# About Nonstandard Neutrosophic Logic

# (Answers to Imamura's "Note on the Definition of Neutrosophic Logic")

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**Abstract**.

In order to more accurately situate and fit the neutrosophic logic into the framework of nonstandard analysis, we present the neutrosophic inequalities, neutrosophic equality, neutrosophic infimum and supremum, neutrosophic standard intervals, including the cases when the neutrosophic logic standard and nonstandard components *T, I, F* get values outside of the classical unit interval *[0, 1],* and a brief evolution of neutrosophic operators.

The paper intends to answer Imamura's criticism that we found benefic in better understanding the nonstandard neutrosophic logic – although the nonstandard neutrosophic logic was never used in practical applications.

## 1. Uselessness of Nonstandard Analysis in Neutrosophic Logic, Set, Probability, et al.

Imamura's discussion [1] on the definition of neutrosphic logic is welcome, but it is useless, since from all neutrosophic papers and books published, from all conference presentations, and from all MSc and PhD theses defended around the world, etc. (more than one thousand) in the last two decades since the first neutrosophic research started *(1998-2018),* and from hundreds of neutrosophic researchers, not even a single one ever used the nonstandard form of neutrosophic logic, set, or probability and statistics in no occasion (extended researches or applications).

All researchers, with no exception, have used the *Standard Neutrosophic Set and Logic* [so no stance whatsoever of *Nonstandard Neutrosophic Set and Logic*], where the neutrosophic components *T, I, F* are subsets of the standard unit interval *[0, 1]*.

Even more, for simplifying the calculations, the majority of researchers have utilized the *Single-Valued Neutrosophic Set and Logic* {when *T, I, F* are single numbers from *[0, 1]*}, on the second place was *Interval-Valued Neutrosophic Set and Logic* {when *T, I, F* are intervals included in *[0, 1]},* and on the third one the *Hesitant Neutrosophic Set and Logic* {when *T, I, F* were discrete finite sets included in *[0, 1]*}.

In this direction, there have been published papers on single-valued "neutrosophic standard sets" [12, 13, 14], where the neutrosophic components are just *standard real numbers,* considering the particular case when $0 \leq T + I + F \leq 1$ (in the most general case $0 \leq T + I + F \leq 3$).

Actually, Imamura himself acknowledges on his paper [1], page 4, that:

> "neutrosophic logic does not depend on transfer, so the use of non-standard analysis is not essential for this logic, and can be eliminated from its definition".

Entire neutrosophic community has found out about this result and has ignored the non-standard analysis in the studies and applications of neutrosophic logic for two decades.

## 2. Applicability of Neutrosophic Logic et al. vs. Theoretical Nonstandard Analysis

Neutrosophic logic, set, measure, probability, statistics and so on were designed with the primordial goal of being applied in practical fields, such as:

Artificial Intelligence, Information Systems, Computer Science, Cybernetics, Theory Methods, Mathematical Algebraic Structures, Applied Mathematics, Automation, Control Systems, Big Data, Engineering, Electrical, Electronic, Philosophy, Social Science, Psychology, Biology, Biomedical, Engineering, Medical Informatics, Operational Research, Management Science, Imaging Science, Photographic Technology, Instruments, Instrumentation, Physics, Optics, Economics, Mechanics, Neurosciences, Radiology Nuclear, Medicine, Medical Imaging, Interdisciplinary Applications, Multidisciplinary Sciences etc. [2],

while nonstandard analysis is mostly a pure mathematics.

Since 1990, when I emigrated from a political refugee camp in Turkey to America, working as a software engineer for Honeywell Inc., in Phoenix, Arizona State, I was advised by American co-workers to do theories that have *practical applications*, not pure-theories and abstractizations as "*art pour art*".

## 3. Theoretical Reason for the Nonstandard Form of Neutrosophic Logic

The only reason I have added the nonstandard form to neutrosophic logic (and similarly to neutrosophic set and probability) was in order to make a distinction between *Relative Truth* (which is truth in some Worlds, according to Leibniz) and *Absolute Truth* (which is truth in all possible Words, according to Leibniz as well) that occur in philosophy.

Another possible reason may be when the neutrosophic degrees of truth, indeterminacy, or falsehood are infinitesimally determined, for example a value infinitesimally bigger than *0.8 (*or $0.8^+$*)*, or infinitesimally smaller than *0.8 (*or $^-0.8$*)*. But these can easily be overcome by roughly using interval neutrosophic values, for example (0.80, 0.81) and (0.79, 0.80) respectively.

I wanted to get the neutrosophic logic as general as possible [6], extending all previous logics (Boolean, fuzzy, intuitionistic fuzzy logic, intuitionistic logic, paraconsistent logic, dialethism), and to have it able to deal with all kind of logical propositions (including paradoxes, nonsensical propositions, etc.).

That's why in 2013 I extended the Neutrosophic Logic to *Refined Neutrosophic Logic* [ from generalizations of *2-valued Boolean logic* to fuzzy logic, also from the *Kleene's and*

*Lukasiewicz's* and *Bochvar's 3-symbol valued logics* or *Belnap's 4-symbol valued logic* to the most general *n-symbol* or *n-numerical* valued refined neutrosophic logic, for any integer $n \geq 1$ ], the largest ever so far, when some or all neutrosophic components *T, I, F* were respectively split/refined into neutrosophic subcomponents: $T_1, T_2, \ldots; I_1, I_2, \ldots; F_1, F_2, \ldots$ which were deduced from our everyday life [3].

## 4. From Paradoxism movement to Neutrosophy branch of philosophy and then to Neutrosophic Logic

I started first from *Paradoxism* (that I founded in 1980's as a movement based on antitheses, antinomies, paradoxes, contradictions in literature, arts, and sciences), then I introduced the *Neutrosophy* (as generalization of Dialectics, neutrosophy is a branch of philosophy studying the dynamics of triads, inspired from our everyday life, triads that have the form:

$<A>$, its opposite $<antiA>$, and their neutrals $<neutA>$, (1)

where $<A>$ is any item or entity [4].
(Of course, we take into consideration only those triads that make sense in our real and scientific world.)

The Relative Truth neutrosophic value was marked as $1$, while the Absolute Truth neutrosophic value was marked as $1^+$ (a tinny bigger than the Relative Truth's value): $1^+ >_N 1$, where $>_N$ is a neutrosophic inequality, meaning $1^+$ is neutrosophically bigger than $1$.
Similarly for Relative Falsehood / Indeterminacy (which falsehood / indeterminacy in some Worlds), and Absolute Falsehood / Indeterminacy (which is falsehood / indeterminacy in all possible worlds).

## 5. Introduction to Nonstandard Analysis [15, 16]

An *infinitesimal number* ($\varepsilon$) is a number $\varepsilon$ such that $|\varepsilon| < 1/n$, for any non-null positive integer *n*. An infinitesimal is close to zero, and so small that it cannot be measured.

The infinitesimal is a number smaller, in absolute value, than anything positive nonzero.

Infinitesimals are used in calculus.

An *infinite number ( ω )* is a number greater than anything:

$$1 + 1 + 1 + \ldots + 1 \text{ (for any finite number terms)} \quad (2)$$

The infinites are reciprocals of infinitesimals.

The set of *hyperreals* (*non-standard reals*), denoted as $R^*$, is the extension of set of the real numbers, denoted as *R*, and it comprises the infinitesimals and the infinites, that may be represented on the *hyperreal number line*

$$1/\varepsilon = \omega/1. \quad (3)$$

The set of hyperreals satisfies the *transfer principle*, which states that the statements of first order in $R$ are valid in $R^*$ as well.

A *monad* (*halo*) of an element $a \in R^*$, denoted by $\mu(a)$, is a subset of numbers infinitesimally close to $a$.

Let's denote by $R_+^*$ the set of positive nonzero hyperreal numbers.

We consider the left monad and right monad, and we have introduced the *binad* [5]:

*Left Monad* { that we denote, for simplicity, by $(^-a)$ or only $^-a$ } is defined as:

$$\mu(^-a) = (^-a) = {}^-a = \{a - x, x \in R_+^* \mid x \text{ is infinitesimal}\}. \quad (4)$$

*Right Monad* { that we denote, for simplicity, by $(a^+)$ or only by $a^+$ } is defined as:

$$\mu(a^+) = (a^+) = a^+ = \{a + x, x \in R_+^* \mid x \text{ is infinitesimal}\}. \quad (5)$$

*Bimonad* { that we denote, for simplicity, by $(^-a^+)$ or only $^-a^+$ } is defined as:

$$\mu(^-a^+) = (^-a^+) = {}^-a^+$$

$$= \{a - x, x \in R_+^* \mid x \text{ is infinitesimal}\} \cup \{a + x, x \in R_+^* \mid x \text{ is infinitesimal}\}$$

$$= \{ a \pm x , x \in R_+^* \mid x \text{ is infinitesimal}\}. \quad (6)$$

The left monad, right monad, and the bimonad are subsets of $R^*$.

## 6. Neutrosophic Strict Inequalities

We recall the neutrosophic inequality which is needed for the inequalities of nonstandard numbers.

Let $\alpha, \beta$ be elements in a partially ordered set $M$.

We have defined the *neutrosophic strict inequality*

$$\alpha >_N \beta \quad (7)$$

and read as

*"α is neutrosophically greater than β"* *(8)*

if

*α in general is greater than β,*

or *α is approximately greater than β,*

or *subject to some indeterminacy* (unknown or unclear ordering relationship between α and β) *or subject to some contradiction (*situation when α is smaller than or equal to β*) α is greater than β.*

It means that in most of the cases, on the set $M$, *$\alpha$ is greater than $\beta$.*

And similarly for the opposite neutrosophic strict inequality $\alpha <_N \beta$.

## 7. Neutrosophic Equality

We have defined the *neutrosophic inequality*

$$\alpha =_N \beta \tag{9}$$

and read as

*"$\alpha$ is neutrosophically equal to $\beta$"* *(10)*

if

*$\alpha$ in general is equal to $\beta$,*

or *$\alpha$ is approximately equal to $\beta$,*

or *subject to some indeterminacy* (unknown or unclear ordering relationship between α and β) *or subject to some contradiction (*situation when α is not equal to β*) α is equal to β.*

It means that in most of the cases, on the set $M$, *$\alpha$ is equal to $\beta$.*

## 8. Neutrosophic (Non-Strict) Inequalities

Combining the neutrosophic strict inequalities with neutrosophic equality, we get the $\geq_N$ and $\leq_N$ neutrosophic inequalities.

Let $\alpha, \beta$ be elements in a partially ordered set $M$.

The *neutrosophic (non-strict) inequality*

$$\alpha \geq_N \beta \tag{11}$$

and read as

*"$\alpha$ is neutrosophically greater than or equal to $\beta$"* *(12)*

if

*$\alpha$ in general is greater than or equal to $\beta$,*

or *$\alpha$ is approximately greater than or equal to $\beta$,*

or *subject to some indeterminacy* (unknown or unclear ordering relationship between α and β) *or subject to some contradiction (*situation when α is smaller than β*) α is greater than or equal to β.*

It means that in most of the cases, on the set $M$, *$\alpha$ is greater than or equal to $\beta$.*

And similarly for the opposite neutrosophic (non-strict) inequality $\alpha \leq_N \beta$.

## 9. Neutrosophically Ordered Set

Let $M$ be a set. $(M, <_N)$ is called a neutrosophically ordered set if:

$$\forall \ \alpha, \beta \in M, \text{ one has: either } \alpha <_N \beta, \text{ or } \alpha =_N \beta, \text{ or } \alpha >_N \beta. \tag{13}$$

## 10. Neutrosophic Nonstandard Inequalities

Let $\mathscr{P}(R^*)$ be the power-set of $R^*$. Let's endow $(\mathscr{P}(R^*), <_N)$ with a neutrosophic inequality

Let $a, b \in R$, where $R$ is the set of (standard) real numbers.

And let $({}^-a), (a^+), ({}^-a^+) \in \mathscr{P}(R^*)$, and $({}^-b), (b^+), ({}^-b^+) \in \mathscr{P}(R^*)$, be the left monads, right monads, and the bimonads of the elements (standard real numbers) $a$ and $b$ respectively. Since all monads are subsets, we may treat the single real numbers $a = [a, a]$ and $b = [b, b]$ as subsets too.

$\mathscr{P}(R^*)$ is a set of subsets, and thus we deal with neutrosophic inequalities between subsets.

i) If the subset $\alpha$ has many of its elements above all elements of the subset $\beta$, then $\alpha >_N \beta$ (partially).

*ii)* If the subset $\alpha$ has many of its elements below all elements of the subset $\beta$, then $\alpha <_N \beta$ (partially).

*iii)* If the subset $\alpha$ has many of its elements equal with elements of the subset $\beta$, then $\alpha =_N \beta$ (partially).

If the subset $\alpha$ verifies *i)* and *iii)* with respect to subset $\beta$, then $\alpha \geq_N \beta$.

If the subset $\alpha$ verifies *ii)* and *iii)* with respect to subset $\beta$, then $\alpha \leq_N \beta$.

If the subset $\alpha$ verifies *i)* and *ii)* with respect to subset $\beta$, then there is no neutrosophic order (inequality) between $\alpha$ and $\beta$.

{ For example, between $({}^-a^+)$ and $a$ there is no neutrosophic order. }

Similarly, if the subset $\alpha$ verifies *i), ii)* and *iii)* with respect to subset $\beta$, then there is no neutrosophic order (inequality) between $\alpha$ and $\beta$.

## 11. Open Neutrosophic Research

The quantity or measure of "many of its elements" of the above *i), ii),* and *iii)* conditions depends on each *neutrosophic application* and on its *neutrosophic experts*.

For the *neutrosophic nonstandard inequalities*, we propose based on the above three conditions the following:

$$(-a) <_N a <_N (a^+) \tag{14}$$

because $\forall x \in R_+^*, a - x < a < a + x$, where $x$ is of course a (nonzero) positive infinitesimal (the above double neutrosophic inequality actually becomes a double classical standard real inequality for each fixed positive infinitesimal).

$$({}^-a) \le_N ({}^-a^+) \le_N (a^+) \tag{15}$$

This double neutrosophic inequality may be justified due to $({}^-a^+) = ({}^-a) \cup (a^+)$, so:

$$({}^-a) \le_N ({}^-a) \cup (a^+) \le_N (a^+) \tag{16}$$

whence the left side of the inequality middle term coincides with the inequality first term, while the right side of the inequality middle term coincides with the third inequality term.

If $a > b$, which is a (standard) classical real inequality, then we have the following neutrosophic nonstandard inequalities:

$$a >_N ({}^-b), \quad a >_N (b^+), \quad a >_N ({}^-b^+); \tag{17}$$

$$({}^-a) >_N b, ({}^-a) >_N ({}^-b), ({}^-a) >_N (b^+), \quad ({}^-a) >_N ({}^-b^+); \tag{18}$$

$$(a^+) >_N b, \ (a^+) >_N ({}^-b), \quad (a^+) >_N (b^+), \quad (a^+) >_N ({}^-b^+); \tag{19}$$

$$({}^-a^+) >_N b, \ ({}^-a^+) >_N ({}^-b), \quad ({}^-a^+) >_N (b^+), \quad ({}^-a^+) >_N ({}^-b^+). \tag{20}$$

If $a \ge b$, which is a (standard) classical real inequality, then we have the following neutrosophic nonstandard inequalities:

$$a \ge_N ({}^-b); \tag{21}$$

$$({}^-a) \ge_N ({}^-b); \tag{22}$$

$$(a^+) \ge_N ({}^-b), \quad (a^+) \ge_N b, \quad (a^+) \ge_N (b^+), \quad (a^+) \ge_N ({}^-b^+); \tag{23}$$

$$({}^-a^+) \ge_N ({}^-b), \quad ({}^-a^+) \ge_N ({}^-b^+). \tag{24}$$

And similarly for $<_N$ and $\le_N$ neutrosophic nonstandard inequalities.

## 12. Neutrosophic Nonstandard Equalities

Let $a$, $b$ be standard real numbers; if $a = b$ that is a (classical) standard equality, then:

$$({}^-a) =_N ({}^-b), \ (a^+) =_N (b^+), \ ({}^-a^+) =_N ({}^-b^+). \tag{25}$$

## 13. Neutrosophic Infimum and Neutrosophic Supremum

As an extension of the classical infimum and classical supremum, and using the neutrosophic inequalities and neutrosophic equalities, we define the neutrosophic infimum ( denoted as $inf_N$ ) and the neutrosophic supremum (denoted as $sup_N$ ).

*Neutrosophic Infimum.*

Let $(S, <_N)$ be a set that is neutrosophically partially ordered, and $M$ a subset of $S$. The neutrosophic infimum of $M$, denoted as $inf_N(M)$ is the neutrosophically greatest element in $S$ that is neutrosophically less than or equal to all elements of $M$:

*Neutrosophic Supremum.*

Let $(S, <_N)$ be a set that is neutrosophically partially ordered, and $M$ a subset of $S$. The neutrosophic supremum of $M$, denoted as $sup_N(M)$ is the neutrosophically smallest element in $S$ that is neutrosophically greater than or equal to all elements of $M$.

## 14. Classical Infimum and Supremum vs. Neutrosophic Infimum and Supremum.

Giving the definitions of neutrosophic components from my book [5]:

> "Let $T, I, F$ be standard or non-standard real subsets of $]^-0, 1^+[$,
> with sup T = t_sup, inf T = t_inf,
> sup I = i_sup, inf I = i_inf,
> sup F = f_sup, inf F = f_inf,
> and n_sup = t_sup+i_sup+f_sup,
> n_inf = t_inf+i_inf+f_inf."

Imamura argues (page 3) that:

> "Subsets of $R^*$, even bounded, may have neither infima nor suprema, because the transfer principle ensures the existences of infima and suprema only for internal sets."

This is true from a classical point of view, yet according to the definitions of the neutrosophic inequalities, the neutrosophic infimum and supremum do exist for the nonstandard intervals, for example:

$$inf_N ( ]^-a, b^+[ ) = {}^-a, \text{ and } sup_N ( ]^-a, b^+[ ) = b^+. \quad (26)$$

Indeed, into my definition above I had to clearly mention that we talk neutrosophically [*mea culpa*] by inserting an "N" standing for neutrosophic ($inf_N$ and $sup_N$):

> Let $T, I, F$ be standard or non-standard real subsets of $]^-0, 1^+[$,
> with $sup_N$ T = t_sup, $inf_N$ T = t_inf,
> $sup_N$ I = i_sup, $inf_N$ I = i_inf,
> $sup_N$ F = f_sup, $inf_N$ F = f_inf,
> and n_sup = t_sup+i_sup+f_sup,
> n_inf = t_inf+i_inf+f_inf.

I was more prudent when I presented the sum of single valued standard neutrosophic components, saying:

*Let T, I, F be single valued numbers, T, I, F* $\in [0, 1]$*, such that* $0 \leq T + I + F \leq 3$.

A friend alerted me: *"If T, I, F are numbers in [0, 1], of course their sum is between 0 and 3."* "Yes, I responded, I afford this tautology, because if I did not mention that the sum is up to *3*, readers would take for granted that the sum $T + I + F$ is bounded by *1*, since that is in all logics and in probability!"

**15. Notations**

Imamura is right when criticizing my confusion of notations between hyperreals (numbers) and monads (subsets). I was rather informal than formal at the beginning.
By ${}^{-}a$ and $b^{+}$ most of times I wanted to mean the subsets of left monad and right monad respectively. Taking an arbitrary positive infinitesimal ε, and writing ${}^{-}a = a\text{-}\varepsilon$ and $b^{+} = b+\varepsilon$ was actually picking up a representative from each class (monad).
Similarly, representations of the monads by intervals were not quite accurate from a classical point of view:

$({}^{-}a) = (a\text{-}\varepsilon, a),$ (27)

$(b^{+}) = (b, b+\varepsilon),$ (28)

$({}^{-}a^{+}) = (a\text{-}\varepsilon, a) \cup (b, b+\varepsilon),$ (29)

but they were rather neutrosophic equalities (approximations):

$({}^{-}a) =_N (a\text{-}\varepsilon, a),$ (30)

$(b^{+}) =_N (b, b+\varepsilon),$ (31)

$({}^{-}a^{+}) =_N (a\text{-}\varepsilon, a) \cup (b, b+\varepsilon).$ (32)

**16. Nonarchimedean Ordered Field**.
At pages 5-6 of note [1], Imamura proposed the following Nonarchimedean Ordered Field K:

> "Let $x, y \in K$. *x* and *y* are said to be *infinitely close* (denoted by $a \approx b$) if $a - b$ is infinitesimal. We say that x is *roughly smaller than y* (and write $x \lesssim y$) if $x < y$ or $x \approx y$."

An ordered field is called nonarchimedian field, if it has non-null infinitesimals.
While it is a beautiful definition to consider that *x* and *y* are *infinitely close* (denoted by $a \approx b$) if $a - b$ is infinitesimal, it produces confusions into the nonstandard neutrosophic logic. Why? Because one cannot distinguish any-longer between ${}^{-}a$, $a$, and $a^{+}$ (which is essential in and the flavor of nonstandard neutrosophic logic, in order to differentiate the *relative truth/indeterminacy/falsehood* from *absolute truth/indeterminacy/falsehood* respectively), since one gets that:

$({}^{-}a) \approx a \approx (a^{+})$ (33)

or with the simplest notations:

${}^{-}a \approx a \approx a^{+}.$ (34)

Proof:

$\forall x \in R_+^*, a-(a-x)=x=$ infinitesimal, whence $a \approx (^-a)$ *(35)*

and $\forall x \in R_+^*, (a+x)-a=x=$ infinitesimal, whence $a^+ \approx a$. *(36)*

For the definition of nonstandard interval $]^-a, b^+[$, Imamura proposes at page 6:

> "For $a, b \in K$ the set $]^-a, b^+[_K$ is defined as follows:
>
> $]^-a, b^+[_K = \{x \in K \mid a \lesssim x \lesssim b\}$."

In nonstandard neutrosophic logic and set, we may have not only $]^-a, b^+[$, but various forms of nonstandard intervals:

$$]\overset{m_1}{a}, \overset{m_2}{b}[ \tag{37}$$

where $m_1$ and $m_2$ stand for: left monads ($^-$), right monads ($^+$), or bimonads ($^{-+}$), in all possible combinations (in total $3 \times 3 = 9$ possibilities).

Yet, Imamura's definition cannot be adjusted for all above nonstandard intervals, for example the nonstandard intervals of the form $]a^+, {}^-b[$, because if one writes:

$$]a^+, b^-[_K = \{x \in K \mid a \lesssim x \lesssim b\} \tag{38}$$

one arrives at proving that

$$]^-a, b^+[_K \subseteq ]a^+, b^-[_K \tag{39}$$

which is obviously false, since: $^-a$ is below $a$ and hence below $a^+$, and in the same way $b^+$ is above $b$ and hence above $^-b$ {one gets a bigger nonstandard interval included in or equal to a smaller nonstandard interval}. This occurs because $^-a \approx a^+$ and $b^+ \approx b^-$ (in Imamura's notation).

## 17. Nonstandard Unit Interval.

Imamura cites my work:

> "by "–a" one signifies a monad, i.e., a set of hyper-real
> numbers in non-standard analysis:
> (–a) = { a − x ∈ R* | x is infinitesimal } ,
> and similarly "b+" is a hyper monad:
> (b+) = { b + x ∈ R* | x is infinitesimal } .
> ([5] p. 141; [6] p. 9)"

But these are inaccurate, because my exact definitions of monads, since my 1998 first world neutrosophic publication {see [5], page 9; and [6], pages 385 - 386}, were:

> "$(^-a) = \{ a - x: x \in R_+^* \mid x$ is infinitesimal },
> and similarly "b+" is a hyper monad:
> $(b^+) = \{ b + x: x \in R_+^* \mid x$ is infinitesimal }"

Imamura says that:

> "The correct definitions are the following:
> $(_{-}a) = \{ a - x \in R_{*} \mid x$ is *positive* infinitesimal $\}$,
> $(b_{+}) = \{ b + x \in R_{*} \mid x$ is *positive* infinitesimal $\}$."

I did not have a chance to see how my article was printed in *Proceedings of the 3rd Conference of the European Society for Fuzzy Logic and Technology* [7], that Imamura talks about, maybe there were some typos, but Imamura can check the *Multiple Valued Logic / An International Journal* [6], published in England in 2002 (ahead of the European Conference from 2003, that Imamura cites) by the prestigious Taylor & Francis Group Publishers, and clearly one sees that it is: $\boldsymbol{R}_{+}^{*}$ (so, $x$ is a *positive* infinitesimal into the above formulas), therefore there is no error.

Then Imamura continues:

> "Ambiguity of the definition of the nonstandard unit interval. Smarandache did not give any explicit definition of the notation $]^{-}0, 1^{+}[$ in [5] (or the notation $⊩^{-}0, 1^{+}⫣$ in [6]). He only said:
> Then, we call $]^{-}0, 1^{+}[$ a non-standard unit interval. Obviously, 0 and 1, and analogously non-standard numbers infinitely small but less than 0 or infinitely small but greater than 1, belong to the non-standard unit interval. ([5] p. 141; [6] p. 9)."

Concerning the notations I used for the nonstandard intervals as ⊩ ⫣ or ] [, it was imperative to employ notations different from the classical *[ ]* or *( )* intervals, since the extremes of the nonstandard unit interval were unclear, vague.
I thought it was easily understood that:

$$]^{-}0, 1^{+}[\ = (^{-}0) \cup [0, 1] \cup (1^{+}). \tag{40}$$

Or, using the previous neutrosophic inequalities, we may write:

$$]^{-}0, 1^{+}[\ = \{x \in R^{*}, {}^{-}0 \leq_{N} x \leq_{N} 1^{+}\}. \tag{41}$$

Imamura says that:

> "Here $^{-}0$ and $1^{+}$ are particular real numbers defined in the previous paragraph:
> $^{-}0 = 0-\varepsilon$ and $1^{+} = 1+\varepsilon$, where ε is a fixed non-negative infinitesimal."

This is untrue, I never said that "ε is a *fixed* non-negative infinitesimal", ε was not fixed, I said that for any real numbers $a$ and $b$ {see again [5], page 9; and [6], pages 385 - 386}:

> *"$(^{-}a) = \{ a - x: x \in R_{+}^{*} \mid x$ is infinitesimal $\}$,*
> *$(b^{+}) = \{ b + x: x \in R_{+}^{*} \mid x$ is infinitesimal $\}$".*

Therefore, once we replace a = 0 and b = 1 we get:

> *$(^{-}0) = \{ 0 - x: x \in R_{+}^{*} \mid x$ is infinitesimal $\}$,*
> *$(1^{+}) = \{ 1 + x: x \in R_{+}^{*} \mid x$ is infinitesimal $\}$.*

Thinking out of box, inspired from the real world, was the first intent, i.e. allowing neutrosophic components (truth / indeterminacy / falsehood) values be outside of the classical (standard) unit real interval *[0, 1]* used in all previous (Boolean, multi-valued etc.) logics if needed in applications, so neutrosophic component values $< 0$ and $> 1$ had to occurs due to the Relative / Absolute stuff, with:

$$^{-}0 <_N 0 \quad \text{and} \quad 1^{+} >_N 1. \qquad (42)$$

Later on, in 2007, I found plenty of cases and real applications in Standard Neutrosophic Logic and Set (therefore, not using the Nonstandard Neutrosophic Logic and Set), and it was thus possible the extension of the neutrosophic set *to Neutrosophic Overset (when some neutrosophic component is > 1), and to Neutrosophic Underset (when some neutrosophic component is < 0), and to Neutrosophic Offset (when some neutrosophic components are off the interval [0, 1], i.e. some neutrosophic component > 1 and some neutrosophic component < 0). Then, similar extensions to respectively Neutrosophic Over/Under/Off Logic, Measure, Probability, Statistics etc.* [8, 17, 18, 19], extending the unit interval [0, 1] to

$$[\Psi, \Omega], \text{ with } \Psi \leq 0 < 1 \leq \Omega, \qquad (43)$$

where $\Psi, \Omega$ are standard real numbers.

Imamura says, regarding the definition of neutrosophic logic that:

> "In this logic, each proposition takes a value of the form (T, I, F), where T, I, F are subsets of the nonstandard unit interval $]^{-}0, 1^{+}[$ and represent all possible values of Truthness, Indeterminacy and Falsity of the proposition, respectively."

Unfortunately, this is not exactly how I defined it.

In my first book {see [5], p. 12; or [6] pp. 386 – 387} it is stated:

> "Let T, I, F be real standard or non-standard subsets of ]-0, 1+["

meaning that T, I, F may also be "real standard" not only real non-standard.

In *The Free Online Dictionary of Computing*, 1999-07-29, edited by Denis Howe from England, it is written:

> Neutrosophic Logic:
> <*logic*> (Or "Smarandache logic") A generalization of fuzzy logic based on Neutrosophy. A proposition is t true, i indeterminate, and f false, where t, i, and f are real values from the ranges T, I, F, with no restriction on T, I, F, or the sum n=t+i+f. Neutrosophic logic thus generalizes:
> - intuitionistic logic, which supports incomplete theories (for 0<n<100, 0<=t,i,f<=100);

- fuzzy logic (for n=100 and i=0, and 0<=t,i,f<=100);
- Boolean logic (for n=100 and i=0, with t,f either 0 or 100);
- multi-valued logic (for 0<=t,i,f<=100);
- paraconsistent logic (for n>100, with both t,f<100);
- dialetheism, which says that some contradictions are true (for t=f=100 and i=0; some paradoxes can be denoted this way).

Compared with all other logics, neutrosophic logic introduces a percentage of "indeterminacy" - due to unexpected parameters hidden in some propositions. It also allows each component t,i,f to "boil over" 100 or "freeze" under 0. For example, in some tautologies t>100, called "overtrue".
*Home*.
["Neutrosophy / Neutrosophic probability, set, and logic", F. Smarandache, American Research Press,
1998].

As Denis Howe said in 1999, the neutrosophic components *t, i, f* are "*real values* from the ranges *T, I, F*", not nonstandard values or nonstandard intervals. And this was because nonstandard ones were not important for the neutrosophic logic (the Relative/Absolute plaid no role in technological and scientific applications and future theories).

## 18. The Logical Connectives ∧, ∨, →

Imamura's critics of my first definition of the neutrosophic operators is history for long ago.

All fuzzy, intuitionistic fuzzy, and neutrosophic logic operators are *inferential approximations,* not written in stone. They are improved from application to application.

Let's denote:

$\wedge_F, \wedge_N, \wedge_P$ representing respectively the fuzzy conjunction, neutrosophic conjunction, and plithogenic conjunction;

similarly

$\vee_F, \vee_N, \vee_P$ representing respectively the fuzzy disjunction, neutrosophic disjunction, and plithogenic disjunction,

and

$\rightarrow_F$, $\rightarrow_N$, $\rightarrow_P$ representing respectively the fuzzy implication, neutrosophic implication, and plithogenic implication.

I agree that my beginning neutrosophic operators (when I applied the same *fuzzy t-norm*, or the same *fuzzy t-conorm*, to all neutrosophic components *T*, *I*, *F*) were less accurate than others developed later by the neutrosophic community researchers. This was pointed out since 2002 by Ashbacher [9] and confirmed in 2008 by Rivieccio [10]. They observed that if on $T_1$ and $T_2$ one applies a *fuzzy t-norm*, on their opposites $F_1$ and $F_2$ one needs to apply the *fuzzy t-conorm* (the opposite of fuzzy t-norm), and reciprocally.

About inferring $I_1$ and $I_2$, some researchers combined them in the same directions as $T_1$ and $T_2$.

Then:

$$(T_1, I_1, F_1) \wedge_N (T_2, I_2, F_2) = (T_1 \wedge_F T_2, I_1 \wedge_F I_2, F_1 \vee_F F_2), \quad (44)$$
$$(T_1, I_1, F_1) \vee_N (T_2, I_2, F_2) = (T_1 \vee_F T_2, I_1 \vee_F I_2, F_1 \wedge_F F_2), \quad (45)$$
$$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, I_1, T_1) \vee_N (T_2, I_2, F_2) = (F_1 \vee_F T_2, I_1 \vee_F I_2, T_1 \wedge_F F_2); \quad (46)$$

others combined $I_1$ and $I_2$ in the same direction as $F_1$ and $F_2$ (since both $I$ and $F$ are negatively qualitative neutrosophic components), the most used one:

$$(T_1, I_1, F_1) \wedge_N (T_2, I_2, F_2) = (T_1 \wedge_F T_2, I_1 \vee_F I_2, F_1 \vee_F F_2), \quad (47)$$
$$(T_1, I_1, F_1) \vee_N (T_2, I_2, F_2) = (T_1 \vee_F T_2, I_1 \wedge_F I_2, F_1 \wedge_F F_2), \quad (48)$$
$$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, I_1, T_1) \vee_N (T_2, I_2, F_2) = (F_1 \vee_F T_2, I_1 \wedge_F I_2, T_1 \wedge_F F_2). \quad (49)$$

Now, applying the neutrosophic conjunction suggested by Imamura:

> "This causes some counterintuitive phenomena. Let A be a (true) proposition with value ({ 1 } , { 0 } , { 0 }) and let B be a (false) proposition with value ({ 0 } , { 0 } , { 1 }).
> Usually we expect that the falsity of the conjunction A ∧ B is { 1 }. However, its actual falsity is { 0 }."

we get:

$$(1, 0, 0) \wedge_N (0, 0, 1) = (0, 0, 1), \quad (50)$$

which is correct (so the falsity is *1*).

Even more, recently, in an extension of neutrosophic set to *plithogenic set* [11] (which is a set whose each element is characterized by many attribute values), the *degrees of contradiction c( , )* between the neutrosophic components *T, I, F* have been defined (in order to facilitate the design of the aggregation operators), as follows:
$c(T, F) = 1$ (or *100%*, because they are totally opposite), $c(T, I) = c(F, I) = 0.5$ (or *50%*, because they are only half opposite), then:

$$(T_1, I_1, F_1) \wedge_P (T_2, I_2, F_2) = (T_1 \wedge_F T_2, 0.5(I_1 \wedge_F I_2) + 0.5(I_1 \vee_F I_2), F_1 \vee_F F_2), \quad (51)$$
$$(T_1, I_1, F_1) \vee_P (T_2, I_2, F_2) = (T_1 \vee_F T_2, 0.5(I_1 \vee_F I_2) + 0.5(I_1 \wedge_F I_2), F_1 \wedge_F F_2). \quad (52)$$
$$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, I_1, T_1) \vee_N (T_2, I_2, F_2)$$
$$= (F_1 \vee_F T_2, 0.5(I_1 \vee_F I_2) + 0.5(I_1 \wedge_F I_2), T_1 \wedge_F F_2). \quad (53)$$

**Conclusion**.

We thank very much Dr. Takura Imamura for his interest and critics of *Nonstandard Neutrosophic Logic*, which eventually helped in improving it. {In the history of mathematics, critics on nonstandard analysis, in general, have been made by Paul Halmos, Errett Bishop, Alain Connes and others.} We hope we'll have more dialogues on the subject in the future.

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