0^{T'} v(r,t')^* v(r,t')dt'$ of electron 2 (cases A2-E2); The eqs.(5.88) link A1 with A2, B1 with B2, C1 with C2, D1 with D2 and E1 with E2;

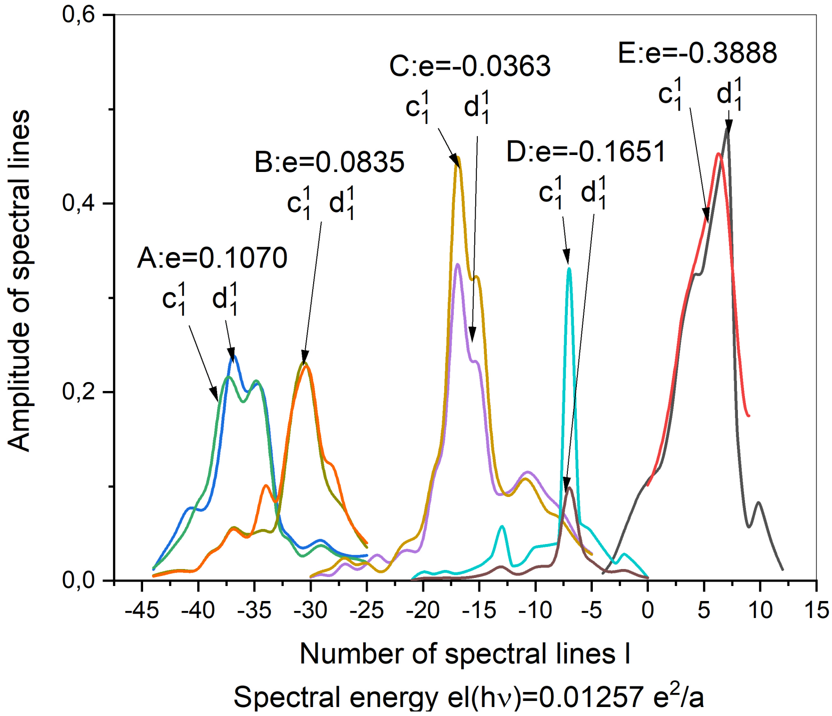

Fig. 11: Spectral distributions of the 1s-coefficient $|c^{l=1}_{n=1}|$ of electron 1 (see the egs.(5.93)) and the 1s-coefficient $|d^{l1=1}_{m=1}|$ of the electron 2 (cases A-E);

Table 10 Energy parameters of two-electron states:
(A-E:states, e=eg1+eg2 in $\acute{e}^2/\acute{a}$),

A: T'=1000, e=0.1070

$eg1$	$ed1$	$ea1$	$R1$	$\Delta R1$
0.0307	-0.0404	0.0711	34.69	2.74
$eg2$	$ed2$	$ea2$	$R2$	$\Delta R2$
0.0763	0.0109	0.0654	38.91	2.97

e	edg	eag	$A(0)$	$\Delta A(0)$
0.1070	-0.0295	0.1365	0.756	0.019

B: T'=1000, e=0.0835

$eg1$	$ed1$	$ea1$	$R1$	$\Delta R1$
0.0383	-0.0532	0.0915	33.41	3.49
$eg2$	$ed2$	$ea2$	$R2$	$\Delta R2$
0.0451	-0.0456	0.0907	34.17	3.49
e	edg	eag	$A(0)$	$\Delta A(0)$
0.0835	-0.0988	0.1823	0.9287	0.1842

C: T'=1000, e=-0.03627

$eg1$	$ed1$	$ea1$	$R1$	$\Delta R1$
0.0488	-0.0662	0.1150	32.68	3.60
$eg2$	$ed2$	$ea2$	$R2$	$\Delta R2$
-0.0851	-0.2466	0.1616	25.65	4.35
e	edg	eag	$A(0)$	$\Delta A(0)$
-0.0363	-0.3128	0.2766	1.1867	0.1333

D: T'=1000, egg=-0.1651

$eg1$	$ed1$	$ea1$	$R1$	$\Delta R1$
0.0865	-0.0275	0.1140	33.00	2.73
$eg2$	$ed2$	$ea2$	$R2$	$\Delta R2$
-0.2516	-0.6000	0.3485	10.53	2.02

e	edg	eag	$A(0)$	$\Delta A(0)$
-0.1651	-0.6274	0.4623	1.5431	0.1001

E: T'=1000, egg=-0.3888

$eg1$	$ed1$	$ea1$	$R1$	$\Delta R1$
-0.1944	-0.4922	0.2978	13.32	1.70
$eg2$	$ed2$	$ea2$	$R2$	$\Delta R2$
-0.1944	-0.4522	0.2978	13.32	1.70
e	edg	eag	$A(0)$	$\Delta A(0)$
-0.3888	-0.9843	0.5955	1.6959	0.1333

Chapter 6

Influence of the spin

6.1 *Context of the problem*

For the description of the influence of orientation on the spectral lines we consider fields, which are divided into two excitation levels. In addition, special shares describing the radiation effects are to be separated in a final presentation. In particular, we restrict the investigation to cases in which two of the selected eigenfunctions are equal, i.e. ${}^{o}K_m = {}^{z}K_m = K_m$. A transition field can be formed using the following solutions of the Dirac equation (see eqs.(5.2/5.3)

$$\begin{gathered} U_1(\mathcal{R}_1,t) = \textstyle\sum_j \; {}^{z}\hat{a}_j(t)\, {}^{z}H_j(\mathcal{R}_1) \\ + \textstyle\sum_l \exp(-\,i\,l\,\omega\,t)\, {}^{z}c_n^{l}\; {}^{z}H_n(\mathcal{R}_1) \\ + \textstyle\sum_{j'} \; {}^{o}\hat{a}_{j'}(t)\, {}^{o}H_{j'}(\mathcal{R}_1) \\ + \textstyle\sum_{l'} \exp(-\,i\,l'\,\omega\,t)\, {}^{o}c_{n'}^{l'}\; {}^{o}H_{n'}(\mathcal{R}_1), \end{gathered} \tag{6.1}$$

$$\begin{gathered} U_2(\mathcal{R}_2,t) = \textstyle\sum_k \; {}^{z}\hat{b}_k(t)\, {}^{z}K_k(\mathcal{R}_2) \\ + \textstyle\sum_l \exp(-\,i\,l1\,\omega\,t)\, {}^{z}d_m^{l1}\; {}^{z}K_m(\mathcal{R}_2) \\ + \textstyle\sum_{k'} \; {}^{o}\hat{b}_{k'}(t)\, {}^{o}K_{k'}(\mathcal{R}_2) \\ + \textstyle\sum_{l'} \exp(-\,i\,l1'\,\omega\,t)\, {}^{o}d_m^{l1'}\; {}^{o}K_m(\mathcal{R}_2). \end{gathered} \tag{6.2}$$

We study appropriate separated products of these functions and define, related to eq.(5.120),

$$
\begin{aligned}
C_1(t) &= ({}^o c_{n'}^{l'})^* ({}^z c_n^l) \exp(i(l'-l)\omega t)\, {}^o, \\
C_2(t) &= ({}^o d_m^{l1'})^* ({}^z d_m^l) \exp(i(l1'-l1)\omega t).
\end{aligned} \tag{6.3}
$$

Here the indexes o and z mean upper and lower level. ${}^o\hat{a}_{j'}(t)$, ${}^z\hat{a}_j(t)$, ${}^o\hat{b}_{k'}(t)$ and ${}^z\hat{b}_k(t)$ follow from the coefficients of the complete system, shown in the eqs.(5.4), by subtracting the separated terms.

According to the eqs.(5.46/5.52), the divergence of the tensor, formed with the separated terms, reads

$$
\begin{aligned}
C_2(t)\tfrac{\partial}{\partial x_\mu^1} T_{4\,\mu}^{d1}(\mathcal{R}_1,t) + C_1(t)\tfrac{\partial}{\partial x_\mu^1} T_{4\,\mu}^{d2}(\mathcal{R}_2,t) = -\tfrac{\acute{a}}{\acute{\alpha}} C_2(t) \textstyle\int (\bar{H}_{n'}(\mathcal{R}_1)\bar{K}_m(\mathcal{R}_2) \\
\times(\overleftarrow{Q}^4_{1\,2} + \overrightarrow{Q}^4_{1\,2} + 2\gamma_4^1\gamma_4^2 \tfrac{\hbar\partial}{i\partial t}) C_1(t))\, H_n(\mathcal{R}_1) K_m(\mathcal{R}_2) dV_2 \\
-\tfrac{\acute{a}}{\acute{\alpha}} C_1(t) \textstyle\int (\bar{H}_{n'}(\mathcal{R}_1)\bar{K}_m(\mathcal{R}_2)(\overleftarrow{Q}^4_{1\,2} + \overrightarrow{Q}^4_{1\,2} \\
+2\gamma_4^1\gamma_4^2 \tfrac{\hbar\partial}{i\partial t}) C_2(t))\, H_n(\mathcal{R}_1) K_m(\mathcal{R}_2) dV_1
\end{aligned} \tag{6.4}
$$

or

$$
\begin{aligned}
C_2(t)\tfrac{\partial}{\partial x_\mu^1} T_{4\,\mu}^{d1}(\mathcal{R},t) + C_1(t)\tfrac{\partial}{\partial x_\mu^2} T_{4\,\mu}^{d2}(\mathcal{R},t) = \\
\tfrac{\acute{a}}{2\acute{\alpha}}\, C_1(t) C_2(t) ((\bar{E}^+_{n'\,m} - \bar{E}^+_{n\,m} - 2\hbar(l'-l)\omega)\, \bar{H}_{n'}(\mathcal{R})\gamma_4^0 H_n(\mathcal{R}) \\
+(\bar{E}^-_{n'\,m} - \bar{E}^-_{n\,m} - 2\hbar(l'-l)\omega)\, \bar{H}_{n'}(\mathcal{R})\gamma_4^0 H_n(\mathcal{R})).
\end{aligned} \tag{6.5}
$$

One obtains for $n' \neq n$ and $l1' - l1 = l' - l$ the eqs.(5.47-5.52). Here we apply the practical dimensions (unit of the energy: $\acute{e}^2/\acute{a}$, unit of the length $\acute{a}$, unit of the time Δt) and get for the divergence of the current

$$\acute{c}\hbar\frac{\partial}{\partial x_\mu}(J^{\pm d1}_{\bar{\mu}}(\mathcal{R})+J^{\pm d2}_{\bar{\mu}}(\mathcal{R}))=\frac{i}{2}\,\acute{e}C_1(t)C_2(t)(E^{\pm}_{n'\,m}-E^{\pm}_{n\,m}$$
$$-2\hbar(l'-l)\omega)\bar{H}_{n'}(\mathcal{R})\gamma^0_4 H_n(\mathcal{R}) \tag{6.6}$$

and the comparison with eq.(5.52) to $(\frac{\acute{a}}{\acute{\alpha}}\,\acute{c}\hbar=\acute{M}\acute{c}^2=\frac{1}{\acute{\alpha}^2}\frac{\acute{e}^2}{\acute{a}})$

$$\begin{gathered}C_2(t)\frac{\partial}{\partial x^1_\mu}T^{d1}_{4\,\mu}(\mathcal{R},t)+C_1(t)\frac{\partial}{\partial x^2_\mu}T^{d2}_{4\,\mu}(\mathcal{R},t)=\\ \frac{\acute{a}}{2\acute{\alpha}}C_1(t)C_2(t)((\bar{E}^{+}_{n'\,m}-\bar{E}^{+}_{n\,m}-2\hbar(l'-l)\omega)\,\bar{H}_{n'}(\mathcal{R})\gamma^0_4 H_n(\mathcal{R})\\ +(\bar{E}^{-}_{n'\,m}-\bar{E}^{-}_{n\,m}-2\hbar(l'-l)\omega)\,\bar{H}_{n'}(\mathcal{R})\gamma^0_4 H_n(\mathcal{R}))\\ =-i\frac{1}{\acute{\alpha}^2}\frac{\acute{e}}{\acute{a}}\,\frac{\partial}{\partial x_\mu}(J^{+d}_{\mu}(\mathcal{R})+J^{-d}_{\mu}(\mathcal{R})).\end{gathered} \tag{6.7}$$

This current causes a radiation if ${}^{o}\bar{H}_{n'}(\mathcal{R}_1)\gamma^1_4\,{}^{z}H_n(\mathcal{R}_1)$ forms a radiation moment and the prerequisite, formulated in section 5.2, are fulfilled. That means, one of the frequency meets the condition

$$\begin{gathered}{}^{o}(E^{+}_{n'\,m})'-\,{}^{z}(E^{+}_{n\,m})'=2(l'-l)\omega'\\ or\\ {}^{o}(E^{-}_{n'\,m})'-\,{}^{z}(E^{-}_{n\,m})'=2(l'-l)\omega'.\end{gathered} \tag{6.8}$$

However, it must taken into account that $J^{+d}_{\mu}(\mathcal{R},t')$ and $J^{-d}_{\mu}(\mathcal{R},t')$ oscillate with the same frequency, but only one moment satisfies the radiation condition. That means, only the oscillation with one of the frequency given in the eqs.(6.8) causes radiation. We discus in section 6.2 witch frequency is selected.

6.2 *The influence of spin orientation*

The orientation must be taken into account, to describe fine structure effects. Therefore we introduce vertical arrows in the functions. The selected terms in the eqs.(6.1/6.2) can be depicted by

$$U(\mathcal{R}_1, t, \uparrow) = \sum_l c_n^l \exp(-il\omega t) H_n(\mathcal{R}_1, \uparrow), \tag{6.9}$$

and

$$V(\mathcal{R}_2, \uparrow, t) = \sum_{l1} d_m^{l1} \exp(-i\omega l1)) K_m(\mathcal{R}_2, \uparrow), \tag{6.10}$$

whereby the direction of the arrow stands for the spin orientation. We consider the known solution of the eq.(5.42), however, replace the operators $\overrightarrow{P}_1$, $\overrightarrow{P}_2$ and $\overrightarrow{Q}^4_{1\,2}$ with

$$\begin{aligned} P_1 &= \hbar\gamma^1_{\bar\mu}(\partial/\partial x^1_{\bar\mu}) - 2\acute{e}^2\gamma^1_4/(\acute{c}r_1) + \acute{M}\acute{c}, \\ P_2 &= \hbar\gamma^2_{\bar\mu}(\partial/\partial x^2_{\bar\mu}) - 2\acute{e}^2\gamma^2_4/(\acute{c}r_2) + \acute{M}\acute{c} \end{aligned} \tag{6.11}$$

and

$$\begin{aligned} Q_{1\,2} &= (\gamma^2_4(\acute{c}\hbar\gamma^1_{\bar\mu}\tfrac{\partial}{\partial x^1_{\bar\mu}} - \gamma^1_4\acute{e}^2(\tfrac{2}{r_1} - \tfrac{1}{r_{1\,2}})) \\ &\quad +\gamma^1_4(\acute{c}\hbar\gamma^2_{\bar\mu}\tfrac{\partial}{\partial x^2_{\bar\mu}} - \acute{e}^2\gamma^2_4(\tfrac{2}{r_2} - \tfrac{1}{r_{2\,1}}))) \\ \hat{Q}_{1\,2} &= Q_{1\,2} + \gamma^1_4\gamma^2_4\tfrac{\hbar\partial}{i\partial t}. \end{aligned} \tag{6.12}$$

The functions $H_n(\mathcal{R}_1, \uparrow)$ and $K_m(\mathcal{R}_2, \uparrow)$ satisfy the eigenvalue equations

$$\begin{aligned} \acute{c}P_1 H_n(\mathcal{R}_1, \uparrow) &= \bar{E}^{H1}_n(\uparrow)\gamma^1_4 H_n(\mathcal{R}_1, \uparrow), \\ \acute{c}P_2 K_m(\mathcal{R}_2, \uparrow) &= \bar{E}^{K2}_m(\uparrow)\gamma^2_4 K_m(\mathcal{R}_2, \uparrow), \\ Q_{1\,2}\Psi^{\pm}_{n\,m}(\mathcal{R}_1, \mathcal{R}_2, \uparrow\uparrow) &= \bar{E}^{\pm}_{n\,m}(\uparrow\uparrow)\gamma^1_4\gamma^2_4\Psi^{\pm}_{n\,m}(\mathcal{R}_1, \mathcal{R}_2, \uparrow\uparrow). \end{aligned} \tag{6.13}$$

Here is

$$\Psi^{\pm}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,\uparrow\uparrow) = \tfrac{1}{\sqrt{2}}(H_n(\mathcal{R}_1,\uparrow)K_m(\mathcal{R}_2,\uparrow) \pm K_m(\mathcal{R}_1,\uparrow)H_j(\mathcal{R}_2,\uparrow)). \qquad (6.14)$$

We look at a few combinations of s- and p-states. For example:

$$H_n(\uparrow) = \phi'(\mathcal{R}_1, \acute{n}, 0, \breve{m} = 1/2, \uparrow) = \left\{ \begin{array}{c} \hat{h}_{\acute{n}\,0}(\uparrow) \\ \check{h}_{\acute{n}\,0}(\uparrow) \end{array} \right\},\ \hat{h}_{\acute{n}\,0}(\uparrow) = \left\{ \begin{array}{c} iY_{0\,0} f^s_{\acute{n}}(r) \\ 0 \end{array} \right\},$$

$$K_m(\uparrow) = \phi'(\mathcal{R}_2, \acute{m}, 1, \breve{m} = 1/2, \uparrow) = \left\{ \begin{array}{c} \hat{k}_{\acute{m}\,1}(\uparrow) \\ \check{k}_{\acute{m}\,1}(\uparrow) \end{array} \right\}, \qquad (6.15)$$

$$\hat{k}_{\acute{m}\,1}(\uparrow) = \left\{ \begin{array}{c} iY_{1\,0}\sqrt{2/3} f^p_{\acute{m}}(r) \\ -iY_{1\,1}\sqrt{1/3} f^p_{\acute{m}}(r) \end{array} \right\}.$$

You find in the eqs.(3.8) the meaning of the arguments of $\phi'(\mathcal{R}_1, \acute{n}, 0, \breve{m})$ for $j = l - \breve{m} = 1/2,$) and also the alternative expression $K_m(\uparrow) = \phi'(\mathcal{R}_2, \acute{m}, 1, \breve{m} = -1/2, \uparrow)$ for j=3/2. In addition, we use again relations like $n = n(\acute{n}, 0, \breve{m})$ and $m = m(\acute{m}, 1, \breve{m})$ between the numbers. With the functions of the eqs.(6.15) you cannot check the fine structure rules with the aid of the basic states because some eigenvalues are equal. Therefore we consider different p1p1-states.

Case A:

With the functions

$$
\begin{aligned}
H_n(\uparrow) = \phi'(\mathcal{R}_1, \acute{n}, 1, \breve{m} = 1/2, \uparrow) &= \left\{ \begin{array}{c} \hat{h}_{\acute{n}\,0}(\uparrow) \\ \check{h}_{\acute{n}\ 0}(\uparrow) \end{array} \right\}, \\
\hat{h}_{\acute{n}\,1}(\uparrow) &= \left\{ \begin{array}{c} iY_{1\,0}\sqrt{2/3} f^p_{\acute{n}}(r) \\ -iY_{1\,1}\sqrt{1/3} f^p_{\acute{n}}(r) \end{array} \right\}, \\
K_m(\uparrow) \ = \phi'(\mathcal{R}_2, \acute{m}, 1, \breve{m} = 1/2, \uparrow) &= \left\{ \begin{array}{c} \hat{k}_{\acute{m}\,1}(\uparrow) \\ \breve{k}_{\acute{m}\,1}(\uparrow) \end{array} \right\}, \\
\hat{k}_{\acute{m}\,1}(\uparrow) &= \left\{ \begin{array}{c} iY_{1\,0}\sqrt{2/3} f^p_{\acute{m}}(r) \\ -iY_{1\,1}\sqrt{1/3} f^p_{\acute{m}}(r) \end{array} \right\}
\end{aligned} \tag{6.16}
$$

and

$$
\begin{aligned}
CD_1(t) = c^l_n d^{l1}_m \exp(-i(l + l1)wt + \phi_1(t)) \\
CD_2(t) = c^l_n d^{l1}_m \exp(-i(l + l1)wt + \phi_2(t))
\end{aligned} \tag{6.17}
$$

we form the expression

$$
\begin{aligned}
W^{\pm}_{n\,m}(\mathcal{R}_1, \mathcal{R}_2, t, \uparrow\uparrow + \uparrow\uparrow) = \tfrac{1}{2}(CD_1(t)(\Psi^{\pm}_{n\,m}(\mathcal{R}_1, \mathcal{R}_2, \uparrow\uparrow) \\
+CD_2(t)\Psi^{\pm}_{n\,m}(\mathcal{R}_1, \mathcal{R}_2, \uparrow\uparrow))
\end{aligned} \tag{6.18}
$$

and use eq.(5.42) or eq.(6.12). We consider the cases of which the frequencies meet one of the conditions given in eq.(6.7) or eq.(6.8). Then the radiation must be taken into account. If $\phi_2(t)$ and $\phi_2(t)$ record this influence, we assume that $|d(\phi_1(t)/dt|$ and $|\phi_2(t))/dt|$ are negligible because the radiation field is very small in the cases examined. Therefore one obtains the eigenvalue using of the following equation

$$
\begin{aligned}
\bar{W}^{\pm}_{n\,m}(\mathcal{R}_1, \mathcal{R}_2, t, \uparrow\uparrow + \uparrow\uparrow)\hat{Q}_{1\,2}W^{\pm}_{n\,m}(\mathcal{R}_1, \mathcal{R}_2, t, \uparrow\uparrow + \uparrow\uparrow) = (c^l_n d^{l1}_m)^* c^l_n d^{l1}_m \\
\times(KK^{\pm}_{n\,m}(\mathcal{R}_1, \mathcal{R}_2, \uparrow\uparrow + \uparrow\uparrow) + GG^{\pm}_{n\,m}(\mathcal{R}_1, \mathcal{R}_2, \uparrow\uparrow + \uparrow\uparrow) \\
+\bar{A}J^{\pm}_{n\,m}(\uparrow\uparrow + \uparrow\uparrow) + HH^{\pm}_{n\,m}(\mathcal{R}_1, \mathcal{R}_2, \uparrow\uparrow + \uparrow\uparrow))
\end{aligned} \tag{6.19}
$$

with

$$
\begin{aligned}
&KK^{\pm}_{nm}(\mathcal{R}_1,\mathcal{R}_2,\uparrow\uparrow+\uparrow\uparrow)\\
&=\tfrac{1}{2}(\bar{H}_n(\mathcal{R}_1,\uparrow)\acute{c}P_1H_n(\mathcal{R}_1,\uparrow)\,\bar{K}_m(\mathcal{R}_2,\uparrow)\gamma_4^2K_m(\mathcal{R}_2,\uparrow)\\
&+\bar{K}_m(\mathcal{R}_1,\uparrow)\acute{c}P_1K_m(\mathcal{R}_1,\uparrow)\,\bar{H}_n(\mathcal{R}_2,\uparrow)\gamma_4^2H_n(\mathcal{R}_2,\uparrow)\\
&+\bar{H}_n(\mathcal{R}_1,\uparrow)\gamma_4^1H_n(\mathcal{R}_1,\uparrow)\,\bar{K}_m(\mathcal{R}_2,\uparrow)\acute{c}P_2K_m(\mathcal{R}_2,\uparrow)\\
&+\bar{K}_m(\mathcal{R}_1,\uparrow)\gamma_4^1K_m(\mathcal{R}_1,\uparrow)\,\bar{H}_n(\mathcal{R}_2,\uparrow)\acute{c}P_2H_n(\mathcal{R}_2,\uparrow)),
\end{aligned}
\tag{6.20}
$$

$$
\begin{aligned}
&GG^{\pm}_{nm}(\mathcal{R}_1,\mathcal{R}_2,\uparrow\uparrow+\uparrow\uparrow)\\
&=\pm\tfrac{1}{2}(\bar{H}_n(\mathcal{R}_1,\uparrow)\acute{c}P_1K_m(\mathcal{R}_1,\uparrow)\,\bar{K}_m(\mathcal{R}_2,\uparrow)\gamma_4^2H_n(\mathcal{R}_2,\uparrow)\\
&+\bar{K}_m(\mathcal{R}_1,\uparrow)\acute{c}P_1H_n(\mathcal{R}_1,\uparrow)\,\bar{H}_n(\mathcal{R}_2,\uparrow)\gamma_4^2K_m(\mathcal{R}_2,\uparrow)\\
&+\bar{H}_n(\mathcal{R}_1,\uparrow)\gamma_4^1K_m(\mathcal{R}_1,\uparrow)\,\bar{K}_m(\mathcal{R}_2,\uparrow)\acute{c}P_2H_n(\mathcal{R}_2,\uparrow)\\
&+\bar{K}_m(\mathcal{R}_1,\uparrow)\gamma_4^1H_n(\mathcal{R}_1,\uparrow)\,\bar{H}_n(\mathcal{R}_2,\uparrow)\acute{c}P_2K_m(\mathcal{R}_2,\uparrow)),
\end{aligned}
\tag{6.21}
$$

$$
\begin{aligned}
\bar{A}J^{\pm}(\uparrow\uparrow+\uparrow\uparrow)=\tfrac{1}{4}(&{}_4({}^{H1\uparrow H1\uparrow}_{nn}AJ^{K2\uparrow K2\uparrow}_{mm})_4\pm{}_4({}^{H1\uparrow K1\uparrow}_{nm}AJ^{K2\uparrow H2\uparrow}_{mn})_4\\
&+{}_4({}^{H2\uparrow H2\uparrow}_{nn}AJ^{K1\uparrow K1\uparrow}_{mm})_4\pm{}_4({}^{H2\uparrow K2\uparrow}_{nm}AJ^{K1\uparrow H1\uparrow}_{mn})_4\\
&+{}_4({}^{K1\uparrow K1\uparrow}_{mm}AJ^{H2\uparrow H2\uparrow}_{nn})_4\pm{}_4({}^{K1\uparrow H1\uparrow}_{mn}AJ^{H2\uparrow K2\uparrow}_{nm})_4\\
&+{}_4({}^{K2\uparrow K2\uparrow}_{mm}AJ^{H1\uparrow H1\uparrow}_{nn})_4\pm{}_4({}^{K2\uparrow H2\uparrow}_{mn}AJ^{H1\uparrow K1\uparrow}_{nm})_4),
\end{aligned}
\tag{6.22}
$$

$$
{}_4({}^{HH}_{nn}AJ^{KK}_{mm})_4=\acute{e}^2\bar{H}_n(\mathcal{R}_1)\gamma_4^1H_n(\mathcal{R}_1)\tfrac{1}{r_{12}}\bar{K}_m(\mathcal{R}_2)\gamma_4^2K_m(\mathcal{R}_2),\ \ etc.
$$

and

$$
\begin{aligned}
& HH^{\pm}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,\uparrow\uparrow+\uparrow\uparrow) \\
&= -(l+l1)w\hbar\bar{\Psi}^{\pm}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,\uparrow\uparrow)\gamma^1_4\gamma^2_4\Psi^{\pm}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,\uparrow\uparrow).
\end{aligned}
\tag{6.23}
$$

The new designations $H1 := H_n(\mathcal{R}_1)$, $H2 : H_n(\mathcal{R}_2)$ etc. indicate the electrons 1 or 2. After integration it reads

$$
\begin{aligned}
& \int KK^{\pm}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,\uparrow\uparrow+\uparrow\uparrow)dV_1dV_2 \\
&= \tfrac{1}{2}(\bar{E}^{H1}_n(\uparrow)+\bar{E}^{K1}_m(\uparrow)+\bar{E}^{H2}_n(\uparrow)+\bar{E}^{K2}_m(\uparrow)), \\
& \int GG^{\pm}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,\uparrow\uparrow+\uparrow\uparrow)dV_1dV_2 = 0, \\
& \int \bar{A}J^{\pm}_{n\,m}(\uparrow\uparrow+\uparrow\uparrow)dV_1dV_2 = \bar{M}^{\pm}_{n\,m}(\uparrow\uparrow+\uparrow\uparrow), \\
& \int HH^{\pm}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,t,\uparrow\uparrow+\uparrow\uparrow)dV_1dV_2 = -(l+l1)w\hbar
\end{aligned}
\tag{6.24}
$$

with

$$
\begin{aligned}
\bar{M}^{\pm}_{n\,m}(\uparrow\uparrow+\uparrow\uparrow) &= \tfrac{1}{2}\big(\,{}_4({}^{H1\uparrow H1\uparrow}_{nn}M^{K2\uparrow K2\uparrow}_{mm})_4 \pm {}_4({}^{H1\uparrow K1\uparrow}_{nm}M^{K2\uparrow H2\uparrow}_{mn})_4 \\
&+ {}_4({}^{H2\uparrow H2\uparrow}_{nn}M^{K1\uparrow K1\uparrow}_{mm})_4 \pm {}_4({}^{H2\uparrow K2\uparrow}_{nm}M^{K1\uparrow H1\uparrow}_{mn})_4\big).
\end{aligned}
\tag{6.25}
$$

Therefore one obtains the equation

$$
\begin{aligned}
& \int \bar{W}^{\pm}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,t,\uparrow\uparrow+\uparrow\uparrow)\hat{Q}_{1\,2}W^{\pm}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,t,\uparrow\uparrow+\uparrow\uparrow)dV_1dV_2 \\
&= (c^l_n d^{l1}_m)^* c^l_n d^{l1}_m(\bar{E}^{\pm HK}_{n\,m}(\uparrow\uparrow+\uparrow\uparrow) - (l+l1)w\hbar\,),
\end{aligned}
\tag{6.26}
$$

what yields the condition

$$\bar{E}^{\pm HK}_{nm}(\uparrow\uparrow + \uparrow\uparrow) = \ (l + l1)w\hbar \tag{6.27}$$

for the eigenvalues

$$\begin{array}{c}\bar{E}^{\pm HK}_{nm}(\uparrow\uparrow + \uparrow\uparrow) = \ \frac{1}{2}(\bar{E}^{H1}_{n}(\uparrow) + \bar{E}^{K1}_{m}(\uparrow) \\ +\bar{E}^{H2}_{n}(\uparrow) + \bar{E}^{K2}_{m}(\uparrow)) + \bar{M}^{\pm}_{nm}(\uparrow\uparrow + \uparrow\uparrow).\end{array} \tag{6.28}$$

The following relations

$$\begin{array}{c}KK^{-}_{nm}(\mathcal{R}_1, \mathcal{R}_2, t, \uparrow\uparrow + \uparrow\uparrow) = KK^{+}_{nm}(\mathcal{R}_1, \mathcal{R}_2, \uparrow\uparrow + \uparrow\uparrow), \\ GG^{+}_{nm}(\mathcal{R}_1, \mathcal{R}_2, t, \uparrow\uparrow + \uparrow\uparrow) = -GG^{-}_{nm}(\mathcal{R}_1, \mathcal{R}_2, \uparrow\uparrow + \uparrow\uparrow) \\ \bar{M}^{\pm}_{nm}(\uparrow\uparrow + \uparrow\uparrow) = \bar{M}^{\pm}_{nm}(*)\end{array} \tag{6.29}$$

are useful for comparing of the results. In our approximation the potential integrals are independent on the orientation and we suppress these informations sometimes.

The orientation can be combined with solutions of the Schrödinger equation if one apply the following notations:

$$\begin{array}{c}\alpha = \alpha' = \left\{\begin{array}{c}1\\0\end{array}\right\}, \ \beta = \beta' = \left\{\begin{array}{c}1\\0\end{array}\right\}, \\ \bar{\alpha} \ = \bar{\alpha}' = \{1\ 0\}, \, \bar{\beta} = \bar{\beta}' = \{0\ 1\}.\end{array} \tag{6.30}$$

For example, it is

$$\begin{array}{c}\hat{h}_n(\mathcal{R}_1, \uparrow) := \alpha h_n(\mathcal{R}_1), \\ \hat{h}_n(\mathcal{R}_2, \downarrow) := \beta' h_n(\mathcal{R}_2).\end{array} \tag{6.31}$$

We use these notations to characterise the function $W^{\pm}_{nm}(\mathcal{R}_1, \mathcal{R}_2, t, \uparrow\uparrow + \uparrow\uparrow)$ and get the following equivalent expression

$$W^{\pm}_{nm}(\mathcal{R}_1\mathcal{R}_2, t', \uparrow\uparrow + \uparrow\uparrow) := \tfrac{1}{2}(CD_1(t') + CD_2(t'))\alpha\alpha'\psi^{\pm}_{nm}(\mathcal{R}_1, \mathcal{R}_2). \tag{6.32}$$

Here

$$\psi^{\pm}_{nm}(\mathcal{R}_1, \mathcal{R}_2) = \frac{1}{\sqrt{2}}(h_n(\mathcal{R}_1)\grave{k}_m(\mathcal{R}_2) \pm \grave{k}_m(\mathcal{R}_1)h_n(\mathcal{R}_2)) \tag{6.33}$$

are the eigenfunctions of the operator $\overrightarrow{Q}^{s}_{1\,2} = Q^{s}_{1\,2}$, defined in eqs.(5.77/6.12). This leads to

$$E^{\pm h\,k}_{nm}(*) = \int \psi^{\pm}_{nm}(\mathcal{R}_1, \mathcal{R}_2)^* Q^{s}_{1\,2}\psi^{\pm}_{nm}(\mathcal{R}_1, \mathcal{R}_2) dV_2 dV_1, \tag{6.34}$$

where $\psi^{\pm}_{nm}$ is formed from the related eigenfunctions $Y_{1\,0}f^{p}_{\grave{m}}$ and $Y_{1\,\pm 1}f^{p}_{\grave{m}}$ of the Schrödinger equation (see eq.(3.18)). $E^{h}_{\acute{n}\,l}(*)$ etc. are eigenvalues of P^{s}_{1}. Hence eq.(6.16) leads to

$E^{H}_{n}(\uparrow) = E^{h}_{\acute{n}\,1}(*) + \Delta E(\acute{n}, l = 1, \uparrow)$ and

$E^{K}_{m}(\uparrow) = E^{k}_{\acute{m}\,1}(*) + \Delta E(\acute{m}, l = 1, \uparrow)$,

the eq.(6.28) reads

$$E_{n\,m}^{-HK}(\uparrow\uparrow + \uparrow\uparrow) = E_{n\,m}^{-h\,k}(*) + \tfrac{1}{2}(\Delta E(\acute{n}(h1), l = 1, \uparrow)) \\ +\Delta E(\acute{n}(h2), l = 1, \uparrow) + \Delta E(\acute{m}(k1), l = 1, \uparrow) + \Delta E(\acute{m}(k2), l = 1, \uparrow)) \tag{6.35}$$

with

$$E_{n\,m}^{-h\,k}(*) = E_{\acute{n}\,0}^{h}(*) + E_{\acute{m}\,1}^{k}(*) + {}_4({}_{nn}^{hh}M_{mm}^{kk})_4 - {}_4({}_{nm}^{hk}M_{mn}^{kh})_4. \tag{6.36}$$

In section 6.3 you find the correction terms $\Delta E(\acute{n}(h1), l = 1, \uparrow)$, etc. Eq.(6.35) fulfils the established symmetry condition. The reason is discussed later and the influence of the potentials is considered in section 6.4.

Case B:

In the next examples the spin orientation is changed and the basic functions are:

$$\begin{aligned} H_n(\downarrow) = \chi'(\mathcal{R}_1, \acute{n}, 1, \downarrow) &= \begin{Bmatrix} \hat{h}_{\acute{n}\,1}(\downarrow) \\ \breve{h}_{\acute{n}\,0}(\downarrow) \end{Bmatrix}, \\ \hat{h}_{\acute{n}\,1}(\downarrow) &= \begin{Bmatrix} Y_{1\,0}\sqrt{1/3}f_{\acute{n}}^{p}(r) \\ Y_{1\,1}\sqrt{2/3}f_{\acute{n}}^{p}(r) \end{Bmatrix} \\ K_m(\downarrow) = \chi'(\mathcal{R}_2, \acute{n}, 1, \downarrow) &= \begin{Bmatrix} \hat{k}_{\acute{m}\,1}(\downarrow) \\ \breve{k}_{\acute{m}\,1}(\downarrow) \end{Bmatrix}, \\ \hat{k}_{\acute{n}\,1}(\downarrow) &= \begin{Bmatrix} Y_{1\,0}\sqrt{1/3}f_{\acute{m}}^{p}(r) \\ Y_{1\,1}\sqrt{2/3}f_{\acute{m}}^{p}(r) \end{Bmatrix}. \end{aligned} \tag{6.37}$$

You find the meaning of the argument also in the eqs.(3.8). The functions, presented in the eqs.(6.18), read now

$$W^{\pm}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,t,\downarrow\downarrow+\downarrow\downarrow)=\tfrac{1}{2}(CD_1(t)(\Psi^{\pm}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,\downarrow\downarrow)$$
$$+CD_2(t)\Psi^{\pm}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,\downarrow\downarrow)), \qquad (6.38)$$

or

$$W^{-}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,t',\,\downarrow\downarrow+\downarrow\downarrow):=\tfrac{1}{2}(CD_1(t')+CD_2(t'))\beta\beta'\psi^{-}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2). \qquad (6.39)$$

That gives
$E^{H}_{n}(\downarrow)=E^{h}_{\acute{n},1}(*)+\Delta E(\acute{n},l=1,\downarrow)$,

$E^{K}_{m}(\downarrow)=E^{k}_{\acute{m},1}(*)+\Delta E(\acute{m},l=1,\downarrow)$ and

eq.(6.28) reads

$$E^{-HK}_{n\,m}(\downarrow\downarrow+\downarrow\downarrow)=\;E^{-h\,k}_{n\,m}(*)\;+\tfrac{1}{2}(\Delta E(\acute{n}(h1),l=1,\downarrow))$$
$$+\Delta E(\acute{n}(h2),l=1,\downarrow)+\Delta E(\acute{m}(k1),l=1,\downarrow)+\Delta E(\acute{m}(k2),l=1,\downarrow)). \qquad (6.40)$$

Case C:

Using the functions depicted in the eqs.(6.16/6.37), you can build the following expression

$$W^{-}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,t,\uparrow\downarrow+\downarrow\uparrow)=\tfrac{1}{\sqrt{2}}CD_1(t)(\Psi^{-a}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,\uparrow\downarrow)$$
$$+CD_2(t)\Psi^{-a}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,\downarrow\uparrow)) \qquad (6.41)$$

with

$$\Psi^{\pm a}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,\uparrow\downarrow)=\tfrac{1}{\sqrt{2}}(H_n(\mathcal{R}_1,\uparrow)K_m(\mathcal{R}_2,\downarrow)\pm K_m(\mathcal{R}_1,\downarrow)H_n(\mathcal{R}_2,\uparrow)),$$
$$\Psi^{\pm a}_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,\downarrow\uparrow)=\tfrac{1}{\sqrt{2}}(H_n(\mathcal{R}_1,\downarrow)K_m(\mathcal{R}_2,\uparrow)\pm K_m(\mathcal{R}_1,\uparrow)H_n(\mathcal{R}_2,\downarrow)). \qquad (6.42)$$

For this approach the equations (6.9/6.10) must be replaced by

$$U(\mathcal{R}_1,\uparrow\downarrow,t) = \frac{1}{\sqrt{2}}\sum_l c_n^l \exp(-ilwt)(H_n(\mathcal{R}_1,\uparrow) + H_n(\mathcal{R}_1,\downarrow)), \tag{6.43}$$

and

$$V(\mathcal{R}_2,\uparrow\downarrow,t) = \frac{1}{\sqrt{2}}\sum_{l1} d_m^{l1} \exp(-il1)wt)(K_m(\mathcal{R}_2,\uparrow) + K_m(\mathcal{R}_2,\downarrow)). \tag{6.44}$$

The combination of $U = H_n(\uparrow) + K_n(\uparrow)$ and $V = K_m(\downarrow) + H_m(\downarrow)$ does not make sense because of ${}_4({}^{HK\uparrow}_{nm}M^{KH\downarrow}_{mn})_4 = {}_4({}^{HK\downarrow}_{nm}M^{KH\uparrow}_{mn})_4 = 0$. However, it is possible to describe the function $W^-_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,t,\uparrow\downarrow+\downarrow\uparrow)$ by

$$W^-_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,t,\uparrow\downarrow+\downarrow\uparrow) := \tfrac{1}{\sqrt{2}}(CD_1(t)\alpha\beta' + CD_2(t)\beta\alpha')\psi^-_{n\,m}(\mathcal{R}_1,\mathcal{R}_2). \tag{6.45}$$

The eq(6.41) leads to

$$\begin{aligned} &\bar{W}^-_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,t,\uparrow\downarrow+\downarrow\uparrow)\hat{Q}_{1\,2}W^-_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,t,\uparrow\downarrow+\downarrow\uparrow) \\ &= (c_n^l d_m^{l1})^* c_n^l d_m^{l1} KK^-_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,\uparrow\downarrow+\downarrow\uparrow) + GG^-_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,\uparrow\downarrow+\downarrow\uparrow) \\ &\quad + \bar{A}J^-_{n\,m}(\uparrow\downarrow+\downarrow\uparrow) + HH^-_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,\uparrow\downarrow+\downarrow\uparrow) \end{aligned} \tag{6.46}$$

with

$$
\begin{aligned}
&KK^{\pm}_{nm}(\mathcal{R}_1,\mathcal{R}_4,\uparrow\downarrow\mp\downarrow\uparrow)\\
&=\tfrac{1}{4}(\bar{H}_n(\mathcal{R}_1,\uparrow)\acute{c}\hat{P}_1H_n(\mathcal{R}_1,\uparrow)\,\bar{K}_m(\mathcal{R}_2,\downarrow)\gamma_4^2K_m(\mathcal{R}_2,\downarrow)\\
&+\bar{K}_m(\mathcal{R}_1,\downarrow)\acute{c}\hat{P}_1K_m(\mathcal{R}_1,\downarrow)\,\bar{H}_n(\mathcal{R}_2,\uparrow)\gamma_4^2H_n(\mathcal{R}_2,\uparrow)\\
&+\bar{H}_n(\mathcal{R}_1,\uparrow)\gamma_4^1H_n(\mathcal{R}_1,\uparrow)\,\bar{K}_m(\mathcal{R}_2,\downarrow)\acute{c}\hat{P}_2K_m(\mathcal{R}_2,\downarrow)\\
&+\bar{K}_m(\mathcal{R}_1,\downarrow)\gamma_4^1K_m(\mathcal{R}_1,\downarrow)\,\bar{H}_n(\mathcal{R}_2,\uparrow)\acute{c}\hat{P}_2H_n(\mathcal{R}_2,\uparrow)\\
&+\bar{H}_n(\mathcal{R}_1,\downarrow)\acute{c}\hat{P}_1H_n(\mathcal{R}_1,\downarrow)\,\bar{K}_m(\mathcal{R}_2,\uparrow)\gamma_4^2K_m(\mathcal{R}_2,\uparrow)\\
&+\bar{K}_m(\mathcal{R}_1,\uparrow)\acute{c}\hat{P}_1K_m(\mathcal{R}_1,\uparrow)\,\bar{H}_n(\mathcal{R}_2,\downarrow)\gamma_4^2H_n(\mathcal{R}_2,\downarrow)\\
&+\bar{H}_n(\mathcal{R}_1,\downarrow)\gamma_4^1H_n(\mathcal{R}_1,\downarrow)\,\bar{K}_m(\mathcal{R}_2,\uparrow)\acute{c}\hat{P}_2K_m(\mathcal{R}_2,\uparrow)\\
&+\bar{K}_m(\mathcal{R}_1,\uparrow)\gamma_4^1K_m(\mathcal{R}_1,\uparrow)\,\bar{H}_n(\mathcal{R}_2,\downarrow)\acute{c}\hat{P}_2H_n(\mathcal{R}_2,\downarrow)),
\end{aligned}
\tag{6.47}
$$

$$
\begin{aligned}
&G^{\pm}_{nm}(\mathcal{R}_1,\mathcal{R}_2,\uparrow\downarrow\mp\uparrow\downarrow)\\
&=\mp\tfrac{1}{4}(\bar{H}_n(\mathcal{R}_1,\uparrow)\acute{c}\hat{P}_1K_m(\mathcal{R}_1,\uparrow)\,\bar{K}_m(\mathcal{R}_2,\downarrow)\gamma_4^2H_n(\mathcal{R}_2,\downarrow)\\
&+\bar{K}_m(\mathcal{R}_1,\downarrow)\acute{c}\hat{P}_1H_n(\mathcal{R}_1,\downarrow)\,\bar{H}_n(\mathcal{R}_2,\uparrow)\gamma_4^2K_m(\mathcal{R}_2,\uparrow)\\
&+\bar{H}_n(\mathcal{R}_1,\uparrow)\gamma_4^1K_m(\mathcal{R}_1,\uparrow)\,\bar{K}_m(\mathcal{R}_2,\downarrow)\acute{c}\hat{P}_2H_n(\mathcal{R}_2,\downarrow)\\
&+\bar{K}_m(\mathcal{R}_1,\downarrow)\gamma_4^1H_n(\mathcal{R}_1,\downarrow)\,\bar{H}_n(\mathcal{R}_2,\uparrow)\acute{c}\hat{P}_2K_m(\mathcal{R}_2,\uparrow)\\
&+\bar{H}_n(\mathcal{R}_1,\downarrow)\acute{c}\hat{P}_1K_m(\mathcal{R}_1,\downarrow)\,\bar{K}_m(\mathcal{R}_2,\uparrow)\gamma_4^2H_n(\mathcal{R}_2,\uparrow)\\
&+\bar{K}_m(\mathcal{R}_1,\uparrow)\acute{c}\hat{P}_1H_n(\mathcal{R}_1,\uparrow)\,\bar{H}_n(\mathcal{R}_2,\downarrow)\gamma_4^2K_m(\mathcal{R}_2,\downarrow))\\
&+\bar{H}_n(\mathcal{R}_1,\downarrow)\gamma_4^1K_m(\mathcal{R}_1,\downarrow)\,\bar{K}_m(\mathcal{R}_2,\uparrow)\acute{c}\hat{P}_2H_n(\mathcal{R}_2,\uparrow)\\
&+\bar{K}_m(\mathcal{R}_1,\uparrow)\gamma_4^1H_n(\mathcal{R}_1,\uparrow)\,\bar{H}_n(\mathcal{R}_2,\downarrow)\acute{c}\hat{P}_2K_m(\mathcal{R}_2,\downarrow)),
\end{aligned}
\tag{6.48}
$$

$$
\begin{aligned}
\bar{A}J^{\pm}(\uparrow\downarrow \mp \downarrow\uparrow)_{n\,m} = \tfrac{1}{8}\big(& {}_4({}^{H1\uparrow H1\uparrow}_{nn} AJ^{K2\downarrow K2\downarrow}_{mm})_4 \pm {}_4({}^{H1\uparrow K1\uparrow}_{nm} AJ^{K2\downarrow H2\downarrow}_{mn})_4 \\
& + {}_4({}^{H2\uparrow H2\uparrow}_{nn} AJ^{K1\downarrow K1\downarrow}_{mm})_4 \pm {}_4({}^{H2\uparrow K2\uparrow}_{nm} AJ^{K1\downarrow H1\downarrow}_{mn})_4 \\
& + {}_4({}^{K1\uparrow K1\uparrow}_{mm} AJ^{H2\downarrow H2\downarrow}_{nn})_4 \pm {}_4({}^{K1\uparrow H1\uparrow}_{mn} AJ^{H2\downarrow K2\downarrow}_{nm})_4 \\
& + {}_4({}^{K2\uparrow K2\uparrow}_{mm} AJ^{H1\downarrow H1\downarrow}_{nn})_4 \pm {}_4({}^{K2\uparrow H2\uparrow}_{mn} AJ^{H1\downarrow K1\downarrow}_{nm})_4 \\
& + {}_4({}^{H1\downarrow H1\downarrow}_{nn} AJ^{K2\uparrow K2\uparrow}_{mm})_4 \pm {}_4({}^{H1\uparrow K1\uparrow}_{nm} AJ^{K2\uparrow H2\uparrow}_{mn})_4 \\
& + {}_4({}^{H2\downarrow H2\downarrow}_{nn} AJ^{K1\uparrow K1\uparrow}_{mm})_\mu \pm {}_4({}^{H2\downarrow K2\downarrow}_{nm} AJ^{K1\uparrow H1\uparrow}_{mn})_4 \\
& + {}_4({}^{K1\downarrow K1\downarrow}_{mm} AJ^{H2\uparrow H2\uparrow}_{nn})_4 \pm {}_4({}^{K1\downarrow H1\downarrow}_{mn} AJ^{H2\uparrow K2\uparrow}_{nm})_4 \\
& + {}_4({}^{K2\downarrow K2\downarrow}_{mm} AJ^{H1\uparrow H1\uparrow}_{nn})_4 \pm {}_4({}^{K2\downarrow H2\downarrow}_{mn} AJ^{H1\uparrow K1\uparrow}_{nm})_4\big)
\end{aligned}
\tag{6.49}
$$

and

$$
\begin{aligned}
& HH^{\pm}_{n\,m}(\mathcal{R}_1, \mathcal{R}_2, \uparrow\downarrow \mp \downarrow\uparrow) \\
&= -\tfrac{1}{2}(l + l1)w\hbar(\bar{\Psi}^{\pm a}_{n\,m}(\mathcal{R}_1, \mathcal{R}_2, \uparrow\downarrow)\gamma^1_4\gamma^2_4\Psi^{\pm a}_{n\,m}(\mathcal{R}_1, \mathcal{R}_2, \uparrow\downarrow) \\
&\quad + \bar{\Psi}^{\pm a}_{n\,m}(\mathcal{R}_1, \mathcal{R}_2, \downarrow\uparrow)\gamma^1_4\gamma^2_4\Psi^{\pm a}_{n\,m}(\mathcal{R}_1, \mathcal{R}_2, \downarrow\uparrow)).
\end{aligned}
\tag{6.50}
$$

Here we have also introduced the expressions for $W^{+}_{n\,m}(\mathcal{R}_1, \mathcal{R}_2, t, \uparrow\downarrow - \downarrow\uparrow)$. The integration yields

$$\begin{gathered}\int KK^-_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,\uparrow\downarrow+\downarrow\uparrow)dV_1dV_2\\ =\tfrac{1}{2}(\bar{E}^{H1}_n(\uparrow)+\bar{E}^{H2}_n(\downarrow)+\bar{E}^{K1}_m(\uparrow)+\bar{E}^{K2}_m(\downarrow)),\end{gathered}$$

$$\int GG^-_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,\uparrow\downarrow+\downarrow\uparrow)dV_1dV_2=0,$$

$$\begin{gathered}\bar{M}^-(\uparrow\downarrow+\downarrow\uparrow)_{n\,m}=\int\bar{A}J^-(\uparrow\downarrow+\downarrow\uparrow)dV_1dV_2\\ =\tfrac{1}{4}(\,{}_4({}^{H1\uparrow H1\uparrow}_{nn}M^{K2\downarrow K2\downarrow}_{mm})_4-\,{}_4({}^{H1\uparrow K1\uparrow}_{nm}M^{K2\downarrow H2\downarrow}_{mn})_4\\ +{}_4({}^{H2\uparrow H2\uparrow}_{nn}M^{K1\downarrow K1\downarrow}_{mm})_4-\,{}_\mu({}^{H2\uparrow K2\uparrow}_{nm}M^{K1\downarrow H1\downarrow}_{mn})_4\\ +{}_4({}^{K1\uparrow K1\uparrow}_{mm}M^{H2\downarrow H2\downarrow}_{nn})_4-\,{}_4({}^{K1\uparrow H1\uparrow}_{mn}M^{H2\downarrow K2\downarrow}_{nm})_4\\ +{}_4({}^{K2\uparrow K2\uparrow}_{mm}M^{H1\downarrow H1\downarrow}_{nn})_4-\,{}_4({}^{K2\uparrow H2\uparrow}_{mn}M^{H1\downarrow K1\downarrow}_{nm})_4).\end{gathered} \tag{6.51}$$

Therefore, integrating eq.(6.46), one obtains the eigenvalue equation

$$\begin{gathered}\int\bar{W}^-_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,t,\uparrow\downarrow+\downarrow\uparrow)\hat{Q}_{1\,2}W^-_{n\,m}(\mathcal{R}_1,\mathcal{R}_2,t,\uparrow\downarrow+\downarrow\uparrow)dV_1dV_2\\ =(c^l_nd^{l1}_m)^*c^l_nd^{l1}_m(\bar{E}^{-HK}_{n\,m}(\uparrow\downarrow+\downarrow\uparrow)-\,(l+l1)w\hbar\,).\end{gathered} \tag{6.52}$$

That means, for

$$\bar{E}^{-HK}_{n\,m}(\uparrow\downarrow+\downarrow\uparrow)=\,(l+l1)w\hbar. \tag{6.53}$$

the integral becomes zero. Here is

$$\begin{gathered}\bar{E}^{-HK}_{n\,m}(\uparrow\downarrow+\downarrow\uparrow)=\,\tfrac{1}{2}(\bar{E}^{H1}_n(\uparrow)+\bar{E}^{K1}_m(\downarrow)\\ +\bar{E}^{H2}_n(\downarrow)+\bar{E}^{K2}_m(\uparrow))+\bar{M}^-_{n\,m}(\uparrow\downarrow+\downarrow\uparrow)\end{gathered} \tag{6.54}$$

and due to the eqs.(6.34/6.41) it becomes

$E^{H}_{n}(\uparrow) = E^{h}_{\acute{n}\,0}(*) + \Delta E(\acute{n}, l = 1, \uparrow)$, $E^{H}_{n}(\downarrow) = E^{h}_{\acute{n}\,0}(*) + \Delta E(\acute{n}, l = 1, \downarrow)$ and

$E^{K}_{m}(\uparrow) = E^{k}_{\acute{m}\,1}(*) + \Delta E(\acute{m}, l = 1, \uparrow)$ and $E^{K}_{m}(\downarrow) = E^{k}_{\acute{m}\,1}(*) + \Delta E(\acute{m}, l = 1, \downarrow)$.

Using the eqs.(6.47-6.50), the eigenvalue energy reads

$$\begin{gathered} E^{-HK}_{n\,m}(\uparrow\downarrow + \downarrow\uparrow) = \; E^{-h\,k}_{n\,m}(*) \; + \tfrac{1}{4}(\Delta E(\acute{n}(h1), l = 1, \uparrow) \\ + \Delta E(\acute{m}(k1), l = 1, \downarrow) + \Delta E(\acute{m}(k2), l = 1, \downarrow) + \Delta E(\acute{n}(h2), l = 1, \uparrow) \\ + \Delta E(\acute{n}(h1), l = 1, \downarrow)) + \Delta E(\acute{m}(k1), l = 1, \uparrow) \\ + \Delta E(\acute{m}(k2), l = 1, \uparrow) + \Delta E(\acute{n}(h2), l = 1, \downarrow)). \end{gathered} \tag{6.55}$$

or

$$\begin{gathered} E^{-HK}_{n\,m}(\uparrow\downarrow + \downarrow\uparrow) = \; E^{-h\,k}_{n\,m}(*) \; + \tfrac{1}{2}(\Delta E(\acute{n}(h), l = 1, \uparrow) \\ + \Delta E(\acute{m}(k), l = 1, \downarrow) + \Delta E(\acute{m}(k), l = 1, \uparrow) + \Delta E(\acute{n}(h), l = 1, \downarrow)) \end{gathered} \tag{6.56}$$

The correction terms remain equal for $E^{+HK}_{n\,m}(\uparrow\downarrow + \downarrow\uparrow)$.

Case D:

Finally we replace eq.(6.41) by

$$\begin{gathered} W^{+}_{n\,m}(\mathcal{R}_1, \mathcal{R}_2, t, \uparrow\downarrow - \downarrow\uparrow) = \tfrac{1}{\sqrt{2}} CD_1(t)(\Psi^{+a}_{n\,m}(\mathcal{R}_1, \mathcal{R}_2, \uparrow\downarrow) \\ - CD_2(t)\Psi^{+a}_{n\,m}(\mathcal{R}_1, \mathcal{R}_2, \downarrow\uparrow)) \end{gathered} \tag{6.57}$$

or

$$W^+_{nm}(\mathcal{R}_1,\mathcal{R}_2,t',\uparrow\downarrow-\downarrow\uparrow) := \tfrac{1}{\sqrt{2}}(CD_1(t')\alpha\beta' - CD_2(t')\beta\alpha')\psi^+_{nm}(\mathcal{R}_1,\mathcal{R}_2). \quad (6.58)$$

Then, analogously to eq.(6.46), it reads

$$\begin{gathered}\bar{W}^+_{nm}(\mathcal{R}_1,\mathcal{R}_2,t,\uparrow\downarrow-\downarrow\uparrow)\hat{Q}_{12}W^+_{nm}(\mathcal{R}_1,\mathcal{R}_2,t,\uparrow\downarrow-\downarrow\uparrow)\\ =(c^l_n d^{l1}_m)^* c^l_n d^{l1}_m(KK^+_{nm}(\mathcal{R}_1,\mathcal{R}_2,t,\uparrow\downarrow-\downarrow\uparrow)+GG^+_{nm}(\mathcal{R}_1,\mathcal{R}_2,t,\uparrow\downarrow-\downarrow\uparrow)\\ +\bar{A}J^+_{nm}(\uparrow\downarrow-\downarrow\uparrow)+HH^+_{nm}(\mathcal{R}_1,\mathcal{R}_2,\uparrow\downarrow-\downarrow\uparrow)).\end{gathered} \quad (6.59)$$

One finds the functions KK^+_{nm} ect. also in the eqs (6.47-6.50) and one obtains by integration of eq.(6.59) the relation

$$\begin{gathered}\int\bar{W}^+_{nm}(\mathcal{R}_1,\mathcal{R}_2,t,\uparrow\downarrow-\downarrow\uparrow)\hat{Q}_{12}W^+_{nm}(\mathcal{R}_1,\mathcal{R}_2,t,\uparrow\downarrow-\downarrow\uparrow)dV_1dV_2\\ =(c^l_n d^{l1}_m)^* c^l_n d^{l1}_m(\bar{E}^{+HK}_{nm}(\uparrow\downarrow-\downarrow\uparrow)-(l+l1)w\hbar).\end{gathered} \quad (6.60)$$

This expression is zero for

$$\bar{E}^{+HK}_{nm}(\uparrow\downarrow-\downarrow\uparrow) = (l+l1)w\hbar \quad (6.61)$$

with

$$\begin{gathered}E^{+HK}_{nm}(\uparrow\downarrow-\downarrow\uparrow) = E^{+hk}_{nm}(*) + \tfrac{1}{2}(\Delta E(\acute{n}(h),l=1,\uparrow)\\ +\Delta E(\acute{n}(h),l=1,\downarrow)+\Delta E(\acute{m}(k),l=1,\downarrow)+\Delta E(\acute{m}(k),l=1,\uparrow)).\end{gathered} \quad (6.62)$$

For $E^{-HK}_{nm}(\uparrow\downarrow - \downarrow\uparrow)$ one obtain equal correction terms. The values $\bar{E}^{+HK}_{nm}()$ or $\bar{E}^{-HK}_{nm}()$ require different frequencies $(l + l1)w/(2\pi)$ to fulfil eq.(6.53) or eq.(6.61). If a frequency in one integral of eq.(6.4) meet this condition, the other integral does not contribute to the radiation. Besides, it is reasonable to assume that only one synchronous oscillation is possible during the same time.

A strong mathematical argumentation for the choice of the sign in $W^{\pm}_{nm}()$ is missing yet. However, some completions exhibit connections. For this reason we add in the operators of the transition process Q^4_{12} (see eqs.(5.41/5.42)) the potentials of the radiation fields: $\breve{A}_{\bar{\mu}}({}^{zo}_{nn'}J_{\bar{\mu}}),\ \breve{A}_4({}^{zo}_{nn'}J_4) = {}^{zo}A_4(\mathcal{R}_i) - {}^{zo}A'_4(\mathcal{R}_i) - ()$ (see eqs.(4.52/4.53)). It depends on the spin orientation. With the modified operators and the functions

$$\begin{aligned} W^{\pm}_{nm}(\mathcal{R}_1, \mathcal{R}_2, \uparrow\uparrow + \uparrow\uparrow)' &= \Psi^{\pm}_{nm}(\mathcal{R}_1, \mathcal{R}_2, \uparrow\uparrow), \\ \bar{W}^{\pm}_{n'm}(\mathcal{R}_1, \mathcal{R}_2, \uparrow\uparrow + \uparrow\uparrow)' &= \bar{\Psi}^{\pm}_{n'm}(\mathcal{R}_1, \mathcal{R}_2, \uparrow\uparrow), \end{aligned} \tag{6.63}$$

we form the expression

$$\begin{aligned} {}^{HK}_{n'm}I^{\pm HK}_{nm}(\uparrow\uparrow + \uparrow\uparrow) = \textstyle\int \bar{W}^{\pm}_{n'm}(\mathcal{R}_1, \mathcal{R}_2, \uparrow\uparrow + \uparrow\uparrow)'(\overrightarrow{Q}^4_{12} \\ + \overleftarrow{Q}^4_{12}) W^{\pm}_{nm}(\mathcal{R}_1, \mathcal{R}_2, t, \uparrow\uparrow + \uparrow\uparrow)' dV_2 \\ + i\acute{e}\bar{H}_{n'}(\mathcal{R}_1, \uparrow)\gamma_\mu H_n(\mathcal{R}_1, \uparrow)\breve{A}_\mu(\mathcal{R}_1, {}^{zo}_{nn'}J_\mu(\mathcal{R}_2, \uparrow\uparrow)) \\ + i\acute{e}\bar{H}_{n'}(\mathcal{R}_1, \uparrow)\gamma_\mu H_n(\mathcal{R}_1, \uparrow)\breve{A}_\mu(\mathcal{R}_1, {}^{oz}_{n'n}J_\mu(\mathcal{R}_2, \uparrow\uparrow)). \end{aligned} \tag{6.64}$$

That reads (case a)

$$
\begin{aligned}
{}^{H\,K}_{n'\,m}I^{\pm H\,K}_{n\,m}(\uparrow\uparrow+\uparrow\uparrow) = \bar{H}_{n'}(\mathcal{R}_1,\uparrow)\gamma_4 H_n(\mathcal{R}_1,\uparrow)(\,E^{\pm h\,k}_{n\,m}(*)\;+\Delta E(\acute{n}(h),l,\uparrow))\\
+i\acute{e}\bar{H}_{n'}(\mathcal{R}_1,\uparrow)\gamma_\mu H_n(\mathcal{R}_1,\uparrow)\breve{A}_\mu({}^{z\,o}_{n\,n'}\breve{J}_\mu(\mathcal{R}_2,\uparrow\uparrow))\\
-(\,E^{\pm h\,k}_{n'\,m}(*)\;+\Delta E(\acute{n}'(h),l,\uparrow))\bar{H}_{n'}(\mathcal{R}_1,\uparrow)\gamma_4 H_n(\mathcal{R}_1,\uparrow)\\
+i\acute{e}\bar{H}_{n'}(\mathcal{R}_1,\uparrow)\gamma_\mu H_n(\mathcal{R}_1,\uparrow)\breve{A}_\mu(\mathcal{R}_1,{}^{o\,z}_{n'\,n}J_\mu(\mathcal{R}_2,\uparrow\uparrow)).
\end{aligned}
\tag{6.65}
$$

Similar considerations result in (case b)

$$
\begin{aligned}
{}^{H\,K}_{n'\,m}I^{\pm H\,K}_{n\,m}(\downarrow\downarrow+\downarrow\downarrow) = \bar{H}_{n'}(\mathcal{R}_1,\downarrow)\gamma_4 H_n(\mathcal{R}_1,\downarrow)(\,E^{\pm h\,k}_{n\,m}(*)\\
+\Delta E(\acute{n}(h),l,\downarrow))+i\acute{e}\bar{H}_{n'}(\mathcal{R}_1,\downarrow)\gamma_\mu H_n(\mathcal{R}_1,\downarrow)\breve{A}_\mu({}^{z\,o}_{n\,n'}\breve{J}_\mu(\mathcal{R}_2,\downarrow\downarrow))\\
-(\,E^{\pm h\,k}_{n'\,m}(*)\;+\Delta E(\acute{n}'(h),l,\downarrow))\bar{H}_{n'}(\mathcal{R}_1,\downarrow)\gamma_4 H_n(\mathcal{R}_1,\downarrow)\\
+i\acute{e}\bar{H}_{n'}(\mathcal{R}_1,\downarrow)\gamma_\mu H_n(\mathcal{R}_1,\downarrow)\breve{A}_\mu(\mathcal{R}_1,{}^{o\,z}_{n'\,n}J_\mu(\mathcal{R}_2,\downarrow\downarrow)).
\end{aligned}
\tag{6.66}
$$

In addition, one obtains

$$
\begin{aligned}
{}^{H\,K}_{n'\,m}I^{\pm H\,K}_{n\,m}(\uparrow\downarrow\mp\downarrow\uparrow)\\
=\tfrac{1}{2}\bar{H}_{n'}(\uparrow)\gamma_4 H_n(\uparrow)(\,E^{\pm h\,k}_{n\,m}(*)\;+\Delta E(\acute{n}(h),l,\uparrow))\\
+\tfrac{i}{2}\acute{e}\bar{H}_{n'}(\uparrow)\gamma_\mu H_n(\uparrow)\breve{A}_\mu({}^{z\,o}_{n\,n'}J_\mu(\downarrow\downarrow))\\
+\tfrac{1}{2}\bar{H}_n(\downarrow)\gamma_4 H_n(\downarrow)(\,E^{\pm h\,k}_{n\,m}(*)\;+\Delta E(\acute{n}(h),l,\downarrow))\\
+\tfrac{i}{2}\acute{e}\bar{H}_{n'}(\downarrow)\gamma_\mu H_n(\downarrow)\breve{A}_\mu({}^{z\,o}_{n\,n'}J_\mu(\uparrow\uparrow))\\
-\tfrac{1}{2}\bar{H}_{n'}(\uparrow)\gamma_4 H_n(\uparrow)(\,E^{\pm h\,k}_{n'\,m}(*)\;+\Delta E(\acute{n}'(h),l,\uparrow))\\
+\tfrac{i}{2}\acute{e}\bar{H}_{n'}(\uparrow)\gamma_\mu H_n(\uparrow)\breve{A}_\mu({}^{o\,z}_{n'\,n}J_\mu(\downarrow\downarrow))\\
-\tfrac{1}{2}\bar{H}_{n'}(\downarrow)\gamma_4 H_n(\downarrow)(\,E^{\pm h\,k}_{n'\,m}(*)\;+\Delta E(\acute{n}'(h),l,\downarrow))\\
+\tfrac{i}{2}\acute{e}\bar{H}_{n'}(\downarrow\;\gamma_\mu H_n(\downarrow)\breve{A}_\mu({}^{o\,z}_{n'\,n}J_\mu(\uparrow\uparrow))
\end{aligned}
\tag{6.67}
$$

and, because of
$\frac{i}{2}\acute{e}\bar{H}_{n'}(\uparrow)\gamma_\mu H_n(\uparrow)\breve{A}_\mu({}^{z\,o}_{n\,n'}J_\mu(\downarrow\downarrow)) = -\frac{i}{2}\acute{e}\bar{H}_{n'}(\uparrow)\gamma_\mu H_n(\uparrow)\breve{A}_\mu({}^{z\,o}_{n\,n'}J_\mu(\uparrow\uparrow))$ etc.,

$$
\begin{aligned}
&{}^{H\,K}_{n'\,m}I^{\pm H\,K}_{n\,m}(\uparrow\downarrow \mp \downarrow\uparrow)\\
&= \tfrac{1}{2}\bar{H}_{n'}(\uparrow)\gamma_4 H_n(\uparrow)(\ E^{\pm h\,k}_{n\,m}(*)\ + \Delta E(\acute{n}(h), l, \uparrow))\\
&\quad -\tfrac{i}{2}\acute{e}\bar{H}_{n'}(\uparrow)\gamma_\mu H_n(\uparrow)\breve{A}_\mu({}^{z\,o}_{n\,n'}J_\mu(\uparrow\uparrow))\\
&+\tfrac{1}{2}\bar{H}_n(\downarrow)\gamma_4 H_n(\downarrow)(\ E^{\pm h\,k}_{n\,m}(*)\ \ + \Delta E(\acute{n}(h), l, \downarrow))\\
&\quad -\tfrac{i}{2}\acute{e}\bar{H}_{n'}(\downarrow)\gamma_\mu H_n(\downarrow)\breve{A}_\mu({}^{z\,o}_{n\,n'}J_\mu(\downarrow\downarrow))\\
&-\tfrac{1}{2}\bar{H}_{n'}(\uparrow)\gamma_4 H_n(\uparrow)(\ E^{\pm h\,k}_{n'\,m}(*)\ + \Delta E(\acute{n}'(h), l, \uparrow))\\
&\quad -\tfrac{i}{2}\acute{e}\bar{H}_{n'}(\uparrow)\gamma_\mu H_n(\uparrow)\breve{A}_\mu({}^{o\,z}_{n'\,n}J_\mu(\uparrow\uparrow))\\
&-\tfrac{1}{2}\bar{H}_{n'}(\downarrow)\gamma_4 H_n(\downarrow)(\ E^{\pm h\,k}_{n'\,m}(*)\ \ + \Delta E(\acute{n}'(h), l, \downarrow))\\
&\quad -\tfrac{i}{2}\acute{e}\bar{H}_{n'}(\downarrow\ \gamma_\mu H_n(\downarrow)\breve{A}_\mu({}^{o\,z}_{n'\,n}J_\mu(\downarrow\downarrow)).
\end{aligned}
\tag{6.68}
$$

The eqs.(6.65/6.66) shows equal signs in front of $\Delta E()$ and $\frac{i}{2}\acute{e}\bar{H}_{n'}()\gamma_\mu H_n()\breve{A}_\mu({}^{z\,o}_{n\,n'}J_\mu()$ for the terms with the same orientation, however, the terms in eq.(6.68) have opposite signs. Maybe this is a hint for the possible radiation frequencies. It could mean that radiation with the frequency, given in eq.(6.60), will only be excited if the denoted terms have opposite sign in ${}^{H\,K}_{n'\,m}I^{+H\,K}_{n\,m}()$. The value

$$
\bar{E}^{-HK}_{n\,m}() = \ (l + l1)w\hbar \tag{6.69}
$$

should be possible in all other cases. Then we get the the desired results. However, this consideration does not exactly explain the fine structure. That requires more knowledge about the solutions and the excitation mechanism of the radiation process. Finally we present the results assuming this hypothesis:

Case A

$$
\begin{aligned}
E^{-HK}_{n\,m}(\uparrow\uparrow + \uparrow\uparrow) = \ & E^{-h\,k}_{n\,m}(*)\ + \Delta E(\acute{n}(h), l = 1, \uparrow)\\
& +\Delta E(\acute{m}(k), l = 1, \uparrow);
\end{aligned}
\tag{6.70}
$$

Case B

$$E^{-HK}_{n\,m}(\downarrow\downarrow + \downarrow\downarrow) = E^{-h\,k}_{n\,m}(*) + \Delta E(\acute{n}(h), l = 1, \downarrow)$$
$$+\Delta E(\acute{m}(k), l = 1, \downarrow); \qquad (6.71)$$

Case C:

$$E^{-HK}_{n\,m}(\uparrow\downarrow + \downarrow\uparrow) = E^{-h\,k}_{n\,m}(*) + \tfrac{1}{2}(\Delta E(\acute{n}(h), l = 1, \uparrow)$$
$$+\Delta E(\acute{n}(h), l = 1, \downarrow) + \Delta E(\acute{m}(k), l = 1, \downarrow) + \Delta E(\acute{m}(k), l = 1, \uparrow); \qquad (6.72)$$

Case D:

$$E^{+HK}_{n\,m}(\uparrow\downarrow - \downarrow\uparrow) = E^{+h\,k}_{n\,m}(*) + \tfrac{1}{2}(\Delta E(\acute{n}(h), l = 1, \uparrow)$$
$$+\Delta E(\acute{n}(h), l = 1, \downarrow) + \Delta E(\acute{m}(k), l = 1, \downarrow) + \Delta E(\acute{m}(k), l = 1, \uparrow). \qquad (6.73)$$

6.3 *The calculation of the correction terms*

We transform the eq.(2.4) with $\xi = U = H_n$ or $\xi = U = H_n$ to determine $E^h_n(\uparrow)$ or $E^k_m(\uparrow)$ in the known way and write

$$(E^h_n + i\acute{e}A^K_4)\hat{\xi} = \acute{c}\overrightarrow{\sigma}(-i\hbar)\,\frac{\partial\,\check{\xi}}{\partial\,x_\mu},$$
$$(E^h_n + i\acute{e}A^K_4 + 2\acute{M}\acute{c}^2)\check{\xi} = \acute{c}\overrightarrow{\sigma}(-i\hbar)\,\frac{\partial\,\hat{\xi}}{\partial\,x_\mu}. \qquad (6.74)$$

$A^K_4 = -2i\acute{e}/r$ represents the electrostatic potential of the nucleus and $\overrightarrow{\sigma} = (\sigma_1,\, \sigma_2,\, \sigma_3)$ is formed with Pauli matrices. This leads to (see [18])

$$\check{\xi} = (1 - \tfrac{E^h_n + i\acute{e}A^K_4}{2\acute{M}\acute{c}^2})\acute{c}\overrightarrow{\sigma}(\tfrac{-i\hbar}{2\acute{M}\acute{c}})\,\frac{\partial\,\hat{\xi}}{\partial\,x_\mu} \qquad (6.75)$$

and therefore becomes

$$(E_n^h + i\acute{e}A_4^K)\hat{\xi} = -\frac{\hbar^2}{2\acute{M}}(1 - \frac{E_n^h + i\acute{e}A_4^K}{2\acute{M}\acute{c}^2})\sigma_{\bar{\mu}}\sigma_{\bar{\nu}}\frac{\partial}{\partial x_{\bar{\mu}}}\frac{\partial \hat{\xi}}{\partial x_{\bar{\mu}}}$$
$$+\frac{\hbar^2}{4\acute{M}^2\acute{c}^2}\sigma_{\bar{\mu}}\frac{\partial}{\partial x_{\bar{\mu}}}i\acute{e}A_4^K\sigma_{\bar{\nu}}\frac{\partial}{\partial x_{\bar{\nu}}}\hat{\xi} \quad (6.76)$$

or (in eq.(6.35): $E_n^h(*, j = 1/2) = E_n^h(*)_0 + \Delta\, E_n^h(*)_0$)

$$(E_n^h(*)_0 + \Delta\, E_n^h(*)_0 + \Delta\, E_n^h(\uparrow) + i\acute{e}A_4)\hat{\xi}_j = -\frac{\hbar^2}{2\acute{M}}\frac{\partial}{\partial x_{\bar{\mu}}}\frac{\partial \hat{\xi}_n}{\partial x_{\bar{\mu}}}I$$
$$+\hbar^2\frac{E_n^h + i\acute{e}A_4^K}{4\acute{M}^2\acute{c}^2}\frac{\partial}{\partial x_{\bar{\mu}}}\frac{\partial \hat{\xi}_n}{\partial x_{\bar{\mu}}} + \frac{i\acute{e}\hbar^2}{4M^2c^2}\frac{\partial A_4^K}{\partial x_{\bar{\mu}}}\frac{\partial \hat{\xi}_n}{\partial x_{\bar{\mu}}}$$
$$-\frac{\acute{e}\hbar^2\sigma_3}{4M^2c^2}\frac{\partial A_4^K}{\partial x_1}\frac{\partial \hat{\xi}_n}{\partial x_2} + \frac{\acute{e}\hbar^2\sigma_3}{4M^2c^2}\frac{\partial A_4^K}{\partial x_2}\frac{\partial \hat{\xi}_n}{\partial x_1} \quad (6.77)$$
$$+\frac{\acute{e}\hbar^2\sigma_2}{4M^2c^2}\frac{\partial A_4^K}{\partial x_1}\frac{\partial \hat{\xi}_n}{\partial x_3} - \frac{\acute{e}\hbar^2\sigma_2}{4M^2c^2}\frac{\partial A_4^K}{\partial x_3}\frac{\partial \hat{\xi}_n}{\partial x_1}$$
$$-\frac{\acute{e}\hbar^2\sigma_1}{4M^2c^2}\frac{\partial A_4^K}{\partial x_2}\frac{\partial \hat{\xi}_n}{\partial x_3} + \frac{\acute{e}\hbar^2\sigma_1}{4M^2c^2}\frac{\partial A_4^K}{\partial x_3}\frac{\partial \hat{\xi}_n}{\partial x_2}.$$

Here the relations

$$\sigma_{\bar{\nu}}\sigma_{\bar{\mu}}\frac{\partial}{\partial x_{\bar{\nu}}}\frac{\partial \hat{\xi}_n}{\partial x_{\bar{\mu}}} = \frac{\partial}{\partial x_{\bar{\mu}}}\frac{\partial \hat{\xi}_n}{\partial x_{\bar{\mu}}}I, \quad (6.78)$$

$$i\sigma_{\bar{\mu}}\sigma_{\bar{\nu}}\frac{\partial A_4^K}{\partial x_{\bar{\mu}}}\frac{\partial \hat{\xi}_n}{\partial x_{\bar{\nu}}} = i\frac{\partial A_4^K}{\partial x_{\bar{\mu}}}\frac{\partial \hat{\xi}_n}{\partial x_{\bar{\mu}}}I$$
$$-\sigma_1(\frac{\partial A_4^K}{\partial x_2}\frac{\partial \hat{\xi}_n}{\partial x_3} - \frac{\partial A_4^K}{\partial x_3}\frac{\partial \hat{\xi}_n}{\partial x_2})$$
$$+\sigma_2(\frac{\partial A_4^K}{\partial x_1}\frac{\partial \hat{\xi}_n}{\partial x_3} - \frac{\partial A_4^K}{\partial x_3}\frac{\partial \hat{h}_n}{\partial x_1}) \quad (6.79)$$
$$-\sigma_3(\frac{\partial A_4^K}{\partial x_1}\frac{\partial \hat{\xi}_n}{\partial x_2} - \frac{\partial A_4^K}{\partial x_2}\frac{\partial \hat{\xi}_n}{\partial x_1})$$

are used. The expressions $\Delta\, E_n^h(*)_0$ and $\Delta\, E_n^h(\uparrow)$ describe the correction of the energy eigenvalues E_n^h of the Schrödinger equation (I:=1)

$$(E_n^h(*)_0 + i\acute{e}A_4^K)h_n = -\frac{\hbar^2}{2\acute{M}}\frac{\partial}{\partial\, x_{\bar{\mu}}}\frac{\partial\, h_n}{\partial\, x_{\bar{\mu}}}. \tag{6.80}$$

After multiplying by $(\hat{\xi}_n)^* = \hat{H}_n$ and integration, the eq.(6.77) can be decomposed into

$$E_n^h(*)_0 = -\frac{\hbar^2}{2\acute{M}} \int (h_n)^* \frac{\partial}{\partial\, x_{\bar{\mu}}}\frac{\partial\, h_n}{\partial\, x_{\bar{\mu}}} dV - i\acute{e} \int A_4^K (h_n)^* h_n dV \tag{6.81}$$

and (see [18] eq.(70.14))

$$\Delta E(\acute{n}, l = j(l), \uparrow) = -\frac{8\acute{\alpha}^2}{\acute{n}^3}(\frac{1}{j+1/2} - \frac{3}{4\acute{n}}). \tag{6.82}$$

For $\phi'(\mathcal{R}_1, \acute{n}, l, \breve{m} \pm 1/2, \uparrow)$ is j=l-1/2 and therefore

$$\Delta E(\acute{n}, l, \uparrow) = -\frac{8\acute{\alpha}^2}{\acute{n}^3}(\frac{1}{l} - \frac{3}{4\acute{n}}),\ \ l \neq 0, \tag{6.83}$$

while for $\chi'(\mathcal{R}_1, \acute{n}, l, \breve{m} = \pm 1/2, \downarrow)$ is j=l+1/2 and

$$\begin{array}{c} \Delta E(\acute{n}, l, \downarrow) = -\frac{8\acute{\alpha}^2}{\acute{n}^3}(\frac{1}{l+1} - \frac{3}{4\acute{n}})\ and \\ \Delta E(\acute{n}, l = 0, *) = -\frac{8\acute{\alpha}^2}{\acute{n}^3}(1 - \frac{3}{4\acute{n}}). \end{array} \tag{6.84}$$

6.4 *The potential integrals*

The charge components related to the eqs.(6.15), have the form

$$\begin{aligned} J_4^{HH}(*) &= i\acute{e} f_n^s(r)^2 (Y_{00})^2, \\ J_4^{HK}(\uparrow) &= i\acute{e}\sqrt{\tfrac{2}{3}}(f_n^s(r) f_m^p(r) Y_{00} Y_{10} \;=\; J_4^{HK}(\uparrow), \\ J_4^{KK}(\uparrow) &= i\acute{e} f_m^p(r)^2 (\tfrac{2}{3}(Y_{10})^2 \;+\; \tfrac{1}{3}(Y_{11})^* Y_{11}) \\ &= i\acute{e} f_m^p(r)^2((Y_{00})^2 \;+\; \tfrac{1}{\sqrt{5}} Y_{00} Y_{20}). \end{aligned} \tag{6.85}$$

Therefore becomes

$$\begin{aligned} {}_4({}^{HH}_{nn} M^{KK}_{mm})_4 &= (Y_{00})^4 \textstyle\int\int \frac{1}{r_{00'}} (f_n^p(r)^2 f_m^p(r')^2 dV dV' \\ &= \textstyle\int_0^\infty r^2 f_n^s(r) f_m^p(r) (\frac{1}{r}\int_0^r (r')^2 f_n^s(r') f_m^p(r') dr' + \int_r^\infty r' f_m^p(r') f_n^s(r') dr') dr \\ {}_4({}^{HK}_{nm} M^{KH}_{mn})_4 &= {}_4({}^{KH}_{mn} M^{HK}_{nm})_4 = \\ &\tfrac{2}{3}(Y_{00})^2 \textstyle\int\int \frac{1}{r_{00'}} (f_n^s(r) f_m^p(r) Y_{10}(\delta,\varphi)\,(f_m^p(r') f_n^s(r') Y_{10}(\delta',\varphi') dV dV' \\ &= \tfrac{2}{9}\textstyle\int_0^\infty r^2 f_n^s(r) f_m^p(r) (\frac{1}{r^2}\int_0^r (r')^3 f_n^s(r') f_m^p(r') dr' + r\int_r^\infty f_m^p(r') f_n^s(r') dr') dr. \end{aligned} \tag{6.86}$$

Using the eqs.(6.16), it reads

$$\begin{aligned} J_4^{HH}(\uparrow) &= i\acute{e} f_n^p(r)^2((Y_{00})^2 \;+\; \tfrac{1}{\sqrt{5}} Y_{00} Y_{20}), \\ J_4^{HK}(\uparrow) &= i\acute{e} f_n^p(r) f_m^p(r)((Y_{00})^2 \;+\; \tfrac{1}{\sqrt{5}} Y_{00} Y_{20}) = J_4^{KH}(\uparrow), \\ J_4^{KK}(\uparrow) &= i\acute{e} f_m^p(r)^2((Y_{00})^2 \;+\; \tfrac{1}{\sqrt{5}} Y_{00} Y_{20}). \end{aligned} \tag{6.87}$$

The eqs.(6.37) give the same results. That means, the orientation has no influence on the potential integrals and we can write in the cases A, B, C, D

$$
\begin{aligned}
{}_4({}^{HH}_{nn}M^{KK}_{mm})_4 &= (Y_{0\,0})^4 \int\int \frac{1}{r_{0\,0'}}(f^p_n(r)^2 f^p_m(r')^2 dV dV' \\
&+\tfrac{1}{5}(Y_{0\,0})^2 \int\int \frac{1}{r_{0\,0'}}(f^p_n(r)^2 Y_{2\,0}(\delta,\varphi)\,(f^p_m(r')^2 Y_{2\,0}(\delta',\varphi')dV dV') \\
&= \int_0^\infty r^2 f^s_n(r) f^p_m(r)(\tfrac{1}{r}\int_0^r\ (r')^2 f^s_n(r') f^p_m(r')dr' + \int_r^\infty r' f^p_m(r') f^s_n(r')dr')dr \\
&+\tfrac{1}{25}\int_0^\infty r^2 f^s_n(r) f^p_m(r)(\tfrac{1}{r^3}\int_0^r\ (r')^4 f^s_n(r') f^p_m(r')dr' + r^2\int_r^\infty \tfrac{1}{r'} f^p_m(r') f^s_n(r')dr')dr \\
&= \ {}_4({}^{KH}_{mn}M^{HK}_{nm})_4.
\end{aligned} \tag{6.88}
$$

Chapter 7

The free solutions with self-field

7.1 *The basic equations*

We assume that no external electromagnetic field exists in the considered area. Then the potential in the Schrödinger equation is reduced to the self-field described by

$$A_4(\mathcal{R},t) = i\acute{e}\int \frac{1}{r_{01}} u(\mathcal{R}_1,t)^*\, u(\mathcal{R}_1,t)dV_1. \tag{7.1}$$

Here u represents a solution of the following Schrödinger equation

$$(i\hbar\tfrac{\partial}{\partial t} + (\hbar^2/(2\acute{M}))\Delta + i\acute{e}A_4)u(\mathcal{R},t) = 0. \tag{7.2}$$

We restrict the investigation to spherically symmetric fields and use again the notations introduce in section 3.2. The practical dimensions are: unit of the energy: $\acute{e}^2/\acute{a}$ and unit of the time Δt. Besides we use in this chapter the length unit $\acute{a}$. Therefore is $d/dr := \acute{a}d/dr$ and eq.(7.2) reads

$$\frac{\acute{e}^2}{\acute{a}}(i\frac{\partial}{\partial t'} + \frac{1}{2r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} \; - \; \acute{a}A_{0\,0}(r,t'))f(r,t')Y_{0\,0} = 0 \tag{7.3}$$

A static solution of eq.(7.3) does not exist. That means, u has the form

$$u(\mathcal{R},t') = Y_{0\,0}f(r,t'). \tag{7.4}$$

This leads to

$$f(r,t') = \sum_n c_n(t')\,h_n(r) \tag{7.5}$$

with $c_n = b_n \exp i\varphi_n(t')$. h_n are, according to eg.(3.49/7.2), the eigenfunctions of the following equation

$$(\frac{\acute{e}^2}{\acute{a}})(E'_n + \frac{1}{2r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r})h_n(r) = 0\ . \tag{7.6}$$

These functions have the form

$$h_n(r) = \sqrt{\frac{2}{R}}\frac{1}{r}\sin(n\varpi' r) \tag{7.7}$$

with $E_n = (\acute{e}^2/\acute{a})E'_n = (\acute{e}^2/\acute{a})(n\varpi)^2$. R is the large radius of the considered sphere. $h_n(r)$ represents a complete system of singularity-free eigenfunctions in the range $0 \leq r \leq R$. Using this system, the eq.(7.3) can be solved. In addition, $u(r,t') = Y_{0\,0} f(r,t')$ describes a solution in the free space if the conditions $u(R) = 0$ and $(du(r)/dr = 0$ are satisfied for r=R. The charge density is given by

$$J_4^{0\,0} = i\acute{e}\rho_{0\,0}(r,t')\,Y_{0\,0}Y_{0\,0} \tag{7.8}$$

with

$$\rho_{0\,0} = (f(r,t')^* f(r,t'). \tag{7.9}$$

Therefore the quasi-static component of the vector potential A_4 follows from

$$A_{0\,0}(r,t') = (I_{0\,0}(\infty,t') - \int_0^r (I\rho(y,t')/y^2)\,dy). \tag{7.10}$$

Here the integral function

$$I\rho(y,t') = \int_0^y x^2 \rho_{0\,0}(x,t')dx \tag{7.11}$$

and $\acute{a}A_{0\,0}(r,t')$ are dimensionless. The eqs.(7.3/7.4) lead to

$$i\tfrac{d}{dt'}c_n(t') = E'_n c_n(t') + M_n(t') \tag{7.12}$$

with

$$M_n(t') \;=\; \acute{a}\int_0^\infty r^2\,A_{0\,0}(r,t')h_n(r)\,f(r,t')\,dr. \tag{7.13}$$

The Fourier transformation of this expression reads ($\omega' = 2\pi/T'$)

$$(mx_n^l) = \frac{1}{T'} \int_{t'_0}^{t'_0+T'} \int_0^\infty \exp(il\omega' (t' - t'_0))\, r^2\, \acute{a}(A_{0\,0}^{0\,0}(r,t') h_n(r)\, f(r,t')\; dr dt'. \quad (7.14)$$

With this equation and the relation

$$c_n(t') = \sum_l d_n^l \exp(-\, i\, l\, \omega' (t' - t'_0)) \quad (7.15)$$

the Fourier-transformation of eq.(7.12) yields

$$\begin{aligned} \textstyle\sum_l \omega' \, l \, d_n^l \exp(-i \; l\, \omega' \; (t' - t'_0)) = {} & E'_n \textstyle\sum_l d_n^l \; \exp(-i\, l\, \omega' \; (t' - t'_0)) \\ & + \textstyle\sum_l mx_n^l \exp(-\, i\, l\, \omega' \; (t' - t'_0)\,). \end{aligned} \quad (7.16)$$

Therefore becomes

$$(\omega' \, l \; - \; E'_n)\, d_n^l \; = \; mx_n^l. \quad (7.17)$$

The eqs.(7.12) must be solved with the aid of numerical methods that include the calculations of $M_n(t')$. The numerical determination of this integral can be

avoided, using the procedure given in appendix D.
According to eq.(7.13) the energy of the electrostatic field has the value

$$ea(t') = \frac{\acute{e}^2}{2\acute{a}} \sum_{n,m,j,k} c_n(t')^* c_m(t') c_j(t')^* c_k(t') ({}_{n\,m}M_{j\,k}) \tag{7.18}$$

with (see section 6.4 and appendix D, $\acute{a}/r_{0\,0'} := 1/r_{0\,0'}$)

$${}_{n\,m}M^1_{j\,k} = \acute{a} \int\int \frac{(Y_{0\,0})^4}{r_{0\,0'}} h_n(r) h_m(r) h_j(r') h_k(r') dV dV'. \tag{7.19}$$

The energy of the Dirac field amounts to

$$ed(t') = \frac{\acute{e}^2}{\acute{a}} \sum_n \frac{1}{2} (n\varpi')^2 \, c_n(t')^* c_n(t') = eg - ea(t'). \tag{7.20}$$

7.2 *The force balance of a free solution*

The force and energy balance of the presented system results from the divergence of the associated energy-momentum tensor. It reads

$$T^d_{\bar{\nu}\,\bar{\mu}} = \frac{\hbar \acute{c}}{2\acute{a}} (\bar{U} \gamma_{\bar{\mu}} \frac{\partial}{\partial x_{\bar{\nu}}} U - \frac{\partial \bar{U}}{\partial x_{\bar{\nu}}} \gamma_{\bar{\mu}} U) \tag{7.21}$$

with

$$U=\begin{pmatrix}\hat{U}\\ \check{U}\end{pmatrix},\bar{U}=(\hat{U}^{*}\;\; -\check{U}^{*}) \tag{7.22}$$

and (see eq.(3.9))

$$\hat{U}=\sum_{n}c_{n}(t')\begin{pmatrix}ih_{n}(r)Y_{0\,0}\\ 0\end{pmatrix},\;\check{U}=\sum_{n}c_{n}(t')\begin{pmatrix}g_{n}(r)\sqrt{\frac{1}{3}}Y_{1\,0}\\ g_{n}(r)\sqrt{\frac{2}{3}}Y_{1\,1}\end{pmatrix}, \tag{7.23}$$

or

$$\begin{array}{c}\hat{U}^{*}\;\;=\sum_{j}c_{j}^{*}(t')(-ih_{j}(r)^{*}Y_{0\,0}\;\;0)\\ \check{U}^{*}=\sum_{j}c_{j}^{*}(t')(g_{j}(r)^{*}\frac{1}{\sqrt{3}}Y_{1\,0}\;\;g_{j}(r)^{*}\sqrt{\frac{2}{3}}(Y_{1\,1})^{*}).\end{array} \tag{7.24}$$

Here is $g_{n}(r)=\frac{\hbar}{2\acute{M}\acute{c}\acute{a}}\frac{\partial h_{n}(r)}{\partial r}$. Using the approximation (see [18], eq.(61.19))

$$\begin{array}{c}\check{U}=-i\frac{\hbar}{2\acute{M}\acute{c}\acute{a}}\overrightarrow{\sigma}\nabla\hat{U},\\ \check{U}^{*}=i\frac{\hbar}{2\acute{M}\acute{c}\acute{a}}\nabla\hat{U}^{*}\overrightarrow{\sigma},\end{array} \tag{7.25}$$

we get

$$\begin{array}{c}T^{d}_{\bar{\nu}\,\bar{\nu}}=\frac{\hbar^{2}}{4i\acute{M}\acute{a}^{2}}(-(\hat{U}^{*}\overrightarrow{\sigma})(\overrightarrow{\sigma}\frac{\partial}{\partial x_{\bar{\nu}}}\nabla\hat{U})+\nabla(\hat{U}^{*}\overrightarrow{\sigma})(\overrightarrow{\sigma}\frac{\partial\hat{U}}{\partial x_{\bar{\nu}}})\\ +(\frac{\partial\hat{U}^{*}}{\partial x_{\bar{\nu}}}\overrightarrow{\sigma})(\overrightarrow{\sigma}\nabla\hat{U})-(\frac{\partial}{\partial x_{\bar{\nu}}}\nabla\hat{U}^{*}\overrightarrow{\sigma})(\overrightarrow{\sigma}\hat{U}))\end{array} \tag{7.26}$$

or (see eq.(2.30))

$$T^{d}_{\bar\nu\,\bar\nu} = i\sum_{n,j} HF_{j\,n}(\,-\,h_j(r)^*\tfrac{\partial}{\partial x_{\bar\nu}}\nabla h_n + \tfrac{\partial h_j(r)^*}{\partial x_{\bar\nu}}\nabla h_n) + \nabla h_j(r)^*\tfrac{\partial h_n}{\partial x_{\bar\nu}} - (\tfrac{\partial}{\partial x_{\bar\nu}}\nabla h_j(r)^*)h_n)$$
$$+\sum_{n,j} HF_{j\,n} rot((h_j(r)^*\tfrac{\partial h_n}{\partial x_{\bar\nu}} + \tfrac{\partial h_j(r)^*}{\partial x_{\bar\nu}}h_n)(0\ 0\ 1)) \tag{7.27}$$

with $HF_{j\,n}(t') = \frac{(\hbar\, Y_{0\,0})^2}{4\acute{M}\acute{a}^2}c_j(t')^*c_n(t') = \frac{(\acute{e}\, Y_{0\,0})^2}{4\acute{a}}c_j^*(t')c_n(t')$. The last expression in eq.(7.27) has no influence on the force balance, since the divergence of the rotation of a vector is zero. Therefore we can replace eq.(7.26) with

$$T^{d'}_{\bar\nu\,\bar\mu} = i\sum_{n,j} HF_{j\,n}(t')(\,-\,h_j(r)^*\tfrac{\partial^2 h_n(r)}{\partial x_{\bar\nu}\partial x_{\bar\mu}} - \tfrac{\partial^2 h_j(r)^*}{\partial x_{\bar\nu}\partial x_{\bar\mu}}h_n(r)$$
$$+\tfrac{\partial(h_j(r)^*}{\partial x_{\bar\nu}}\tfrac{\partial h_n(r)}{\partial x_{\bar\mu}} + \tfrac{\partial h_j(r)^*}{\partial x_{\bar\mu}}\tfrac{\partial h_n(r)}{\partial x_{\bar\nu}}), \tag{7.28}$$

or

$$T^{d'}_{\bar\nu\,\bar\mu} = i\sum_{n,j} HF_{j\,n}(t')(\,-\,\tfrac{x_{\bar\nu}}{r}\tfrac{x_{\bar\mu}}{r}(h_j(r)^*\tfrac{r\partial}{\partial r}[\tfrac{1}{r}\tfrac{\partial h_n(r)}{\partial r}] - \tfrac{r\partial}{\partial r}[\tfrac{1}{r}\tfrac{\partial h_j(r)^*}{\partial r}]h_n(r)$$
$$-(\tfrac{h_j(r)^*}{r}\tfrac{\partial h_n(r)}{\partial r} + \tfrac{\partial h_j(r)^*}{\partial r}\tfrac{h_n(r)}{r})\delta_{\bar\nu\,\bar\mu} + 2\tfrac{\partial h_j(r)^*}{\partial r}\tfrac{\partial h_n(r)}{\partial r}). \tag{7.29}$$

The last column of this tensor is

$$T^{d'}_{\bar\nu\,4} = \tfrac{2}{\acute{\alpha}}\sum_{n,j} HF_{j\,n}(t')\tfrac{x_{\bar\nu}}{r}(h_j(r)^*\tfrac{\partial h_n(r)}{\partial r} - \tfrac{\partial h_j(r)^*}{\partial r}h_n(r)). \tag{7.30}$$

Here the relations $\frac{\partial F_n(r)}{\partial x_{\bar\nu}} = \frac{x_{\bar\nu}}{r}\frac{\partial F_n(r)}{\partial r}$ are applied. One also has to consider the contribution of the electrostatic field, given by (see eq.2.19)/2.29/3.53))

$$T^{e}_{\bar\nu\,\bar\mu} = \tfrac{\acute{e}^2}{\acute{a}^2}\sum_{n,j} c_j(t')^*c_n(t')(Y_{0\,0})^2\,(-\tfrac{x_{\bar\nu}}{r}\tfrac{x_{\bar\mu}}{r} + \tfrac{1}{2}\delta_{\bar\nu\,\bar\mu})(\tfrac{\partial A_{0\,0}(r,t')}{\partial r})^2 =$$
$$\tfrac{4}{\acute{a}}\sum_{n,j} HF_{j\,n}(t')\,(-\tfrac{x_{\bar\nu}}{r}\tfrac{x_{\bar\mu}}{r} + \tfrac{1}{2}\delta_{\bar\nu\,\bar\mu})(\tfrac{\partial A_{0\,0}(r,t')}{\partial r})^2 \tag{7.31}$$
$$T^{e}_{\bar\nu\,4} \cong 0.$$

The eqs.(7.28/7.30) exhibit spherical symmetry when the terms are transformed into a spherical system. For this reason we introduce the following four dimensional unit-vectors

$$\begin{aligned} e_r &= (\tfrac{x_1}{r}, \tfrac{x_2}{r}, \tfrac{x_3}{r}, 0), \\ e_\vartheta &= (cos(\alpha)\, cos(\vartheta),\, sin(\varphi)\, cos(\vartheta),\, sin(\vartheta),\, 0), \\ e_\varphi &= (-sin(\varphi),\, cos(\varphi),\, 0,\, 0), \\ e_4 &= (0,\, 0,\, 0,\, 1), \end{aligned} \tag{7.32}$$

which are associated with the following transformation matrix

$$\hat{\alpha} = \begin{pmatrix} \alpha_{1\,1} = \frac{x_1}{r} & \alpha_{1\,2} = \frac{x_2}{r} & \alpha_{1\,3} = \frac{x_3}{r} & \alpha_{1\,4} = 0 \\ \alpha_{2\,1} = cos(\alpha)\, cos(\vartheta) & \alpha_{2\,2} = sin(\varphi)\, cos(\vartheta) & \alpha_{2\,3} = sin(\vartheta) & \alpha_{2\,4} = 0 \\ \alpha_{3\,1} = -sin(\varphi) & \alpha_{3\,2} = cos(\varphi) & \alpha_{3\,3} = 0 & \alpha_{3\,4} = 0 \\ \alpha_{4\,1} = 0 & \alpha_{4\,2} = 0, & \alpha_{4\,3} = 0 & \alpha_{4\,4} = 1 \end{pmatrix}. \tag{7.33}$$

The sum of eq.(7.28/29) and eq.(7.30) can be described by

$$T^{d'}_{\nu\,\mu} + T^{e}_{\nu\,\mu} = \alpha_{1\,\nu}\alpha_{1\,\mu} T^{p}_{r\,r}(r, t') + \delta_{\nu\,\mu} T^{q}_{r\,r}(r, t') \tag{7.34}$$

with

$$
\begin{aligned}
T^p_{r\,r} = \ & i\textstyle\sum_{n,j} HF_{j\,n}\big(- h_j(r)^* \frac{\partial^2 h_n(r)}{\partial^2 r} - \frac{\partial^2 h_j(r)^*}{\partial^2 r} h_n(r) \\
&+\tfrac{1}{r}(h_j(r)^* \tfrac{\partial h_n(r)}{\partial r} + \tfrac{\partial h_j(r)^*}{\partial r} h_n(r)) \\
&+2\tfrac{\partial h_j(r)^*}{\partial r}\tfrac{\partial h_n(r)}{\partial r}\big) - \ \tfrac{4}{\acute{a}} \textstyle\sum_{n,j} HF_{j\,n}(\tfrac{\partial A_{0\,0}(r,t')}{\partial r})^2,
\end{aligned}
\tag{7.35}
$$

$$
T^q_{r\,r} = \ -i\textstyle\sum_{n,j} H_{j\,n}(\frac{h_j(r)^*}{r}\frac{\partial h_n(r)}{\partial r} + \frac{\partial h_j(r)^*}{\partial r}\frac{h_n(r)}{r}) + \ \frac{4}{\acute{a}}\textstyle\sum_{n,j} HF_{j\,n}\,(\frac{\partial A_{0\,0}(r,t')}{\partial r})^2
$$

and

$$
\begin{aligned}
T^p_{r\,4} &= (2/\acute{\alpha})\textstyle\sum_{n,j} HF_{j\,n}(t')(h_j(r)^* \frac{\partial h_n(r)}{\partial r} - \frac{\partial h_j(r)^*}{\partial r} h_n(r)) \\
T^p_{4\,4} &= (2/\acute{\alpha}^2)\textstyle\sum_{n,j} HF_{j\,n}(t')(h_j(r)^* h_n(r).
\end{aligned}
\tag{7.36}
$$

We transform eq.(7.34) with the aid of the matrix $\hat{\alpha}$ into the following tensor

$$
\begin{aligned}
T^g_{\bar{\nu}\,\bar{\mu}} = \ & \alpha_{\bar{\nu}\,\lambda}\alpha_{\bar{\mu}\,\kappa}(\alpha_{1\,\lambda}\alpha_{1\,\kappa}T^p_{r\,r}(r,t') + \delta_{\lambda\,\kappa}T^q_{r\,r}(r,t')) \\
&= \delta_{\bar{\nu}\,1}\,\delta_{\bar{\mu}\,1}\,T^p_{r\,r}(r,t') + \delta_{\bar{\nu}\,\bar{\mu}}T^q_{r\,r}(r,t'), \\
T^g_{\nu\,4} = \ & \delta_{\nu\,4}\alpha_{\nu\,1}T^p_{r\,4}(r,t').
\end{aligned}
\tag{7.37}
$$

That means, the sum $T^g_{\nu\,\mu} = T^{d'}_{\nu\,\mu} + T^e_{\nu\,\mu}$ obtains in the system with the unit-vectors defined in the eqs.(7.32) the form

$$
T^g = \begin{pmatrix} T^g_{r\,r} & 0 & 0 & T^g_{r\,4} \\ 0 & T^g_{\vartheta\,\vartheta} & 0 & 0 \\ 0 & 0 & T^g_{\varphi\,\varphi} & 0 \\ T^g_{4\,r} & 0 & 0 & T^g_{4\,4} \end{pmatrix}.
\tag{7.38}
$$

Here are

$$
\begin{array}{c}
T^g_{r\,r} = i\sum_{n,j} HF_{j\,n}(- h_j(r)^*\frac{\partial^2 h_n(r)}{\partial r\partial r} - \frac{\partial^2 h_j(r)^*}{\partial r\partial r}h_n(r) + 2\frac{\partial (h_j(r)^*}{\partial r}\frac{\partial h_n(r)}{\partial r}) \\
- \frac{4}{\acute{a}}\sum_{n,j} HF_{j\,n}\,(\frac{\partial A_{0\,0}(r,t')}{\partial r})^2, \\
T^g_{\vartheta\,\vartheta} = T^g_{\varphi\,\varphi} = -i\sum_{n,j} HF_{j\,n}\frac{1}{r}(h_j(r)^*\frac{\partial h_n(r)}{\partial r} + \frac{\partial h_j(r)^*}{\partial r}h_n(r)) \\
+ \frac{4}{\acute{a}}\sum_{n,j} HF_{j\,n}(\frac{\partial A_{0\,0}(r,t')}{\partial r})^2, \\
T^g_{r\,4} = (2/\acute{\alpha})\sum_{n,j} HF_{j\,n}(t')\,(h_j(r)^*\frac{\partial h_n(r)}{\partial r} - \frac{\partial h_j(r)^*}{\partial r}h_n(r)), \\
T^g_{4\,r} = T^g_{r\,4}, \\
T^g_{4\,4} = -(2/\acute{\alpha}^2)\sum_{n,j} HF_{j\,n}(t')h_j(r)^*h_n(r).
\end{array} \tag{7.39}
$$

All terms are independent of the direction of the radial unit-vector due to the spherical symmetry. Therefore the r-component of the four-dimensional divergence of this tensor reads

$$
(divT^g)_r = \frac{\partial T^g_{r\,r}}{\acute{a}\partial r} + \frac{2}{\acute{a}r}T^g_{r\,r} - \frac{1}{\acute{a}r}(T^g_{\vartheta\,\vartheta} + T^g_{\varphi\,\varphi}) + \frac{\partial T^g_{r\,4}}{\partial x_4} \tag{7.40}
$$

respectively

$$
\begin{array}{c}
(divT^g)_r = i\sum_{n,j} HF_{j\,n}(t')(- h_j(r)^*(\frac{\partial}{\partial r}[(\frac{\partial^2 h_n(r)}{\partial r\partial r} + \frac{2}{r}\frac{\partial h_n(r)}{\partial r})] \\
-\frac{\partial}{\partial r}[(\frac{\partial^2 h_j(r)^*}{\partial r\partial r} + \frac{2}{r}\frac{\partial h_j(r)^*}{\partial r})]h_n(r) + (\frac{\partial h_j(r)^*}{\partial r})(\frac{\partial^2 h_n(r)}{\partial r\partial r} + \frac{2}{r}\frac{\partial h_n(r)}{\partial r}) \\
+(\frac{\partial^2 h_j(r)^*}{\partial r\partial r} + \frac{2}{r}\frac{\partial h_j(r)^*}{\partial r})\frac{\partial}{\partial r}h_n(r)) \\
- \frac{4}{\acute{a}^2}\sum_{n,j} HF_{j\,n}\,(Y_{0\,0})^2\,\frac{\partial A_{0\,0}(r,t')}{\partial r}(\frac{\partial^2 A_{0\,0}(r,t')}{\partial r\partial r} + \frac{2}{r}\frac{\partial A_{0\,0}(r,t')}{\partial r}) \\
-i\frac{(\acute{e}Y_{0\,0})^2}{2\acute{c}\bar{\Delta}t}\sum_{n,j}\frac{\partial (c^*_j(t')c_n(t'))}{\partial t'}(h_j(r)^*\frac{\partial h_n(r)}{\partial r} - \frac{\partial h_j(r)^*}{\partial r}h_n(r))
\end{array} \tag{7.41}
$$

or due to eq.(3.54)

$$
\begin{aligned}
(divT^g)_r =\ & i\frac{\acute{e}^2\,(Y_{0\,0})^2}{4\acute{a}}\sum_{n,j} c_j^*(t')c_n(t')(2(E_n' \\
-E_j')(h_j(r)^*(&\frac{\partial h_n(r')}{\partial r} - \frac{\partial h_j(r)^*}{\partial r}h_n(r)) - \frac{4}{\acute{a}^2}\sum_{n,j} HF_{j\,n}\frac{\partial A_{0\,0}(r,t')}{\partial r}\varrho_{0\,0}(r,t') \\
-i\frac{\acute{\alpha}\acute{e}^2\,(Y_{0\,0})^2}{2\acute{a}} &\sum_{n,j}\frac{\partial(c_j^*(t')c_n(t'))}{\partial t'}(h_j(r)^*\frac{\partial h_n(r)}{\partial r} - \frac{\partial h_j(r)^*}{\partial r}h_n(r)).
\end{aligned}
\tag{7.42}
$$

The constants follow from $1/(\acute{c}\Delta t) = \acute{\alpha}/\acute{a}$, $\exp(-iEt) = \exp(-iE't')$ and $\acute{e}^2\Delta t/\acute{a}\hbar = 1$. The numerical results recommend the approach $c_n = b_n(t')\exp i(-E_n't' - \varphi_n(t'))$. This leads with $HF_{j\,n}(t') = \frac{(\acute{e}\,Y_{0\,0})^2}{4\acute{a}}c_j^*(t')c_n(t'))$ to the following force balance equation

$$
\begin{aligned}
(divT^g)_r + \frac{\partial T_{4\,r}^e}{\partial x_4} =\ & i\sum_{n,j} HF_{j\,n}(2(E_n' - E_j')(h_j(r)^*\frac{\partial h_n(r)}{\partial r} - \frac{\partial h_j(r)^*}{\partial r}h_n(r)) \\
+\frac{4}{\acute{a}^2}\sum_{n,j} HF_{j\,n}&\frac{\partial A_{0\,0}(r,t')}{\partial r}\varrho_{0\,0} + 2\acute{\alpha}\sum_{n,j} HF_{j\,n}(\frac{-i}{b_j(t')b_n(t')}\frac{\partial(b_j(t')b_n(t'))}{\partial t'} \\
-((E_n' + \frac{\partial\varphi_n(t')}{\partial t'} - E_j' - &\frac{\partial\varphi_j(t')}{\partial t'}))(h_j(r)^*\frac{\partial h_n(r)}{\partial r} - \frac{\partial h_j(r)^*}{\partial r}h_n(r)) = 0.
\end{aligned}
\tag{7.43}
$$

Hence becomes

$$
\begin{aligned}
\frac{4}{\acute{a}^2}\sum_{n,j} HF_{j\,n}\frac{\partial A_{0\,0}(r,t')}{\partial r}\varrho_{0\,0} &= 2\acute{\alpha}\sum_{n,j} HF_{j\,n}\,(\frac{i}{b_j(t')b_n(t')}\frac{\partial(b_j(t')b_n(t'))}{\partial t'} \\
-(E_n' - E_j')(-1+i) - \frac{\partial\varphi_n(t')}{\partial t'} &+ \frac{\partial\varphi_j(t')}{\partial t'})(h_j(r)^*\frac{\partial h_n(r)}{\partial r'} - \frac{\partial h_j(r)^*}{\partial r}h_n(r)).
\end{aligned}
\tag{7.44}
$$

The left side looks like a part of a Lorentz force density and the right side is the local change in the radial current. Here you can see that the expansion rate of the mean radius becomes constant because the limit of $\frac{\partial A_{0\,0}(r,t')}{\partial r}\varrho_{0\,0}$ is zero.

Finally we mention the energy law which follows from the divergence of the fourth line. That reads

$$
(divT^g)_4 = \frac{\partial T_{4\,r}^g}{\partial r} + \frac{2}{r}T_{4\,r}^g + \frac{\partial T_{4\,4}^g}{\partial x_4} = 0 \tag{7.45}
$$

or

$$
\begin{aligned}
(divT^g)_4 = \ & (4/\acute{\alpha})\, \Sigma_{n,j}\, HF_{j\,n}(t')\, (h_j(r)^* h_n(r) 2(E'_j - E'_n) \\
& +4/\acute{\alpha})\, \Sigma_{n,j}\, HF_{j\,n}(t') h_j(r)^* h_n(r)\, (\tfrac{i}{b_j(t')b_n(t')} \tfrac{\partial(b_j(t')b_n(t'))}{\partial t'} \\
& +((E'_n - E'_j + \tfrac{\partial\varphi_n(t')}{\partial t'} - \tfrac{\partial\varphi_j(t')}{\partial t'}) = 0
\end{aligned}
\tag{7.46}
$$

respectively

$$
\begin{aligned}
(divT^g)_4 = \ & 4/\acute{\alpha})\, \Sigma_{n,j}\, HF_{j\,n}(t') h_j(r)^* h_n(r)\, (\tfrac{i}{b_j(t')b_n(t')} \tfrac{\partial(b_j(t')b_n(t'))}{\partial t'} \\
& + \tfrac{\partial\varphi_n(t')}{\partial t'} - \tfrac{\partial\varphi_j(t')}{\partial t'}) = 0.
\end{aligned}
\tag{7.47}
$$

In this approximation the energy law represents the charge conservation law, shown above.

7.3 *A "free" solution in an external field*

The determination of a "free" solution in an external field requires a great effort. However, it should be possible using modern computer systems. Here we discuss only the problems. Examples are the dynamics of the tensor of an electron in the field of an accelerator or a television tube [27]. It means that vector potentials have to be replaced by

$$A_\mu = A^e_\mu + A^s_\mu, \tag{7.48}$$

where $A^e_\mu(x_1, x_2, x_3)$ represents a static external field and $A^s_\mu(x_1, x_2, x_3, t)$ the self field. We suppose that J^e_μ, the source of A^e_μ, be on the outside of the space considered. Hence is $J_\mu = J^s_\mu(x_1, x_2, x_3, t) = i\acute{e}\bar{U}\gamma_\mu U$. The complete energy-momentum tensor in this area is formed by the following contributions

$$\hat{T}_{\nu\,\mu} = T^d_{\nu\,\mu} + {}^{ss}T^w_{\nu\,\mu} + {}^{ss}T^e_{\nu\,\mu} + {}^{es}T^w_{\nu\,\mu} + {}^{es}T^e_{\nu\,\mu} + {}^{ee}T^e_{\nu\,\mu} \tag{7.49}$$

with

$$\begin{aligned} T^d_{\nu\,\mu} &= \tfrac{\acute{c}\hbar}{2}(\bar{U}\gamma_\mu \tfrac{\partial U}{\partial x_\nu} - \tfrac{\partial \bar{U}}{\partial x_\nu}\gamma_\mu U), \\ {}^{ss}T^w_{\nu\mu} &= -A^s_\nu\, J^s_\mu, \ {}^{es}T^w_{\nu\mu} = -A^e_\nu\, J^s_\mu, \\ {}^{ss}T^e_{\nu\mu} &= \tfrac{1}{4\pi}(F^s_{\nu\sigma}F^s_{\mu\sigma} - \tfrac{1}{4}\delta_{\nu\mu}F^s_{\sigma\lambda}F^s_{\sigma\lambda}), \\ {}^{es}T^e_{\nu\mu} &= \tfrac{1}{4\pi}(F^e_{\nu\sigma}F^s_{\mu\sigma} - \tfrac{1}{4}\delta_{\nu\mu}F^e_{\sigma\lambda}F^s_{\sigma\lambda}), \\ {}^{ee}T^e_{\nu\mu} &= \tfrac{1}{4\pi}(F^e_{\nu\sigma}F^e_{\mu\sigma} - \tfrac{1}{4}\delta_{\nu\mu}F^e_{\sigma\lambda}F^e_{\sigma\lambda}). \end{aligned} \tag{7.50}$$

The divergence of all lines from $\hat{T}_{\nu\,\mu}$ must be zero. That is also independently fulfils for ${}^{ee}T^e_{\nu\,\mu}$ and we neglect this tensor. Using the eqs.(2.34-2.36) one obtains

$$\frac{\partial}{\partial x_\mu}({}^{ss}T^e_{\nu\mu} + {}^{es}T^e_{\nu\mu}) = -J_\mu(\frac{\partial A^s_\mu}{\partial x_\nu} - \frac{\partial A^s_\nu}{\partial x_\mu}) - J_\mu(\frac{\partial A^e_\mu}{\partial x_\nu} - \frac{\partial A^e_\nu}{\partial x_\mu}), \tag{7.51}$$

$$\frac{\partial}{\partial x_\mu}T^d_{\nu\mu} = J_\mu((\frac{\partial A^e_\mu}{\partial x_\nu}) + (\frac{\partial A^s_\mu}{\partial x_\nu})) \tag{7.52}$$

and

$$\frac{\partial}{\partial x_\mu}({}^{ss}T^w_{\nu\mu} + {}^{es}T^w_{\nu\mu}) = -J_\mu((\frac{\partial A^e_\nu}{\partial x_\mu}) + (\frac{\partial A^s_\nu}{\partial x_\mu})), \tag{7.53}$$

which also results in the eq.(2.38). Therefore the divergence condition $\partial\hat{T}_{\nu\,\mu}/\partial x_\mu = 0$ leads for $\nu = \bar{\nu}$ to

$$\frac{\partial}{\partial x_\mu}(T^g_{\bar{\nu}\mu}) = J_\mu(\frac{\partial A^e_\mu}{\partial x_{\bar{\nu}}} - \frac{\partial A^e_{\bar{\nu}}}{\partial x_\mu}) \tag{7.54}$$

with

$$T^g_{\nu\mu} = T^d_{\nu\mu} + {}^{ss}T^w_{\nu\mu} + {}^{ss}T^e_{\nu\mu}. \tag{7.55}$$

$T^g_{\nu\mu}$ represents all components of the energy-momentum tensor of the electron system. Consequently describes eq.(7.54) its general response to an external electromagnetic field. Due to $\hbar\acute{c} = \acute{e}^2/\acute{\alpha}$, $\hbar\frac{\partial U}{\partial x_4} = -\acute{M}\acute{c}U$, you get

$$T^d_{\bar{\nu}\,4} \cong \tfrac{\acute{e}^2}{2\acute{a}\acute{\alpha}}(\hat{U}^* I \tfrac{\partial\hat{U}}{\partial x_{\bar{\nu}}} - \tfrac{\partial\hat{U}^*}{\partial x_{\bar{\nu}}} I\hat{U}), \tag{7.56}$$

$$T^d_{\bar{\nu}\,4} = \tfrac{\hbar\acute{c}}{2}(\bar{U}\gamma_{\bar{\nu}}\tfrac{\partial U}{\partial x_4} - \tfrac{\partial\bar{U}}{\partial x_4}\gamma_{\bar{\nu}}U) \cong i\acute{M}\acute{c}^2\tfrac{1}{\acute{e}}J_{\bar{\nu}} \tag{7.57}$$

and

$$T^d_{4\,\bar{\nu}} \cong \tfrac{\acute{e}^2}{2\acute{a}\acute{\alpha}}(\hat{U}^* I \tfrac{\partial\hat{U}}{\partial x_{\bar{\nu}}} - \tfrac{\partial\hat{U}^*}{\partial x_{\bar{\nu}}} I\hat{U}) + i\tfrac{\acute{e}^2}{2\acute{a}\acute{\alpha}}\mathbf{rot}(\hat{U}^*\overrightarrow{\sigma}\hat{U})_{\bar{\nu}}. \tag{7.58}$$

The integral over the momentum eqs.(7.54) reads

$$\int \tfrac{\partial}{\partial x_\mu} T^d_{\bar{\nu}\mu} dV + \int \tfrac{\partial}{\partial x_\mu}({}^{ss}T^w_{\bar{\nu}\mu} +^{ss} T^e_{\bar{\nu}\mu})dV = \int J_\mu(\tfrac{\partial A^e_\mu}{\partial x_{\bar{\nu}}} - \tfrac{\partial A^e_{\bar{\nu}}}{\partial x_\mu})dV. \tag{7.59}$$

One obtains for the motion of the electron system the equation

$$\begin{array}{c} \acute{M}\frac{d\bar{v}_{\bar{\nu}}}{dt} \cong \int J_\mu(\frac{\partial A^e_\mu}{\partial x_{\bar{\nu}}} - \frac{\partial A^e_{\bar{\nu}}}{\partial x_\mu})dV \ with \\ \bar{v}_{\bar{\nu}} = \frac{\acute{c}}{\acute{e}}\int J_{\bar{\nu}}dV, \end{array} \tag{7.60}$$

because of eq.(7.57), the approximation

$$\int \frac{\partial}{\partial x_\mu}({}^{ss}T^w_{\bar{\nu}\mu} + {}^{ss}T^e_{\bar{\nu}\mu})dV \cong \int J_\mu(\frac{\partial A^e_\mu}{\partial x_{\bar{\nu}}})dV \cong 0 \tag{7.61}$$

and

$$\int \frac{\partial}{\partial x_\mu}T^d_{\bar{\nu}\mu}dV = \frac{\partial}{\partial x_4}\int T^d_{\bar{\nu}4}dV = \frac{\partial}{\partial x_4}\int T^d_{4\ \bar{\nu}}dV. \tag{7.62}$$

The right side of eq.(7.60? has the form of an integral description of the Lorentz force $J_\mu(\frac{\partial A^e_\mu}{\partial x_{\bar{\nu}}} - \frac{\partial A^e_{\bar{\nu}}}{\partial x_\mu}) = -iJ_4\mathbf{E} - i\overrightarrow{J} \times \mathbf{H}$.

It makes sense to show, how strongly the eq.(7.60) deviates from the following basic equation of the electron optics

$$\frac{d}{dt}[\frac{\acute{M}\overrightarrow{v}}{\sqrt{1-(v/\acute{c})^2}}] = \acute{e}(\mathbf{E} + \overrightarrow{v} \times \mathbf{H}), \tag{7.63}$$

which describes the movement of a point charge. Here we also use the time unit Δt and the relations $E(t)dt/\hbar = E'(t')dt',\ \hbar/\Delta t = \acute{e}^2/\acute{a}$. If $\overrightarrow{\mathcal{R}}(t') = (y_1(t'),\ y_2(t'),\ y_3(t'))$ represents the solution vector of eq.(7.63), one obtains

$$\overrightarrow{\mathcal{R}}(t') - \overrightarrow{\mathcal{R}}(t'_0) = \int_{t'_0}^{t'}(\tfrac{dy_1}{dt'}, \tfrac{dy_2}{dt'}, \tfrac{dy_3}{dt'})dt' \tag{7.64}$$

and $v_{\bar{\mu}} = \frac{dy_{\bar{\mu}}}{dt'}$. We put the function

$$\hat{U}(\mathcal{R}, t') = \exp(-i(\int E'(t')dt' \ +\ v_{\bar{\mu}}(t')x_{\bar{\mu}}))W(\mathcal{R}, t') \tag{7.65}$$

in eq.(3.37). That is

$$\begin{array}{c}
(i\frac{\hbar\partial}{\partial t} + i\acute{e}A_4 - \frac{1}{2\acute{M}}(i\hbar\frac{\partial}{\partial x_{\bar{\mu}}} + \frac{\acute{e}}{\acute{c}}A_{\bar{\mu}})^2 + \frac{\acute{e}\hbar}{2\acute{M}\acute{c}}\sigma_{\bar{\mu}}\mathbf{H}_{\bar{\mu}})\hat{U} \\
= W(i\frac{\hbar\partial}{\partial t} - \frac{1}{2\acute{M}}(i\hbar\frac{\partial}{\partial x_{\bar{\mu}}})^2 - \frac{\acute{e}}{\acute{M}\acute{c}}A_{\bar{\mu}}i\hbar\frac{\partial}{\partial x_{\bar{\mu}}})\exp(-i(\int E'(t')dt' \ +\ v_{\bar{\mu}}(t')x_{\bar{\mu}})) \\
+\frac{\hbar^2}{2\acute{M}}\frac{\partial W}{\partial x_{\bar{\mu}}}\frac{\partial}{\partial x_{\bar{\mu}}}\exp(-i(\int E'(t')dt' \ +\ v_{\bar{\mu}}(t')x_{\bar{\mu}})) \\
+\exp(-i(\int E'(t')dt' \ +\ v_{\bar{\mu}}(t')x_{\bar{\mu}}))(i\frac{\hbar\partial}{\partial t} + i\acute{e}A_4 \\
-\frac{1}{2\acute{M}}(i\hbar\frac{\partial}{\partial x_{\bar{\mu}}} + \frac{\acute{e}}{\acute{c}}A_{\bar{\mu}})^2 + \frac{\acute{e}\hbar}{2\acute{M}\acute{c}}\sigma_{\bar{\mu}}\mathbf{H}_{\bar{\mu}})W = 0
\end{array} \tag{7.66}$$

respectively

$$\begin{array}{c}
(i\frac{\hbar\partial}{\partial t} + i\acute{e}A_4 - \frac{1}{2\acute{M}}(i\hbar\frac{\partial}{\partial x_{\bar{\mu}}} + \frac{\acute{e}}{\acute{c}}A_{\bar{\mu}})^2 + \frac{\acute{e}\hbar}{2\acute{M}\acute{c}}\sigma_{\bar{\mu}}\mathbf{H}_{\bar{\mu}})\hat{U} \\
= \frac{\acute{e}^2}{\acute{a}}W(\mathcal{R}, t')(\frac{i\partial}{\partial t'}\exp(-i(\int E'(t')dt' \ + i\,v_{\bar{\mu}}(t')x_{\bar{\mu}})) \\
+\frac{1}{2}\exp(-i\int E'(t')dt')\ \frac{\acute{a}^2\partial^2}{\partial x_{\bar{\mu}}\partial x_{\bar{\mu}}}\exp(-iv_{\bar{\mu}}(t')x_{\bar{\mu}}) \\
+i\frac{\acute{a}}{\acute{e}}A_4\exp(-i(\int E'(t')dt' \ +\ v_{\bar{\mu}}(t')x_{\bar{\mu}})) \\
+\exp(-i(\int E'(t')dt' \ +\ v_{\bar{\mu}}(t')x_{\bar{\mu}}))(i\frac{\partial}{\partial t'} + i\frac{\acute{a}}{\acute{e}}A_4)))W \\
+\exp(-i(\int E'(t')dt' \ +\ v_{\bar{\mu}}(t')x_{\bar{\mu}}))(- \frac{1}{2\acute{M}}(i\hbar\frac{\partial}{\partial x_{\bar{\mu}}} + \frac{\acute{e}}{\acute{c}}A_{\bar{\mu}})^2 \\
+\frac{\acute{e}\hbar}{2\acute{M}\acute{c}}\sigma_{\bar{\mu}}\mathbf{H}_{\bar{\mu}} + i\acute{\alpha}\acute{e}\acute{a}v_{\bar{\mu}}A_{\bar{\mu}})W(\mathcal{R}, t') \\
-i\frac{\acute{e}^2}{2}v_{\bar{\mu}}\exp(-i(\int E'(t')dt' \ +\ v_{\bar{\mu}}(t')x_{\bar{\mu}}))\frac{\partial}{\partial x_{\bar{\mu}}}W(\mathcal{R}, t') = 0
\end{array} \tag{7.67}$$

or

$$
\begin{aligned}
&(i\frac{\hbar\partial}{\partial t} + i\acute{e}A_4 - \frac{1}{2\acute{M}}(i\hbar\frac{\partial}{\partial x_{\bar{\mu}}} + \frac{\acute{e}}{\acute{c}}A_{\bar{\mu}})^2 + \frac{\acute{e}\hbar}{2\acute{M}\acute{c}}\sigma_{\bar{\mu}}\mathbf{H}_{\bar{\mu}})\hat{U} \\
&= \frac{\acute{e}^2}{\acute{a}}(E'(t') - \frac{\acute{a}^2}{2}v_{\bar{\mu}}(t')v_{\bar{\mu}}(t') + x_{\bar{\mu}}\frac{\partial v_{\bar{\mu}}}{\partial t'} + i\frac{\acute{a}}{\acute{e}}A_4)\hat{U}(\mathcal{R},t') \\
&+\exp(-i(\textstyle\int E'(t')dt' \ + \ v_{\bar{\mu}}(t')x_{\bar{\mu}}))(i\frac{\partial}{\partial t'} + i\frac{\acute{a}}{\acute{e}}A_4 \\
&-\frac{1}{2\acute{M}}(i\hbar\frac{\partial}{\partial x_{\bar{\mu}}} + \frac{\acute{e}}{\acute{c}}\acute{a}A_{\bar{\mu}})^2 + \frac{\acute{e}\hbar}{2\acute{M}\acute{c}}\sigma_{\bar{\mu}}\mathbf{H}_{\bar{\mu}} + i\acute{\alpha}\acute{e}\acute{a}v_{\bar{\mu}}A_{\bar{\mu}})W(\mathcal{R},t') \\
&-i\frac{\acute{e}^2}{2}v_{\bar{\mu}}\exp(-i(\textstyle\int E'(t')dt' \ + \ v_{\bar{\mu}}(t')x_{\bar{\mu}}))\frac{\partial}{\partial x_{\bar{\mu}}}W(\mathcal{R},t') = 0.
\end{aligned}
\tag{7.68}
$$

Here the relationships $\hbar^2/\acute{M} = \acute{e}^2\acute{a}$, $\hbar/(\acute{M}\acute{c}) = \acute{\alpha}\acute{a}$ and $A_\mu = A^e_\mu(x_1,x_2,x_3) + A^s_\mu(x_1,x_2,x_3,t')$ are used. For a purely electrostatic external field ($A^e_{\bar{\mu}} = 0,\ A^s_{\bar{\mu}} \cong 0$) it becomes

$$
\begin{aligned}
&(i\frac{\hbar\partial}{\partial t} + i\acute{e}A_4 - \frac{1}{2\acute{M}}(i\hbar\frac{\partial}{\partial x_{\bar{\mu}}})^2)\hat{U} \\
&= \frac{\acute{e}^2}{\acute{a}}(E'(t') - \frac{1}{2}v_{\bar{\mu}}(t')v_{\bar{\mu}}(t') + x_{\bar{\mu}}\frac{\partial v_{\bar{\mu}}}{\partial t'} + i\frac{\acute{a}}{\acute{e}}A_4)\hat{U}(\mathcal{R},t') \\
&+\exp(-i(\textstyle\int E'(t')dt' \ + \ v_{\bar{\mu}}(t')x_{\bar{\mu}}))(i\frac{\partial}{\partial t'} + i\frac{\acute{a}}{\acute{e}}A_4 + \frac{\partial^2}{2\partial x_{\bar{\mu}}\partial x_{\bar{\mu}}})W(\mathcal{R},t') \\
&-i\frac{\acute{e}^2}{2}v_{\bar{\mu}}\exp(-i(\textstyle\int E'(t')dt' \ + \ v_{\bar{\mu}}(t')x_{\bar{\mu}}))\frac{\partial}{\partial x_{\bar{\mu}}}W(\mathcal{R},t') = 0.
\end{aligned}
\tag{7.69}
$$

We replace the energy $E'(t')$ by

$$
E'(t') = \frac{1}{2}v_{\bar{\mu}}(t')^2 - i\frac{\acute{a}}{\acute{e}}A^e_4(\overrightarrow{\mathcal{R}}(t'))
\tag{7.70}
$$

and obtain instead of the eqs.(7.68/7.69) the expressions

$$
\begin{aligned}
&(i\frac{\hbar\partial}{\partial t} + i\acute{e}A_4 - \frac{1}{2\acute{M}}(i\hbar\frac{\partial}{\partial x_{\bar{\mu}}} + \frac{\acute{e}}{\acute{c}}\acute{a}A_{\bar{\mu}})^2 + \frac{\acute{e}\hbar}{2\acute{M}\acute{c}}\sigma_{\bar{\mu}}\mathbf{H}_{\bar{\mu}})\hat{U} \\
&= \frac{\acute{e}^2}{\acute{a}}(x_{\bar{\mu}}\frac{\partial v_{\bar{\mu}}}{\partial t'} + i\frac{\acute{a}}{\acute{e}}(A^e_4(\mathcal{R}) - A^e_4(\overrightarrow{\mathcal{R}}(t')) + A^s_4(\mathcal{R},t)))\hat{U}(\mathcal{R},t') \\
&+\exp(-i(\textstyle\int E'(t')dt' \ + \ v_{\bar{\mu}}(t')x_{\bar{\mu}}))(- \frac{1}{2\acute{M}}(i\hbar\frac{\partial}{\partial x_{\bar{\mu}}} + \frac{\acute{e}}{\acute{c}}\acute{a}A_{\bar{\mu}})^2 \\
&+\frac{\acute{e}\hbar}{2\acute{M}\acute{c}}\sigma_{\bar{\mu}}\mathbf{H}_{\bar{\mu}} + i\acute{\alpha}\acute{e}\acute{a}v_{\bar{\mu}}A_{\bar{\mu}})W(\mathcal{R},t') \\
&-\frac{\acute{e}^2}{2}v_{\bar{\mu}}\exp(-i(\textstyle\int E'(t')dt' \ + \ v_{\bar{\mu}}(t')x_{\bar{\mu}}))\frac{\partial}{\partial x_{\bar{\mu}}}W(\mathcal{R},t') = 0
\end{aligned}
\tag{7.71}
$$

or

$$
\begin{aligned}
&(i\frac{\hbar\partial}{\partial t} + i\acute{e}A_4 + \frac{1}{2\acute{M}}(\hbar\frac{\partial}{\partial x_{\bar{\mu}}})^2)\hat{U} \\
&= \frac{\acute{e}^2}{\acute{a}}(x_{\bar{\mu}}\frac{\partial v_{\bar{\mu}}}{\partial t'} + i\frac{\acute{a}}{\acute{e}}(A_4^e(\mathcal{R}) - A_4^e(\overrightarrow{\mathcal{R}}(t')) + A_4^s(\mathcal{R},t)))\hat{U}(\mathcal{R},t') \\
&+ \exp(-i(\int E'(t')dt' \ + v_{\bar{\mu}}(t')x_{\bar{\mu}}))(i\frac{\partial}{\partial t'} + \frac{\partial^2}{2\partial x_{\bar{\mu}}\partial x_{\bar{\mu}}})W(\mathcal{R},t') \\
&-i\frac{\acute{e}^2}{2}v_{\bar{\mu}}\exp(-i(\int E'(t')dt' \ + v_{\bar{\mu}}(t')x_{\bar{\mu}})\frac{\partial}{\partial x_{\bar{\mu}}}W(\mathcal{R},t') = 0.
\end{aligned}
\tag{7.72}
$$

According to this expressions, the energy $E'(t')$ depends on the coordinates of the solution $\overrightarrow{\mathcal{R}}(t') = (y_1(t')\ y_2(t')\ y_3(t'))$ of eq.(7.63). The energy law of the movement of such point charge has in $\acute{e}^2/\acute{a}$ the form [28,29]

$$
\begin{aligned}
&-i\frac{\acute{a}}{\acute{e}}(A_4^e(y_1(t'), y_2(t'), y_3'(t')) \\
&-A_4^e(y_1(t_0'), y_2(t_0'), y_3(t_0')) = \frac{1}{2}(v^2(t') - v^2(t_0')).
\end{aligned}
\tag{7.73}
$$

Due to

$$
\begin{aligned}
-i\frac{\acute{a}}{\acute{e}}\int\frac{dA_4^e}{dy_{\bar{\mu}}}\frac{dy_{\bar{\mu}}}{dt'}dt' &= -i\frac{\acute{a}}{\acute{e}}(A_4^e(y_1(t'), y_2'(t'), y_3(t')) \\
&-A_4^e(y_1(t_0'), y_2(t_0'), y_3(t_0')),
\end{aligned}
\tag{7.74}
$$

it becomes

$$
-i\frac{\acute{a}}{\acute{e}}\int\frac{dA_4^e}{dy_{\bar{\mu}}}\frac{dy_{\bar{\mu}}}{dt'}dt' = \frac{1}{2}\acute{M}(v(t')^2 - v(t_0')^2).
\tag{7.75}
$$

Hence the right side of eq.(7.70) is independent of time and we get $E'(t) = E'(t_0)$. Eq.(7.63) must be solved to determine the linked energy-momentum tensor and to check the difference to eq.(7.60). It is

$$\frac{\acute{\alpha}\acute{c}}{\acute{a}}\frac{d}{dt'}[\frac{\acute{M}\overrightarrow{v}(\overrightarrow{\mathcal{R}}(t'))}{\sqrt{1-(|v(\overrightarrow{\mathcal{R}}(t'))|/\acute{c})^2}}]=\acute{e}(\bigtriangledown A_4^e(\overrightarrow{\mathcal{R}}(t'))+\overrightarrow{v}(\overrightarrow{\mathcal{R}}(t'))\times\mathbf{H}(\overrightarrow{\mathcal{R}}(\mathbf{t'})), \quad (7.76)$$

or for $|\overrightarrow{v}| \ll \acute{c}$

$$\acute{M}\frac{d}{dt'}\overrightarrow{v}(\overrightarrow{\mathcal{R}}(t'))\cong\frac{\acute{a}\acute{e}}{\acute{\alpha}\acute{c}}(\bigtriangledown A_4^e(\overrightarrow{\mathcal{R}}(t'))+\overrightarrow{v}(\overrightarrow{\mathcal{R}}(t'))\times\mathbf{H}(\overrightarrow{\mathcal{R}}(\mathbf{t'})). \quad (7.77)$$

This must be compared to the integrals given in the eqs.(7.61/7.62). Using the approximation $T^d_{\bar{\nu}\bar{\mu}} \cong 0$ on the surface of the area, it reads

$$\int\frac{\partial}{\partial x_\mu}T^d_{\bar{\nu}\mu}dV\cong\frac{\acute{c}\hbar}{2}\frac{d}{dt}\int(\bar{U}\gamma_4\frac{\partial}{\partial x_{\bar{\nu}}}-\frac{\partial}{\partial x_{\bar{\nu}}}\bar{U}\gamma_4U)dV \quad (7.78)$$

or (see eq.(7.58))

$$\int\frac{\partial}{\partial x_\mu}T^d_{\bar{\nu}\mu}dV\cong\frac{\acute{\alpha}^2\acute{c}^2}{2a}\frac{d}{dt'}\int(\hat{U}^*I\frac{\partial\hat{U}}{\partial x_{\bar{\nu}}}-\frac{\partial\hat{U}^*}{\partial x_{\bar{\nu}}}I\hat{U})dV+i\frac{\acute{e}^2\acute{c}^2}{2\acute{a}}\frac{d}{dt'}\int\mathbf{rot}(\hat{U}^*\overrightarrow{\sigma}\hat{U})_{\bar{\nu}}dV. \quad (7.79)$$

With the eq.(7.65) and $\hbar\acute{c}=\acute{e}^2/\acute{\alpha}$ one obtains for the tensor components

$$T^d_{\bar{\nu}\mu}=-i\frac{\acute{e}^2}{\acute{\alpha}}v_{\bar{\nu}}(t')\bar{W}\gamma_\mu W+\frac{\acute{e}^2}{2\acute{\alpha}}(\bar{W}\gamma_\mu\frac{\partial W}{\partial x_{\bar{\nu}}}-\frac{\partial\bar{W}}{\partial x_{\bar{\nu}}}\gamma_\mu W) \quad (7.80)$$

and for the integrals over the divergences

$$\int \frac{\partial}{\partial x_\mu}(T^d_{\bar{\nu}\mu})dV = \frac{\acute{e}^2}{\acute{\alpha}}\frac{dv_{\bar{\nu}}(t')}{dt}\int \bar{W}\gamma_4 W dV - i\frac{\acute{e}^2}{\acute{\alpha}}v_{\bar{\nu}}(t')\int \frac{\partial(\bar{W}\gamma_\mu W)}{\partial x_\mu}dV$$
$$+\frac{\acute{e}^2}{2\acute{\alpha}\acute{a}}\int (\bar{W}\gamma_\mu \frac{\partial^2 W}{\partial x_\mu \partial x_{\bar{\nu}}} + \frac{\partial \bar{W}}{\partial x_\mu}\gamma_\mu \frac{\partial W}{\partial x_{\bar{\nu}}} - \frac{\partial \bar{W}}{\partial x_{\bar{\nu}}}\gamma_\mu \frac{\partial W}{\partial x_\mu} - \frac{\partial^2 \bar{W}}{\partial x_\mu \partial x_{\bar{\nu}}}\gamma_\mu W)dV. \quad (7.81)$$

If the integration region V is great enough for the assumption $T^d_{\bar{\nu}\bar{\mu}} \cong 0$ on its surface, eq.(7.81) can be approximated by ($\partial/\partial x_4 = -i\frac{\acute{\alpha}}{a}\partial/\partial t'$)

$$\int \frac{\partial}{\partial x_\mu}(T^d_{\bar{\nu}\mu})dV = \frac{\acute{e}^2}{\acute{a}}\frac{dv_{\bar{\nu}}(t')}{dt'}\int \bar{W}\gamma_4 W dV + \frac{\acute{e}^2}{2\acute{a}}\frac{\partial}{\partial t'}\int (\bar{W}\gamma_4 \frac{\partial W}{\partial x_{\bar{\nu}}} - \frac{\partial \bar{W}}{\partial x_{\bar{\nu}}}\gamma_4 W)dV. \quad (7.82)$$

Then, it is possible the eq.(7.59) to replace by

$$\acute{c}\frac{\acute{e}^2}{\acute{a}}\frac{v_{\bar{\nu}}(t')}{\partial t'} + D_{\bar{\nu}} = \int J_\mu(\frac{\partial A^e_\mu}{\partial x_{\bar{\nu}}} - \frac{\partial A^e_{\bar{\nu}}}{\partial x_\mu})dV \quad (7.83)$$

with

$$D_{\bar{\nu}} \cong \frac{\acute{e}^2}{2\acute{a}}\frac{\partial}{\partial t'}\int (\bar{W}\gamma_4 \frac{\partial W}{\partial x_{\bar{\nu}}} - \frac{\partial \bar{W}}{\partial x_{\bar{\nu}}}\gamma_4 W)dV - \int J_\mu(\frac{\partial A^s_\mu}{\partial x_{\bar{\nu}}} - \frac{\partial A^s_{\bar{\nu}}}{\partial x_\mu})dV \quad (7.84)$$

where $D_{\bar{\nu}}$ depends on the internal dynamics of the electron field.
Eq.(7.83) describes the deviation between the influence of the Lorentz force on a point charge and the balance equation of the linked field tensor. Eq.(7.60) and eq.(7.76) are equal for $\frac{1}{\acute{a}}\int J_\mu(\frac{\partial A^e_\mu}{\partial x_{\bar{\nu}}} - \frac{\partial A^e_{\bar{\nu}}}{\partial x_\mu})dV = \acute{e}(\mathbf{E} + \overrightarrow{v} \times \mathbf{H})$ and $D_{\bar{\nu}} - \frac{1}{\acute{a}}\int J_\mu(\frac{\partial A^s_\mu}{\partial x_{\bar{\nu}}} - \frac{\partial A^s_{\bar{\nu}}}{\partial x_\mu})dV = 0$. That means, $\frac{d}{dt}\frac{\acute{M}}{\sqrt{1-(v/\acute{c})^2}} \cong M\frac{d\overrightarrow{v}}{dt'}$ for $(v/\acute{c})^2 << 1$. Then in eq.(7.60),

one can put $\bar{v} = v$. Such test requires the solution of the linked differential equations (see eq.(7.71)). In addition, one can assume that the field of a strong positive charge captures and brings together the "free" electron field as long as the balance of which momentums allows it. Internal vibration should be the consequence. If the "free" electron field is influenced by external fields, it must be checked whether the final results of such capturing processes correlate with paths of point charges. Then the field model also explains the known results of the electron optics. The basic equations of the field theory always guarantee the unit of the electron field. It is an important point for the interpretation. In this picture the dynamics of an isolated electron field with a short distance to a proton must likely lead to vibrating hydrogen atom.

The components of the fourth line of the tensors are

$$\begin{array}{c} T^d_{4\,\mu} = -\frac{\acute{e}^2}{\acute{a}}(\frac{1}{\acute{\alpha}^2} + E' + \frac{\partial v_{\bar{\nu}}(t')x_{\bar{\nu}}}{\partial t'})(\bar{U}\gamma_\mu U) \\ -i\frac{\acute{e}^2}{2\acute{a}}(\bar{W}\gamma_\mu \frac{\partial W}{\partial t'} - \frac{\partial W}{\partial t'}\gamma_\mu W) \end{array} \tag{7.85}$$

and the divergence is

$$\begin{array}{c} \frac{\partial T^d_{4\,\mu}}{\partial x_\mu} = -\frac{\acute{e}^2}{\acute{a}^2}(\frac{1}{\acute{\alpha}^2} + E' + \frac{\partial v_{\bar{\nu}}(t')x_{\bar{\nu}}}{\partial t'})\frac{\partial}{\partial x_\mu}(\bar{U}\gamma_\mu U) - \frac{\acute{e}^2}{\acute{a}^2}\frac{\partial v_{\bar{\mu}}}{\partial t'}(\bar{U}\gamma_{\bar{\mu}}U) \\ -i\frac{\acute{e}^2}{2\acute{a}^2}\frac{\partial}{\partial x_\mu}(\bar{W}\gamma_\mu \frac{\partial W}{\partial t'} - \frac{\partial W}{\partial t'}\gamma_\mu W). \end{array} \tag{7.86}$$

Because of $(\frac{\partial}{\partial x_\mu}(\bar{U}\gamma_\mu U) = 0)$ and $\frac{\partial}{\partial x_\mu}({}^{es}T^w_{4\,\mu} + {}^{es}T^e_{4\,\mu}) = -i\frac{\acute{\alpha}}{\acute{a}}J_\mu(\frac{\partial A^s_\mu}{\partial t'})$,
it becomes

$$\begin{array}{c} \frac{\partial}{\partial x_\mu}T^g_{4\mu} = -\frac{\acute{e}^2}{\acute{a}^2}\frac{\partial v_{\bar{\mu}}}{\partial t'}(\bar{U}\gamma_{\bar{\mu}}U) \\ -i\frac{\acute{e}^2}{2\acute{a}^2}\frac{\partial}{\partial x_\mu}(\bar{W}\gamma_\mu \frac{\partial W}{\partial t'} - \frac{\partial W}{\partial t'}\gamma_\mu W) - i\frac{\acute{\alpha}}{\acute{a}}J_\mu(\frac{\partial A^s_\mu}{\partial t'}). \end{array} \tag{7.87}$$

This must be compensated through the divergence

$$-\frac{\partial}{\partial x_\mu}({}^{es}T^w_{4\mu} + {}^{es}T^e_{4\mu}) = -i\frac{\acute{\alpha}}{\acute{a}} J_{\bar{\mu}} (\frac{\partial A^e_4}{\partial x_{\bar{\mu}}}). \tag{7.88}$$

7.4 *A few numerical results*

Numerical calculations show that the charge distributions of a "free" solution always and permanently expands for all initial distributions. That means, it is (see eq. 7.13))

$$ea(t') = \frac{1}{2}\sum_n a_n(t')^* M_n(t') \ \sim \ 1/RR(t') \tag{7.89}$$

where

$$RR(t') = \int_0^R r^3 \rho_{0\,0}(r,t'))dr \tag{7.90}$$

describes the mean radius. Consequently is $ea(t'_0) > ea(t'_0 + t')$ and $ed(t'_0) < ed(t'_0 + t')$. Besides, the calculations exhibit that only spectral coefficients $a_n(t')$ contribute to the Dirac energy ed in a small range around $a_m(t')$ if t$'$ approaches infinity $(t' \to \infty)$. This underlines the shapes in Fig.12. The charge distributions are located in an expanding shell, in which the empty inner cavity expands too. Therefore, the spectral coefficients a_n are enriched in a small distribution around the limit function

$$h_m(r) = \sqrt{\frac{2}{R}}\frac{1}{r}\sin(m\omega r). \tag{7.91}$$

.

The value of m follows from the condition $ed \longrightarrow eg$, (eg:= total energy), what leads to $m = \sqrt{2\,eg}/\omega$ and to the solution

$$f(r,t' \to \infty) = \sum_{n=m-N}^{n=m+N} a_n(t')\,h_n(r),\;\; m >> N. \tag{7.92}$$

When approaching this limit, the coefficients can be described by

$$a_n(t') = \; b_n(t')\; \exp(-\,i\,(\varphi_n(t') + E_n' t')), \tag{7.93}$$

where b_n represents the amount. If you insert this sum in eq.(7.2), you obtain by multiplying with $a_n(t')^* h_n(r)$ and integrating the equations

$$i\frac{da_n(t')}{dt'} \;=\; ((\frac{d\varphi_n(t')}{dt'} + E_n')b_n(t') \;+ i\frac{b_n(t')}{dt'})(\exp(-\,i\,\varphi_n(t') - iE_n'\,t') \tag{7.94}$$

and

$$(\tfrac{d\varphi_n(t')}{dt'} + E_n')b_n(t')^2 \;+ ib_n(t')\tfrac{b_n(t')}{dt'} = E_n' b_n(t')^2 \;+ a_n(t')^* M_n(t'). \tag{7.95}$$

The split in real and imaginary part yields

$$b_n(t')^2 \tfrac{d\varphi_n(t')}{dt'} = Re(a_n(t')^* M_n(t')) \tag{7.96}$$

respectively

$$b_n(t')\tfrac{b_n(t')}{dt'} = Im(a_n(t')^* M_n(t')). \tag{7.97}$$

These equations result in

$$\varphi_n(t'+dt') = \varphi_n(t') + \tfrac{dt'}{b_n(t)^2} Re(a_n(t')^* M_n(t')) \tag{7.98}$$

and

$$b_n(t'+dt') = b_n(t') + \tfrac{dt'}{b_n} Im(a_n(t')^* M_n(t')). \tag{7.99}$$

The changes are very small due to eq.(7.89). Eq.(7.94) can therefore be approximated by

$$i\frac{da_n(t')}{dt'} \cong (E'_n) b_n(t') \exp(-iE'_n t'). \tag{7.100}$$

These limits of the coefficients explain the behaviour of the expanded cloud which is independent of the self-field. However, its influence must be taken into account when the field of an opposite charge causes an attraction of the cloud.

Assuming such a behaviour, one can also given a weak explanation of the scattering effect, caused by a double slit screen (see Fig. 13). Numerical calculations show that the charge distribution forms in a rest system an expanding spherical shell. If the influence of the Lorentz transformation is neglected, one obtains also the shown scape of the charge cloud. However, one dimension is compress by the Lorentz contraction for a fast moving electron field. Due to the charge conservation law one can expect that an electron field with high speed and a special direction could completely pass the scree. To show that, one has to solve the linked time-dependent Schrödinger equation with the potential of the screen. The screen potential can be described by the field of many static dipole moments with a proper arrangement. This changes the shape of the cloud in time and internal vibrations are the consequence. An exact solution ensures that the charge conservation and the unit of the field are given. The direction of the charge centre behind the screen depends on the path and the speeds of the of the incident charge cloud. This could result in scattering patterns on a detector at a great distance. In addition, the extended electron field must be brought together by the external electromagnetic field in the vicinity of the detector. But, the last considerations are only hypotheses and guidelines for further investigations. It should be possible to determine the behaviour in the rest system of the electron when the screen with the slit moves. For this aim one can use the following solution function

$$u(\mathcal{R}, t') = \sum_{n\,m} Y_{n\,m} f_{n\,m}(r, t') \tag{7.101}$$

with

$$f_{n\,m}(r, t') = \sum_{n\,m} c_{n\,m}(t')\, h_{n\,m}(r). \tag{7.102}$$

This requires the calculation of eigenfunctions $h_{n\,m}(r)$ for a large R (see Appendix C).

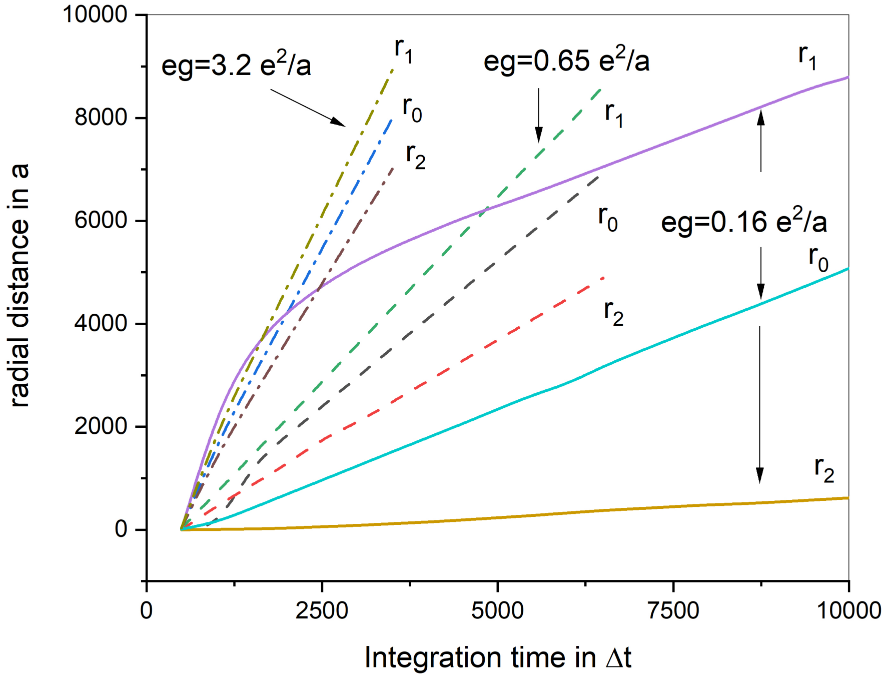

Fig. 12: Expansion of the charge cloud:

$RR(t') = r_0(t/\Delta t)$ mean radius (see eq.7.90),

$r_1(t/\Delta t)$ low limit radius ($\int_0^{r1} \varrho_{0\,0}(r) r^2 dr < 0.001$),

$r_2(t/\Delta t)$ high limit radius ($\int_0^{r2} \varrho_{0\,0}(r) r^2 dr = 0.999$).

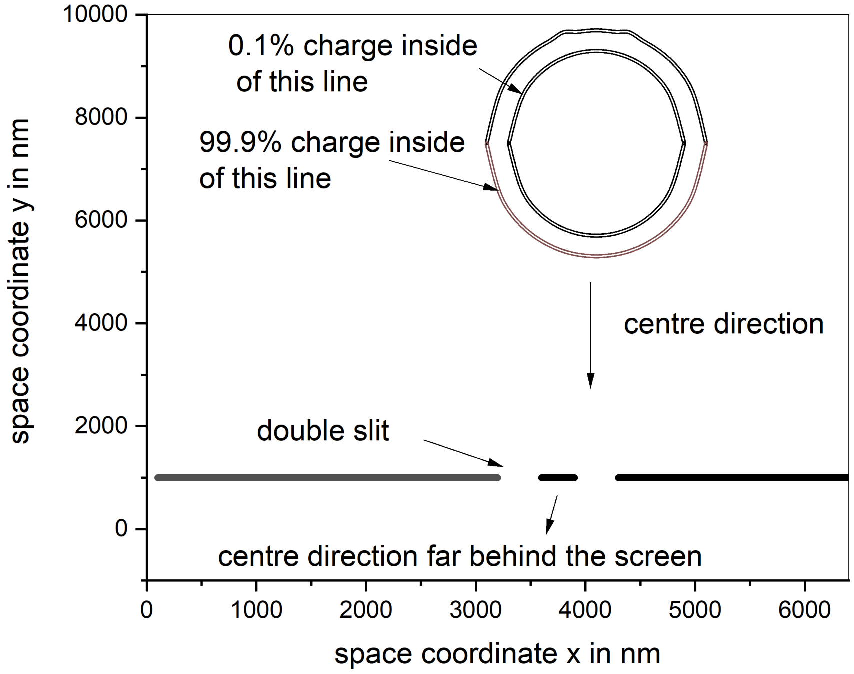

Fig 13: Symbolic sketch of the double slit experiment:

The come in sphere-shell contains the most of the charge of a single electron which moves in the arrow direction. The screen can be described by special potentials that influence the dynamic solution of the Schrödinger equation. Possible interpretation: Different paths of incident clouds cause different escape directions. A sum of individual clouds at different times of incoming could, after an appropriate time, produce a pattern on a screen far behind the slit. Such screen must be able to capture the clouds.

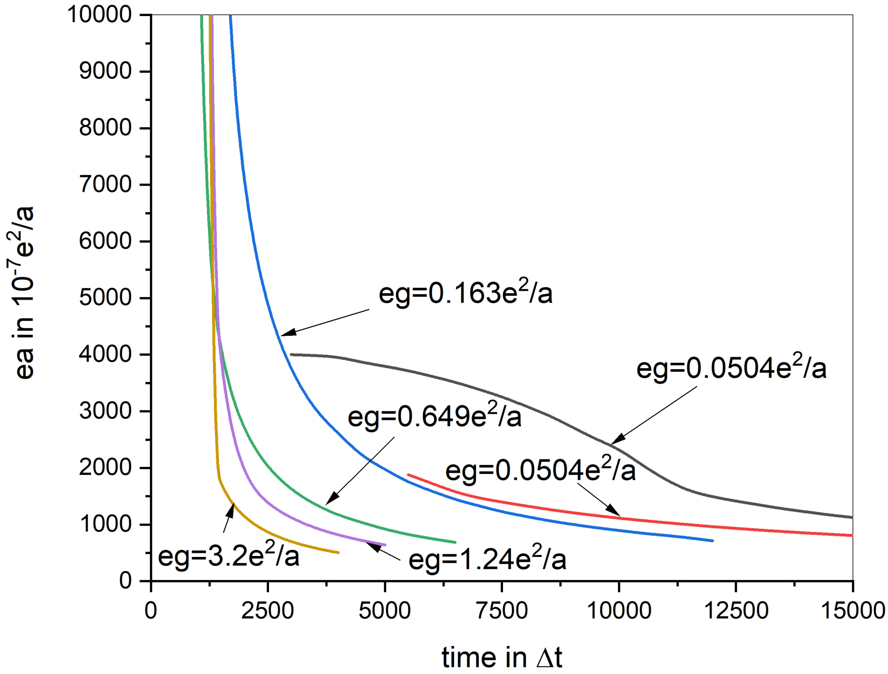

Fig. 14: Change of electrostatic energies ea of ”free” solutions.

Appendix A

Numerical calculations of solution functions

The time integration of eq.(3.55) in a small step yields

$$\begin{aligned} a_k^0(t'_0 + \bar{\Delta}\, t') \;=\; a_k^0(t'_0) \\ -i\,\bar{\Delta}\, t'\,(E_k^0\, a_k^0(t'_0) \;+\; \textstyle\int_0^R r^2 A_{0\,0}^{0\,0}(r, t'_0)(h_k^0(r))^*\; f_{0\,0}^0(r, t'_0)dr). \end{aligned} \tag{A.1}$$

With the aid of this equation, the time development of $a_k^0(t')$ and thus also of $f_{0\,0}^0(r, t')$ can be determined step by step. For this reason the integrals

$${}^0M_k^x \;=\; \textstyle\int_0^R r^2 A_{0\,0}^{0\,0}(r, t'_0)(h_k^0(r))^*\; f_{0\,0}^0(r, t'_0)dr \tag{A.2}$$

must be calculated. Therefore, we put $g(r) = A_{0\,0}^{0\,0}(r, t'_0)(h_k^0(r))^*\; f_{0\,0}^0(r, t'_0)$ in eq.(3.56) for different k and fixed t'_0 and obtain $f_{0\,0}^0(r, t'_0 + \bar{\Delta}\, t')$ according to the eq.(A1). A few small programs were implemented for the calculation of the function $f_{0\,0}^0(r, t')$ step by step. The size of $\bar{\Delta}\, t'$ must be so small that all conservation laws are satisfied. The start function $f_{0\,0}^0(r, t'_0)$ is mostly unknown and arbitrary. It makes sense to start with a defined deviation from a static state. The total energy without rest energy

$$eg = - \int T_{44} dV \; - \; Mc^2 \tag{A.3}$$

has a minimum in static cases. Applying eqs.(3.33/3.34), it becomes

$$\begin{array}{c} T^d_{44} + T^w_{44} \; = \; \frac{c}{2} \bar{U}^0 (-\gamma_{\bar{\mu}} \hbar \frac{\partial}{\partial x_{\bar{\mu}}} + i \frac{\acute{e}}{c} A^K_4 \gamma_4 - Mc) U^0 \\ - \frac{c}{2} \, (-\hbar \frac{\partial \bar{U}^0}{\partial x_{\bar{\mu}}} \gamma_{\bar{\mu}} - i \frac{\acute{e}}{c} A^K_4 \bar{U}_0 \gamma_4 + Mc \,)) U^0 . \end{array} \tag{A.4}$$

The integral over the share

$$\begin{array}{c} ed \; = \; \frac{c}{2} \, \int \bar{U}^0 (-\gamma_{\bar{\mu}} \hbar \frac{\partial}{\partial x_{\bar{\mu}}} + i \frac{\acute{e}}{c} A^K_4 \gamma_4) U^0 dV \\ - \frac{c}{2} \, \int (-\hbar \frac{\partial \bar{U}^0}{\partial x_{\bar{\mu}}} \gamma_{\bar{\mu}} - i \frac{\acute{e}}{c} A^K_4 \bar{U}_0 \gamma_4 \,) U^0 dV \end{array} \tag{A.5}$$

can with the aid of the eqs.(3.26) in the used dimension be replaced by

$$ed(t') \; = \; \frac{\acute{e}^2}{\acute{a}} \sum_k (E^0_k)' (a^0_k(t'))^* a^0_k(t'). \tag{A.6}$$

For the quasi-static electromagnetic energy $ea(t') = - \int T^e_{44} dV$ one gets

$$ea(t') \; \cong \; \frac{\acute{e}^2}{2\acute{a}} \int_0^R r^2 A^{0\,0}_{0\,0}(r,t') (f^0_{0\,0}(r,t'))^* \; f^0_{0\,0}(r,t') dr. \tag{A.7}$$

The entire energy of the system

$$eg \; = \; ed + ea \tag{A.8}$$

is constant if the radiation losses are negligible. Then it must be

$$\frac{d}{dt}ea(t) \; = \; -\frac{d}{dt}ed(t). \tag{A.9}$$

Checking this condition is a test of the accuracy of the numerical results. The total eg is smaller than the Hamilton energy because of

$$eh(t) \; = \; ed(t) + 2\, ea(t). \tag{A.10}$$

Appendix B

Calculation of potentials

According to eq.(3.53) and eq.(3.68), the potentials $A^{0\,0}_{0\,0}(r,t')$ and $A^{1\,1}_{0\,0}(r,t')$ have the form

$$A(r) = (I(\infty,t') - \int\limits_0^r 1/y^2 \int\limits_0^y x^2 \rho(x,t'))dx\,dy). \tag{B.1}$$

The integral

$$I(r,t') = \int_0^r y^{-2} Is(y,t')dy \tag{B.2}$$

can be calculated approximately using the Simpson's rule in the form

$$\begin{aligned} I(m\,\Delta\,r) \;=\; I((m-2)\,\Delta\,r) \;+ Is((m-2)\Delta\,r)/(\Delta\,r\,(m-2))^2 \\ +4\,Is((m-1)\Delta\,r)/(\Delta\,r\,(m-1))^2 + Is(m\Delta\,r)/(\Delta\,r\,m)^2\,. \end{aligned} \tag{B.3}$$

$Is(y) := \int\limits_0^y x^2 \rho(x,t'))dx$ is

$$\begin{aligned} Is(m\,\Delta\,r) \;=\; Is((m-2)\,\Delta\,r) \;+ \tfrac{1}{3}((((m-2)\,\Delta\,r\,)^2 \rho((m-2)\Delta\,r) \\ +4\,((m-1)\,\Delta\,r)^2\,\rho((m-1)\,\Delta\,r) \;+\; (m\Delta\,r)^2\,\rho(m\Delta\,r)\,)\;\; for\; m = 2,3,..\,. \end{aligned} \tag{B.4}$$

The starting values are

$$\begin{aligned} Is(0) &= 0,\; Is(\Delta r) \cong (\Delta r)^3 \rho(0)/3, \\ I(0) = 0,&\;\; I(\Delta r) \cong (\Delta r)^2 \rho(0)/6\,. \end{aligned} \tag{B.5}$$

A similar procedure is possible to determine

$$A^{1\,0}_{1\,0}(r,t') = r(I^{1\,0}_{1\,0}(\infty,t') - \frac{1}{\sqrt{3}}\int\limits_0^r 1/y^4 \int\limits_0^y x^3 \rho^{1\,0}_{1\,0}(x,t'))dx\,dy). \tag{B.6}$$

Due to $I(r) = \int_0^r y^{-4} \int_0^y x^3 \rho(x,t')dxdy$, eqs.(B3/B4) have to be replaced by

$$\begin{aligned} I(m\,\Delta\,r) \;&=\; I((m-2)\,\Delta\,r) \;+ Is((m-2)\Delta\,r)/(\Delta\,r\,(m-2))^4 \\ &+4\,Is((m-1)\Delta\,r)/(\Delta\,r\,(m-1))^4 + Is(m\Delta\,r)/(\Delta\,r\,m)^4 \end{aligned} \tag{B.7}$$

and

$$\begin{aligned} Is(m\,\Delta\,r) \;&=\; Is((m-2)\,\Delta\,r) \;+ \tfrac{1}{3}(((m-2)\,\Delta\,r\,)^3 \rho((m-2)\Delta\,r) \\ +4\,((m-1)\,\Delta\,r)^3\,\rho((m-1)\,\Delta\,r) \;&+\; (m\Delta\,r)^3\,\rho(m\Delta\,r)\,)\;\; for\;\, m = 2,3,.., \end{aligned} \tag{B.8}$$

with the starting values

$$\begin{aligned} Is(0) &= 0,\; Is(\Delta r) \cong (\Delta r)^4 \rho(0)/4, \\ I(0) = 0,&\;\; I(\Delta r) \cong (\Delta r)\,\rho(0)/4\,. \end{aligned} \tag{B.9}$$

In case of

$$A^{1\,1}_{2\,0}(r,t') = r^2\,(I^{1\,1}_{2\,0}(\infty,t') - \int\limits_0^r 1/y^6 \int\limits_0^y x^4 \rho^{1\,1}_{2\,0}(x,t))dx\,dy) \tag{B.10}$$

the integral $I(r) = \int_0^r y^{-6} \int_0^y x^6 \rho'(x,t')dxdy$ leads to

$$\begin{aligned} I(m\,\Delta\,r) \;&=\; I((m-2)\,\Delta\,r) \;+ Is((m-2)\Delta\,r)/(\Delta\,r\,(m-2))^6 \\ &+4\,Is((m-1)\Delta\,r)/(\Delta\,r\,(m-1))^6 + Is(m\Delta\,r)/(\Delta\,r\,m)^6 \end{aligned} \tag{B.11}$$

and

$$\begin{aligned} Is(m\,\Delta\,r) \;&=\; Is((m-2)\,\Delta\,r) \;+ \tfrac{1}{5}(((m-2)\,\Delta\,r\,)^5 \rho'((m-2)\Delta\,r) \\ &+4\,((m-1)\,\Delta\,r)^5\,\rho'((m-1)\,\Delta\,r) \;+\; (m\Delta\,r)^5\,\rho'(m\Delta\,r)\,)\;\; for\;\; m = 2, 3, . \end{aligned} \tag{B.12}$$

The starting values are

$$\begin{aligned} Is(0) \;&= 0,\;\; Is(\Delta\,r) \cong\; (\Delta r)^2\,\rho'(0)/14, \\ I(0) &= 0,\;\;\; I(\Delta\,r) \cong\; (\Delta r)^7\,\rho'(0)/7\,. \end{aligned} \tag{B.13}$$

Here the modification $\rho'(r,t') = r^{-2}\rho(r,t')$ is introduced because $\rho'(r.t')$ is not singular on r=0. Other numerical methods can also be applied that take singularities into account.

Appendix C

Calculation of eigenfunctions

C.1 *Eigenfunctions of s-states*

To find eigenfunctions h_k^0 and eigenvalues $E_k^0 := (E_k^0)'$ of eq.(3.49) in the range $0 \leq r \leq R$, the following equation

$$E_k^0 h_k^0(r) = -((0.5/r^2)(\partial/\partial r)(r^2 \partial/\partial r) + 2/r) h_k^0(r) \tag{C.1}$$

must be solved numerically. After reorientation and integration of eq.(C1) it gives

$$h_k^0(r) = -2 \int_0^r y^{-2} \int_0^y x^2 (1/x + E_k^0) h_k^0(x) dx dy. \tag{C.2}$$

This expression is similar to the integral in eq.(B1). If we replaced $\rho(r)$ by $g(r) = (1/r + E_k^0) h_k^0(r)$, we can write

$$h_n^0(mdr) \cong h_k^0(0) - I_k^0(m\Delta r). \tag{C.3}$$

Here the integral $I_k^0(m\Delta\, r)$ is given by

$$\begin{aligned} I_k^0(m\,\Delta\, r) \;=\; I_k^0((m-2)\,\Delta\, r) \;+ I S_k^0((m-2)\Delta\, r)/(\Delta\, r\,(m-2))^2 \\ +4\, I S_k^0((m-1)\Delta\, r)/(\Delta\, r\,(m-1))^2 + I S_k^0(m\Delta\, r)/(\Delta\, r\, m)^2 \end{aligned} \tag{C.4}$$

with

$$\begin{aligned} I S_k^0(m\,\Delta\, r) \;=\; I S_k^0((m-2)\,\Delta\, r) \\ +\tfrac{1}{3}(f_k^0((m-2)\Delta\, r)(m-2)\,\Delta\, r\,(1+E_k^0(m-2)\Delta\, r)) \\ +4\, f_k^0((m-1)\,\Delta\, r)\,(m-1)\,\Delta\, r)\,(1+E_k^0(m-1)\Delta\, r)) \\ +f_k^0(m\Delta\, r)\;m\Delta\, r)\,(1+E_k^0 m\Delta\, r)))\;\; for\;\, m=2,3,.. \end{aligned} \tag{C.5}$$

and the starting values by

$$\begin{aligned} I_k^0(0) \;=0,\;\; I_k^0(\Delta\, r) = \;(1+E_k^0\Delta r/3)\Delta r\, h_k^0(0), \\ I S(0) = \;0,\;\; I S(1) = \;(0.5\Delta r + E_k^0/3)(\Delta r)^2\, h_k^0(0), \\ h_k^0(\Delta\, r) \;=\; (1+E_k^0\Delta r/3)\Delta r\, h_k^0(0)\,. \end{aligned} \tag{C.6}$$

In eq.(C3) the missing value can be described by the extrapolation

$$h_k^0(m\Delta\, r) \;=\; 2\, h_k^0((m-1)\Delta\, r) \;-\; h_k^0((m-2)\Delta\, r)\,. \tag{C.7}$$

Using these formulas, a simple program was realised for calculating of eigenfunctions in the mentioned range. The functions $h_k^0(r)$ must satisfy the condition

$h_k^0(R) = 0$. This can be achieved by varying the input parameter E_k^0, while the staring value $h_k^0(0)$ follows from the normalising integral

$$(h_k^0(r))^2 = 1/(\int_0^R r^2 (\hat{h}_k^0(r))^2 \, dr). \tag{C.8}$$

Here $\hat{h}_k^0(r)$ represents a function that satisfies the condition $\hat{h}_k^0(R) = 0$. The number of zero points in interval $0 \leq r \leq R$ must be given by the index k, where k includes the last point r=R. For k=n the functions $h_k^0(r)$ are identical to the known radial functions $F_{n\,0}(r)$ of the ns-states (Fig. 1) as long as $F_{n\,0}(R)$ and whose derivations near R are negligible. With R=100 that is given up to k=5. For $k > 5$ up to k=120 the functions $h_k^0(r)$ were calculated numerically and stored in a file. The orthogonality of these functions is proven. By numerical calculations it could be demonstrated that the integrals $\int_0^R r^2 f_k^0(r) f_n^0(r) dr = \delta(k-n) + \Delta I^0(k,n)$ show only slightly deviations $\Delta I^0(k,n)$ from the ideal value $\delta(k-n)$. $\Delta I^0(k,n)$ is for $n \neq k$ less than 0.00001 and for k=n less than 0.0000001. Therefore the eigenfunctions $h_k^0(r)$ with the associated eigenvalues E_k^0 can be used to solve eq.(3.50). $h_k^0(r)$ and E_k^0 depend on the choice of R, how mentioned above. However, solutions $f_{0\,0}^0(r,t)$ of eq.(3.50) are independent on R as long as R is large enough that the functions $f_{0\,0}^0(r,t)$ and whose derivations near r=R are approximately zero during the entire calculation time. The asymptotic solutions for $E_k^0 > 0$ and large r have the form $f_k^0(r) = g_k^0 sin(\sqrt{2\,E_k^0} r + \phi_k^0)/r$. Amount g_k^0 and phase ϕ_k^0 result from numerical calculations.

C.2 *Eigenfunctions of p-states*

The radial eigenfunctions $h_k^1(r)$ with the eigenvalue $E_k^0 := (E_k^0)'$ of p-states in a sphere from a radius R must, according to eq.(3.74), fulfil the following equation

$$E_k^1\, h_k^1(r) \;=\; -\,((0.5/r^2)(\partial/\partial r)(r^2\partial/\partial r) - 1/r^2 + 1/r)h_k^1(r). \tag{C.9}$$

This leads to

$$0.5/r^3(\partial/\partial r)(r^4\partial/\partial r)(h_k^1(r)/r) \;=\; -(E_k^1\; + 1/r)h_k^1(r) \tag{C.10}$$

and thus to

$$(h_k^1(r)/r) \;-\; I_k^1(0) \;=\; -2\int_0^r y^{-4}\int_0^y x^3(E_k^1\; + 1/x)h_k^1(x)\,dx\,dy. \tag{C.11}$$

A numerical approximation of the internal integral of eq.(C12) is given by

$$IS_k^1(y) \;=\; IS_k^1(y-\Delta y) \;+\int_{y-\Delta y}^{y} x^3(E_k^1\; + 1/x)h_k^1(x)\,dx. \tag{C.12}$$

With the limit of $h_k^1(r) = rB_k^1(r)$ for $r \longrightarrow 0$ the eq.(C11) yields

$$IS_k^1(y) \;= -\int_0^y x^3(E_k^1 + \frac{1}{x})h_k^1(x)dx \cong \,-\,B_k^1(0)\,(0.2\,E_k^1\,y^5 +\, 0.25\,y^4). \tag{C.13}$$

Using this approximation the second integration can be described by

$$I_k^1(r) \;=\; I_k^1(r-\Delta r) + 2\int_{r-\Delta r}^{r} y^{-4}\, IS_k^1(y)\, dy. \tag{C.14}$$

and therefore becomes

$$h_k^1(r) = \; r(B_k^1(0) \;-\; I_k^1(r)). \tag{C.15}$$

In addition if r is large enough, the following transformation (see eq.(C10))

$$h_k^1(r) = \; -2\int_0^r y^{-2}\int_0^y x^2(E_k^1 + 1/x - 1/x^2)h_k^1(x)dxdy \tag{C.16}$$

can be used. With

$$g(x) = (E_k^1 + 1/x - 1/x^2)h_k^1(x), \tag{C.17}$$

the methods of the s-case can also be applied in this range. Therefore, three different integration areas were introduced in the program. The selected examples require an integration border of R=200 and 250 eigenfunctions. k again stands for the number of zero points. The well-known description of h_k^1 can be applied up to $k = 8$, because all values $h_k^1(r > 0.8R)$ are small enough for these cases. The numerical accuracy around the values $E_k^1 \cong 0$ (k=6 until 15) is critical. Here the error coefficients $\Delta I^1(k,n)$ come close to 0.0001, while for other combinations of n and k these coefficients have values of about 0.00001 or 0.0000001 for k=n.

Appendix D

The potential integral of a free solution

The numerical determination of the potential integral formulated in section 7.1 can be replaced by the following analytical calculation of

$$ {}_{n\,m}M_{j\,k} = \int\int \frac{(Y_{0\,0})^4}{r_{0\,0'}} h_n(r)h_m(r)h_j(r')h_k(r')dV dV'. \tag{D.1}$$

This reads

$$ {}_{n\,m}M_{j\,k} = \int_0^R r^2 h_n(r)h_m(r) \int_r^\infty \frac{1}{s^2} \int_0^s (r')^2 h_j(r')h_k(r')dr'dsdr. \tag{D.2}$$

The internal integral

$$ H_{j\,k}(s) = \frac{1}{R} \int_0^s (\cos((j-k)\omega\, r') - \cos((j+k)\omega\, r'))dr' \tag{D.3}$$

yields

$$H_{jk} = \begin{pmatrix} \frac{1}{R}(s - \frac{1}{2j\omega} sin(2j\omega s) & for & s \leq R, j = k \\ 1 & for & s > R, j = k \\ \frac{1}{R}(\frac{1}{(j-k)\omega} sin((j-k)\omega s) - \frac{1}{(j+k)\omega} sin((j+k)\omega s) & for & s \leq R, j \neq k \\ 0 & for & s > R, j \neq k \end{pmatrix} \quad \text{(D.4)}$$

and therefore is

$$\int_r^\infty \frac{1}{s^2} H_{jk}(s) ds = \begin{pmatrix} \frac{1}{R}(1 - ln(\frac{r}{R}) - \frac{sin(2j\omega r)}{2j\omega r} + Ci(2j\omega R) - Ci(2j\omega r)) \\ for\, j = k \\ \frac{1}{R}((\frac{sin((j-k)\omega r)}{(j-k)\omega r} + Ci((j-k)\omega r) - Ci((j-k)\omega R)) \\ -\frac{sin((j+k)\omega r)}{(j+k)\omega r} - Ci((j+k)\omega r) + Ci((j+k)\omega R))) \\ for\, j \neq k \end{pmatrix}. \quad \text{(D.5)}$$

Finally $n\,mM_{j\,k}$ is given by

$$\begin{aligned} {}_{n\,m}M_{j\,k} &= \tfrac{1}{2R\pi}(\tfrac{1}{\beta} - \tfrac{1}{\alpha}) Si((\beta-\alpha)\pi) + \tfrac{R}{2\pi}(\tfrac{1}{\beta} + \tfrac{1}{\alpha}) Si((\beta+\alpha)\pi) \\ &+ \tfrac{1}{2R\pi}(\tfrac{1}{\beta} - \tfrac{1}{\delta}) Si((\beta-\delta)\pi) + \tfrac{R}{2\pi}(\tfrac{1}{\beta} + \tfrac{1}{\delta}) Si((\beta+\delta)\pi) \\ &+ \tfrac{1}{2R\pi}(\tfrac{1}{\gamma} - \tfrac{1}{\alpha}) Si((\gamma-\delta)\pi) + \tfrac{1}{2R\pi}(\tfrac{1}{\gamma} + \tfrac{1}{\alpha}) Si((\gamma+\alpha)\pi) \\ &+ \tfrac{1}{2R\pi}(\tfrac{1}{\gamma} - \tfrac{1}{\delta}) Si((\gamma-\delta)\pi) + \tfrac{1}{2R\pi}(\tfrac{1}{\gamma} + \tfrac{1}{\delta}) Si((\gamma+\delta)\pi). \end{aligned} \quad \text{(D.6)}$$

Here we have introduce the notations $\alpha = j - k$, $\beta = n - m$, $\gamma = m + n, \delta = j + k$, and $\omega = \pi/R$ and the integral-functions $Si(x) = \int_0^x \frac{1}{y} \sin(y) dy$ and $Ci(x) = \int_x^\infty \frac{1}{y} \cos(y) dy$. In addition, one must taken into account the following limits:

$$
\begin{aligned}
\alpha \to 0, \beta \to 0: \; {}_{n\,n}M_{j\,j} \;=\; & \tfrac{2}{R} + \tfrac{1}{2R\pi}(\tfrac{2}{\delta}Si(\delta\,\pi) + \tfrac{2}{\gamma}Si(\delta\,\pi) + \\
& +(\tfrac{1}{\gamma} - \tfrac{1}{\delta})Si((\gamma-\delta)\pi) + (\tfrac{1}{\gamma} + \tfrac{1}{\delta})Si((\gamma+\delta)\pi)) + ... \\
\beta \to 0: \; {}_{n\,n}M_{j\,k} \;=\; & \tfrac{1}{2R\pi}(\tfrac{2}{\alpha}Si(\alpha\,\pi) + \tfrac{2}{\delta}Si(\delta\,\pi) \\
& + (\tfrac{1}{\gamma} - \tfrac{1}{\alpha})Si((\gamma-\delta)\pi) + (\tfrac{1}{\gamma} + \tfrac{1}{\alpha})Si((\gamma+\alpha)\pi) \\
& +(\tfrac{1}{\gamma} - \tfrac{1}{\delta})Si((\gamma-\delta)\pi) + (\tfrac{1}{\gamma} + \tfrac{1}{\delta})Si((\gamma+\delta)\pi))) + ... \\
\alpha \to 0: \; {}_{n\,m}M_{j\,j} \;=\; & \tfrac{1}{2R\pi}(\tfrac{2}{\beta}Si(\beta\,\pi) + \tfrac{2}{\gamma}Si(\delta\,\pi) \\
& + (\tfrac{1}{\beta} - \tfrac{1}{\delta})Si((\beta-\delta)\pi) + (\tfrac{1}{\beta} + \tfrac{1}{\delta})Si((\beta+\delta)\pi) \\
& +(\tfrac{1}{\gamma} - \tfrac{1}{\delta})Si((\gamma-\delta)\pi) + (\tfrac{1}{\gamma} + \tfrac{1}{\delta})Si((\gamma+\delta)\pi)) + ... \;.
\end{aligned}
\tag{D.7}
$$

The step from eq.(D5) to eq.(D6) requires the integrals:

$$
\begin{aligned}
& \tfrac{1}{\alpha\pi}\textstyle\int_0^R rsin(\alpha\omega r)h_n(r)h_m(r)dr \\
= \; & \tfrac{1}{2\pi R\alpha}\,(Si((\alpha+\beta)\pi) + Si((\alpha-\beta)\pi) \\
& -Si((\alpha-\gamma)\pi) - Si((\alpha+\gamma)\pi)),
\end{aligned}
\tag{D.8}
$$

$$
\begin{aligned}
& \tfrac{1}{2\pi\delta R}\textstyle\int_0^R rsin(\omega\delta r)h_n(r)h_m(r)dr \\
= \; & \tfrac{1}{2\pi R\delta}\,(Si((\delta-\gamma)\pi) + Si((\delta\gamma)\pi) \\
& - Si((\delta+\beta)\pi) - Si((\delta-\beta)\pi)),
\end{aligned}
\tag{D.9}
$$

$$
\begin{aligned}
& \tfrac{1}{R}\textstyle\int_0^R r^2Ci(\omega\alpha r)h_n(r)h_m(r)dr \\
= \; & \tfrac{1}{2\pi R\beta}\,(Si((\alpha+\beta)\pi) + Si((\alpha-\beta)\pi) \\
& - \tfrac{1}{2\pi R\gamma}\,(Si((\alpha+\gamma)\pi) + Si((-\alpha+\gamma)\pi))
\end{aligned}
\tag{D.10}
$$

or

$$
\begin{aligned}
&\tfrac{1}{R}\textstyle\int_0^R r^2 Ci(\omega\delta r) h_n(r) h_m(r) dr \\
&= \tfrac{1}{2\pi R\beta}\left(Si((\beta+\delta)\pi) + Si((\beta-\delta)\pi)\right. \\
&- \tfrac{1}{2\pi R\gamma}\left(Si((\delta+\gamma)\pi) + Si((\delta-\gamma)\pi)\right).
\end{aligned} \tag{D.11}
$$

Appendix E

General Notations

Conventions:

$x_1 = x,\ x_2 = y,\ x_3 = z,\ x_4 = ict$	four dimensional coordinates
$x_{\bar{\mu}} x_{\bar{\mu}} := \sum\limits_{\bar{\mu}=1}^{3} x_{\bar{\mu}} x_{\bar{\mu}}$	sum convention for three dimension
$x_{\mu} x_{\mu} := \sum\limits_{\mu=1}^{4} x_{\mu} x_{\mu}$	sum convention for four dimension
$\mathbf{R} = (x_1, x_2, x_3)$	three dimensional space coordinate
$r_{0\,1} = \lvert \mathbf{R} - \mathbf{R}_1 \rvert$	distance
$\delta(n) = 1$ for n=0 and 0 for $n \neq 0$	trigger function
$u(x) \frac{\overleftarrow{\partial}}{\partial x} = \frac{\partial u(x)}{\partial x}$	special definition of a derivation;
$\gamma_{\bar{\mu}} = \begin{pmatrix} 0 & -i\sigma_{\bar{\mu}} \\ i\sigma_{\bar{\mu}} & 0 \end{pmatrix}$	special matrix
$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	Pauli-matrix
$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	
$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	
$\overrightarrow{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$	

$\gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$	special matrix;
$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	

Constants:

$\acute{e}$	elementary charge, $\acute{e} = 1.60\,10^{-19} A\,s$
$\acute{c}$	speed of light
h	Planck constant, $h = 6.6262 \times 10^{-34} W s^2$
$\hbar = h/(2\pi)$	used form, $\hbar = \acute{e}^2/(\acute{\alpha}\acute{c}) = \Delta t\, \acute{e}^2/\acute{a}$
$\acute{M}$	rest mass of the electron, $\acute{M} = 0.91096 \times 10^{-30}$ kg (= constant of the Dirac equation)
$\acute{M}\,\acute{c}^2$	rest energy of the electron, $\acute{M}\,\acute{c}^2 \cong 0.5 MeV$
$\acute{a} = \hbar^2/(\acute{M}\acute{e}^2)$	Bohr-radius, $\acute{a} = 5.292 * 10^{-11} m$
$\acute{e}/\acute{a}^2$	used energy unit, $\acute{e}/(\acute{a})^2 = 27.21\, eV$
$\Delta t = \acute{a}/(\acute{c}\,\acute{\alpha})$	used time unit, $\Delta t = 0.24167 * 10^{-16} s$
t	time in s
t'	time in Δt
$\acute{\alpha} = \hbar/(\acute{a}\,\acute{M}\acute{c}^2)$	fine structure constant, $\acute{\alpha} = 0.0072974$;

Functions of the one electron field:

$U(\mathbf{R}, t) = \sum_p c_p(t) H_p(\mathbf{R})$	solution of the Dirac equation
$U^z(\mathbf{R}, t) = \sum_p c_p^z(t) H_p^z(\mathbf{R})$	share of the solution
z	number of excitation level
$\bar{U}(\mathbf{R}, t) = \sum_p (c_p(t))^* \bar{H}_p(\mathbf{R})$	solution of the conjugated Dirac equation

$H_p = \begin{pmatrix} \hat{h}_p \\ \check{h}_p \end{pmatrix} = \begin{pmatrix} h_p^1 \\ h_p^2 \\ h_p^3 \\ h_p^4 \end{pmatrix}$	eigenfuction of the Dirac equation
$\bar{H}_p = ((h_p^1)^* \;\; (h_p^1)^* - (h_p^1)^* - (h_p^1)^*)$	conjucated complex function
$\hat{h}_p \cong \begin{pmatrix} 1 \\ 0 \end{pmatrix} h_p$	special relation between the functions
$u = \sum_p a_p(t) h_p$	solution of the Schroedinger equation
$Y_{j\,m}$	spherical area functions;

Field functions:

$J_4^K = -i\acute{e}\varrho^K$	charge component of the nucleus
$J_\mu = i\acute{e}\bar{U}\gamma_\mu U$	component of the electron current:
	$\mu = \bar{\mu}$: normal current
	$\mu = 4$: charge component
$J_\mu = i\acute{e}\sum_{z\,z-s} \bar{U}^z \gamma_\mu U^{z+s}$	distribution of the current:
	z and z-s: excitation levels
$A_\mu(J_\mu)$	component of the vector potential of the self field
$J_\mu^K,\; A_\mu^K(J_\mu^K)$	current and vector potential of the nucleus (eq.(300)
E	electric field strength
H	magnetic field strength;

Field tensors:

$T_{\nu\,\mu} = T_{\nu\mu}^d + T_{\nu\mu}^w + \Delta\, T_{\nu\mu}^m + T_{\nu\mu}^e$	complete field tensor of one electron and the related electromagnetic field
$T_{\nu\,\mu}^d$	tensor component of the Dirac field
${}^{z\;z-s}T_{\nu\,\mu}^d$	special component of the tensor: z and z-s excitation levels

$T^{w}_{\nu\mu}$	interaction tensor
$\Delta T^{m}_{\nu\mu}$	transformation tensor
$T^{e}_{\nu\mu}$	tensor of the electromagnetic self field;

Notations of the two electron system:

$P_j = \overrightarrow{P}_j$	Dirac operator react to $\mathcal{R}_j$
$P^s_j = \overrightarrow{P}^s_j$	Schröedinger operator react to $\mathcal{R}_j$
$Q_{1\,2} = \overrightarrow{Q}^4_{1\,2}$	Dirac operator react to $\mathcal{R}_1, \mathcal{R}_2$
$Q^s_{1\,2} = \overrightarrow{Q}^s_{1\,2}$,	Schroedinger operator react to $\mathcal{R}_1, \mathcal{R}_2$
$U(\mathcal{R}_1, t, \uparrow)$	solution of the Dirac equation of electron 1
$V(\mathcal{R}_2, t, \uparrow)$	solution of the Dirac equation of electron 2
$\Psi^{\pm}(\mathcal{R}_1, \mathcal{R}_2)$	eigenfunctions of $Q_{1\,2}$
$\psi^{\pm}(\mathcal{R}_1, \mathcal{R}_2)$	eigenfunctions of $Q^s_{1\,2}$
$A^{H\,H}_{\mu}$, $A^{U\,U}_{\mu}$	vector potentials
$({}^{HH}_{n'n}M^{KK}_{m'm})_{\mu} =$ $\int (1/r_{1\,2})\bar{H}_{n'}(\mathcal{R}_1)\gamma_{\mu}H_n(\mathcal{R}_1)$ $\times \bar{K}_{m'}(\mathcal{R}_2)\gamma_{\mu}K_m(\mathcal{R}_2)dV_2dV_1$	electro-magnetic energy integral
$W^{\pm}_{n\,m}(\mathcal{R}_1, \mathcal{R}_2, t, \uparrow\downarrow + \downarrow\uparrow)$	A Combination of eigenfunctions $\Psi^{\pm}_{n_-,m}(\mathcal{R}_1, \mathcal{R}_2)$ with different orientation;

Appendix F

References

1. Selleri, F., "Die Debatte um die Quantentheorie", Vieweg, Braunschweig 1990;

2. Baumann, K. und Sexl, R. U., "Die Deutung der Quantentheorie" (1987, Vieweg, Braunschweig 1990;

3. Planck, M., "Determinismus oder Indeterminismus", Vortrag, published: Johann Ambrosius Barth/Verlag/Leipzig;

4. Einstein, A., "Physik und Realität", Journal of Franklin Institute, Vol. 221, March, 1936, No. 3;

5. Heisenberg, W., "Physikalische Prinzipien der Quantentheorie", Bibliographisches Institut, Mannheim 1958;

6. M. Born, P. Jordan,"Elementare Quantenmechanik", Springer 1930;

7. Heisenberg, W., "Quantentheorie und Philosophie", Reclam, Stuttgart 1979;

8. Meissner, W., "Wie tot ist Schrödingers Katze?", B.I Wiss.-Verl. 1992;

9. Schröedinger, E., "Die gegenwärtige Situation in der QM", in: Die Deutung der Quantentheorie, Vieweg, Braunschweig 1984;

10. Max Born, "Problems of Atomic Dynamics", Dover Publications 2004;

11. E. Schrödinger, 1.-4. Mitteilung, "Quantisierung als Eigenwertproblem", Ann. Phys. 79-81, 1926;

12: D.ter Haar, "Quantentheorie", Einführung und Originaltexte, Akademie Ver-

lag/Pergamon Press/Vieweg+Sohn 1970;
13. Einstein, A., "Grundzüge der Relativitätstheorie", WTB 58, Akademie-Verlag, Berlin 1970;
14. Schmutzer, E.: "Symmetrie und Erhaltungssätze der Physik", WTB Ban4 75, Akademie-Verlag Berlin 1979;
15. Bjorken, J. P., Drell, S. D.: "Relativistic Quantum Mechanics", McGraw-Hill, Inc. 1964;
16. Achieser, A.I. und Brestetzki, W. B.: "Quantenelectrodynamik", B.G. Teubner Verlagsgesellschaft, Leipzig 1963;
17. Bogoljubov, N. N., Sirkov, D. V., "Introduction to the Theory of Quantized Fields", 2. Ed., New York, John Wiley+Sons, Inc. 1063;
18. Dawydow, A. S. : "Quantenmechanik, Deutscher Verlag der Wissenschaften", Berlin 1970;
19. Mie, G., "Grundlagen einer Theorie der Materie", Ann. d. Physik V. 37, 1912;
20. Sommerfeld, A. "Vorlesung ber Theoretische Physik", V. 3: Geest u. Portig K.-G., Leipzig 1961;
21. Landau, L. D., Lifschitz, E. E., "Lehrb. Theort. Physik" V. 2, 1963; V. 3, 1971, V. 4, 1980, Akademie-Verlag, Berlin;
22. Iwanenko, D., Sokolow, A.: "Klassische Feldtheorie" Akademie Verlag Berlin 1963;
23. Wantke, K.-D., "Einige Eigenschaften d. Grundgleichungen d. elektromag. Wechselw.", Vortr. Magnus-Haus, Deut. Phys. Ges. 2018;
24. Weizel, W.,"Lehrbuch der Theoret. Physik", Springer 1949, V. 2 "Struktur der Materie";
25. Grimsehl, V. 4 (Herausgeber: W. Schallreuter): "Struktur der Materie, B. G. Teubner Verlagsgesellschaft", Leipzig 1975;
26. Zuhrt, H. : "Elektromagnetische Strahlungsfelder", Springer-Verlag 2013;

27. Müller, C., "Foundations of the Mathematical Theory of Electromagnetic Waves", Springer 1969;
28. Wantke, K.-D., "Ein Beitrag z. Elektronenoptik d. Bildwandl.", Techn. Mitteil. d. RFZ 13 (1969), H.3;
29. Glaser, W, "Grundlagen der Elektronenoptik", Springer 1952.