Entropy of Cantor Dust and Fuzzy Dark Matter

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Abstract

Fuzzy Dark Matter (FDM) models are of class of theories where Dark Matter (DM) is interpreted as condensate of ultralight scalar or pseudo-scalar particles. Few years back we conjectured that a spacetime endowed with minimal fractality enables a natural association of FDM to *Cantor Dust*, an early Universe phase induced by topological condensation of continuous dimensions. Appealing to the maximal entropy of Cantor Dust, this brief note recovers the mass of FDM particles in reasonable agreement with simulations and cosmological data. An intriguing finding is that, at least in principle, the entropy of Cantor Dust can also account for the non-vanishing photon mass in Proca-Stueckelberg models.

Key words: Dark Matter, Fuzzy Dark Matter, topological condensation, fractal spacetime, Cantor Dust, early Universe Cosmology, Proca-Stueckelberg models.

Within the broad database of existing DM proposals, ultralight dark matter (UDM) represents a class of models in which DM consists of bosonic fields of nearly vanishing mass ($\leq 10^{-11}$ eV), which are characterized by their spin and parity (scalar, pseudoscalar or vector). FDM lies at the lowest mass range of UDM, where the ultralight bosons are expected to have masses on the order of $O(10^{-22})$ eV. The hypothetical existence of such bosons is believed to supply long awaited clues on some open challenges of particle physics and cosmology, such for example the strong CP problem of Quantum Chromodynamics and the reality of axions. Closely related to FDM, models based on self-interacting Bose-Einstein condensates provide a bridge to superfluid DM theories, which gained popularity in recent years and have been the subject of many astrophysical studies [1-5].

It was conjectured some time ago that a spacetime endowed with minimal fractality enables a natural association of FDM to *Cantor Dust*, an early Universe phase induced by topological condensation of continuous dimensions [6-11]. According to this conjecture, the minimal deviation from the standard spacetime dimensionality develops above the Fermi scale and assumes the form

$$\varepsilon_{\min} = \frac{m^2}{M_{Pl}^2} = D_0 - d << 1 \tag{1}$$

where M_{Pl} stands for the Planck scale, D_0 is the dimension of the embedding space (1, 2, 3 for a line, surface, or volume, respectively) and d the scale-dependent dimension, with $d \le D_0$. Since dimensional deviation ε runs with the energy scale, it is inherently related to the concept of *Kolmogorov entropy* and its derivative formulations, which equally applies to the formation of Cantor Dust [12].

With reference to [12-14] and the Appendix section, we introduce the following assumptions:

1) Maximal thermodynamic entropy depends on the cosmological era and takes the set of values listed below (in natural units, $k = 1$):

- *Primordial Black Holes (PBH)*: $S_{PBH} \propto 10^{103}$

- *Cosmic Microwave Background (CMB) and Photons*: $S_{CMB} \propto S_{\gamma} \propto 10^{88}$

- *Baryons*: $S_B \propto 10^{80}$

- *Relic neutrinos*: $S_\nu \propto 10^{88}$

2) Maximal topological entropy ($q=0$) is of the same order of magnitude as the thermodynamic entropy ($q=1$), that is, $S_1 = O(S_0)$.

3) FDM predominantly displays a large-scale filamentary structure which takes the form of spiderweb clusters.

By (A6) and (A7), these assumptions imply $D_0 \approx 1$ such that

$$S_{\max} \propto \varepsilon_{\min}^{-1} \tag{2}$$

and (1) leads to

$$\boxed{m \propto S_{\max}^{-1/2} M_{Pl}} \tag{3}$$

The table below displays the actual versus predicted values of FDM and photon masses computed using (3). The FDM mass is derived from the SBH entropy while the photon mass from the CMB/photon entropy and compared to [17].

Ultralight boson	Predicted	Actual
FDM	$O(10^{-23})$	$O(10^{-22})$
Photon	$O(10^{-16})$	$< 10^{-18}$

Tab. 1: Actual versus predicted mass values (eV)

Combining these results with the findings of [15] indicates that:

a) FDM and photons acquire minimal mass from dimensional condensation of Cantor Dust,

b) By contrast, SM particles/luminous matter do not acquire mass from dimensional condensation, but from successive bifurcations driven by the Feigenbaum scenario of transition to chaos [15-16].

<u>APPENDIX: Entropy of chaotic processes [12]</u>

Let a generic UV to IR trajectory be described by the n - dimensional phase-space flow $x(\tau)$. Here, τ denotes the evolution parameter ("time") corresponding to the running scale μ

$$\tau = \log(\mu / \mu_0) \tag{A1}$$

The random behavior of the flow near the strange attractor can be characterized by dividing the phase-space into n- dimensional hypercubes of side r, which are sampled at discrete time intervals $\Delta\tau$. The generalized *K-entropy* of order $q \neq 1$ is given by the equation

$$K_q(X) = -\lim_{r\to 0}\lim_{\Delta\tau\to 0}\lim_{N\to\infty}\frac{1}{N\Delta\tau}\frac{1}{q-1}\ln\sum_{i_1,i_2,\ldots i_N}^{M(r)} p_{i_1,i_2,\ldots i_N}^{q} \tag{A2}$$

where $X = x_i$ is the discrete random variable, that is, $x_i = x(\tau = i\Delta\tau)$, and $p_{i_1,i_2,\ldots,i_M}$ stands for the joint probability that the trajectory $x(\tau = \Delta\tau)$ is in i_1, $x(\tau = 2\Delta\tau)$ is in i_2 and $x(\tau = M\Delta\tau)$ is in i_M. The K-entropy defines the asymptotic scenario where $r \to 0$ and the phase-space is sampled with an infinite number of steps ($N \to \infty$) at vanishing time intervals $(\Delta\tau \to 0)$. In the special case $N\Delta\tau = 1$ and when the joint probability is constant across all hypercubes ($M(r) = const.,\ \ p_{i_1,i_2,\ldots,i_M} = p_i$), (A2) turns into the *Rényi entropy* in the logarithm base b, which assumes the form

$$S_q(X) = \frac{1}{1-q}\log_b\left(\sum_{i=1}^{M} p_i^q\right) \tag{A3}$$

Furthermore, (A3) reduces to the familiar *thermodynamic entropy* when $q \to 1$ and Boltzmann constant is set to $k = 1$

$$S(X) = -\sum_{i=1}^{M} p_i \ln p_i \tag{A4}$$

Finally, the concept of *generalized dimension* of order q is introduced in conjunction with (A3) according to

$$D_q = \lim_{r \to 0} \frac{1}{1-q} \frac{\log_b (\sum_{i=1}^{M} p_i^q)}{\log r} \tag{A5}$$

A particularly straightforward expression of (A3) is obtained for the null order $q = 0$ and the natural logarithm base. It is referred to as *topological entropy* and is given by

$$S_0(r) = \ln \sum_{i=1}^{M} p_i^0 = \ln M \tag{A6}$$

It is known that the *box-counting dimension* of a fractal object of normalized size r is defined as

$$D_0 \approx \frac{\ln M}{\ln r} \Rightarrow M \approx r^{D_0} = \varepsilon^{-D_0} \tag{A7}$$

in which M denotes the number of covering boxes and $\varepsilon = r^{-1}$ is the normalized size of the box. The dimension of ordinary Euclidean space corresponds to integer and positive-definite values of the box-counting dimension, $D_0 = k$, $k = 0,1,2\ldots$.

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