Self-consistent hydrodynamic model of electron vortex fluid in metals

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Abstract We propose a system of self-consistent equations for electron fluid in metal, which describes vortex flows and frozen-in internal electromagnetic field. It is shown that in case of ideal fluid the proposed model describes the electrodynamics of superconductor, and in vortex-less case it leads to the well known London equations. However, the speed of fluctuations propagation is equal to the speed of electron sound, and not the speed of light, which is more adequate from a physical point of view. The normal metal is described by similar equations, but taking into account damping processes. The main peculiarities of the proposed equations are illustrated with the analysis of electron sound waves.

1. Introduction

In recent decades, much attention has been paid to the description of fluid dynamics by vector fields including vectors of speed and vorticity, which satisfy symmetric Maxwell-type equations [1–9]. In particular, a similar approach is used to describe the plasma motion within the framework of a hydrodynamic two-fluid model [10–13]. However, in all mentioned works an additional equation for the vortex motion is obtained by taking the "curl" operator from the Euler equation and, therefore, the resulting equation is not independent. Recently, we have developed an alternative approach based on the droplet model of fluid introduced by Helmholtz [14] and obtained a closed system of Maxwell-type equations for the vortex fluid taking into account the rotation and twisting of vortex tubes [15,16]. Here, we apply these equations to develop the hydrodynamic description of vortex flows of electron fluid in metal.

2. System of equations for electron fluid in a superconductor

From the hydrodynamic point of view a superconductor is an electron-ion system in which the ions are stationary while the electronic component is a charged ideal liquid without dissipation. The system of hydrodynamic equations describing the electron fluid in a superconductor can be represented [16] in the following form

$$\begin{aligned}
&\frac{1}{s_s}\left(\frac{\partial}{\partial t}+(\boldsymbol{v}_s\cdot\nabla)\right)\boldsymbol{v}_s+\nabla u_s+\nabla\times\boldsymbol{w}_s=\alpha_s\boldsymbol{E}_s,\\
&\frac{1}{s_s}\left(\frac{\partial}{\partial t}+(\boldsymbol{v}_s\cdot\nabla)\right)u_s+\nabla\cdot\boldsymbol{v}_s=0,\\
&\frac{1}{s_s}\left(\frac{\partial}{\partial t}+(\boldsymbol{v}_s\cdot\nabla)\right)\boldsymbol{w}_s+\nabla\xi_s-\nabla\times\boldsymbol{v}_s=\alpha_s\boldsymbol{B}_s,\\
&\frac{1}{s_s}\left(\frac{\partial}{\partial t}+(\boldsymbol{v}_s\cdot\nabla)\right)\xi_s+(\nabla\cdot\boldsymbol{w}_s)=0.
\end{aligned}\tag{1}$$

Here index *s* means that values correspond to superconducting electrons. Parameter s_s is the speed of sound in the superconducting electron fluid, $\boldsymbol{v}_s$ is the local flow velocity, u_s is a quantity proportional to the enthalpy per unit mass, $\boldsymbol{w}_s$ is a quantity characterizing the rotation of the vortex tubes, and ξ_s is a quantity that characterizes the twisting of the vortex tubes. The parameter α_s is defined as

$$\alpha_s = \frac{e}{m_0 s_s},$$

where e is the charge and m_0 is the mass of electron. The variable u_s is

$$u_s = \frac{1}{s_s} h_s,$$
$$dh_s = \frac{s_s^2}{n_{0s}} dn_s,$$

where h_s is the enthalpy per unit mass, n_s is the electron concentration (n_{0s} is equilibrium electron concentration). The variable $\boldsymbol{w}_s$ is

$$\boldsymbol{w}_s = 2 s_s \boldsymbol{\Theta}_s,$$
$$\boldsymbol{\omega}_s = \frac{d\boldsymbol{\Theta}_s}{dt},$$

where $\boldsymbol{\Theta}_s$ is the angular vector of rotation of the vortex tube, $\boldsymbol{\omega}_s$ is the angular velocity of the vortex tube rotation. The value ξ_s characterizes the twisting of the vortex tube

$$|\xi_s| = s_s \beta_s,$$

where β_s is the twisting angle of the vortex tube.

The internal electric and magnetic fields are generated only due to deviations of electron fluid parameters from equilibrium values [16]. They described by the following system of equations:

$$\begin{aligned}
&\nabla \cdot \boldsymbol{E}_s = -4\pi e n_s, \\
&\nabla \cdot \boldsymbol{B}_s = -4\pi e g_s, \\
&\left(\frac{\partial}{\partial t} + (\boldsymbol{v}_s \cdot \nabla)\right)\boldsymbol{B}_s + s_s \nabla \times \boldsymbol{E}_s = 4\pi e n_s \boldsymbol{w}_s, \\
&\left(\frac{\partial}{\partial t} + (\boldsymbol{v}_s \cdot \nabla)\right)\boldsymbol{E}_s - s_s \nabla \times \boldsymbol{B}_s = 4\pi e n_s \boldsymbol{v}_s.
\end{aligned} \tag{2}$$

The equations (1) and (2) form the self-consistent system describing the vortex electron fluid.

3. System of linearized equations

Neglecting the convective derivatives and linearizing the terms

$$\begin{aligned}
&4\pi e n \boldsymbol{w}_s \approx 4\pi e n_{0s} \boldsymbol{w}_s, \\
&4\pi e n \boldsymbol{v}_s \approx 4\pi e n_{0s} \boldsymbol{v}_s,
\end{aligned} \tag{3}$$

in field equations we obtain the following system of linearized equations

$$\begin{aligned}
&\frac{1}{s_s}\frac{\partial}{\partial t}\boldsymbol{v}_s+\nabla u_s+\nabla\times\boldsymbol{w}_s=\alpha_s\boldsymbol{E}_s,\\
&\frac{1}{s_s}\frac{\partial}{\partial t}u_s+\nabla\cdot\boldsymbol{v}_s=0,\\
&\frac{1}{s_s}\frac{\partial}{\partial t}\boldsymbol{w}_s+\nabla\xi_s-\nabla\times\boldsymbol{v}_s=\alpha_s\boldsymbol{B}_s,\\
&\frac{1}{s_s}\frac{\partial}{\partial t}\xi_s+\nabla\cdot\boldsymbol{w}_s=0,
\end{aligned}\tag{4}$$

and

$$\begin{aligned}
&\nabla\cdot\boldsymbol{E}_s=-4\pi e n_s,\\
&\nabla\cdot\boldsymbol{B}_s=-4\pi e g_s,\\
&\frac{\partial}{\partial t}\boldsymbol{B}_s+s_s\nabla\times\boldsymbol{E}_s=4\pi e n_{0s}\boldsymbol{w}_s,\\
&\frac{\partial}{\partial t}\boldsymbol{E}_s-s_s\nabla\times\boldsymbol{B}_s=4\pi e n_{0s}\boldsymbol{v}_s.
\end{aligned}\tag{5}$$

For linearized equations, the following relations hold:

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\boldsymbol{v}_s^2+\boldsymbol{w}_s^2+u_s^2+\xi_s^2\right)+s_s\nabla\cdot\left(\left(\boldsymbol{v}_s\times\boldsymbol{w}_s\right)+u_s\boldsymbol{v}_s+\xi_s\boldsymbol{w}_s\right)=-\frac{e}{m_0}\left(\boldsymbol{v}_s\cdot\boldsymbol{E}_s+\boldsymbol{w}_s\cdot\boldsymbol{B}_s\right),\tag{6}$$

$$\frac{1}{8\pi}\frac{\partial}{\partial t}\left(\boldsymbol{E}_s^2+\boldsymbol{B}_s^2\right)+\frac{s_s}{4\pi}\nabla\cdot\left(\boldsymbol{E}_s\times\boldsymbol{B}_s\right)=en_{0s}\left(\boldsymbol{v}_s\cdot\boldsymbol{E}_s+\boldsymbol{w}_s\cdot\boldsymbol{B}_s\right).\tag{7}$$

The value

$$\frac{1}{2}\left(\boldsymbol{v}_s^2+\boldsymbol{w}_s^2+u_s^2+\xi_s^2\right)\tag{8}$$

represents the density of mechanical energy per unit mass. The value

$$s_s\left(\left(\boldsymbol{v}_s\times\boldsymbol{w}_s\right)+u_s\boldsymbol{v}_s+\xi_s\boldsymbol{w}_s\right)\tag{9}$$

is the mechanical energy flux density. The value

$$\frac{1}{8\pi}\left(\boldsymbol{E}_s^2+\boldsymbol{B}_s^2\right)\tag{10}$$

is the volume density of the electromagnetic energy of the internal field, and the value

$$\frac{s_s}{4\pi}\left(\boldsymbol{E}_s\times\boldsymbol{B}_s\right)\tag{11}$$

is the flux density of the electromagnetic energy of the internal field.

4. Sound waves in a superconducting condensate

Let us consider small fluctuations of electron fluid parameters near the equilibrium state

$$
\begin{aligned}
&n_s = n_{0s} + \tilde{n}_s, \\
&\boldsymbol{v}_s = \tilde{\boldsymbol{v}}_s, \\
&g_s = \tilde{g}_s, \\
&\boldsymbol{w}_s = \tilde{\boldsymbol{w}}_s.
\end{aligned} \tag{12}
$$

Then we get

$$
\begin{aligned}
&\frac{1}{s_s}\frac{\partial \tilde{\boldsymbol{v}}_s}{\partial t} + \frac{s_s}{n_{0s}}\nabla \tilde{n}_s + \left[\nabla \times \tilde{\boldsymbol{w}}_s\right] = -\frac{e}{m_0 s_s}\tilde{\boldsymbol{E}}_s, \\
&\frac{1}{n_{0s}}\frac{\partial \tilde{n}_s}{\partial t} + \left(\nabla \cdot \tilde{\boldsymbol{v}}_s\right) = 0, \\
&\frac{1}{s_s}\frac{\partial \tilde{\boldsymbol{w}}_s}{\partial t} + \frac{s_s}{n_{0s}}\nabla \tilde{g}_s - \left[\nabla \times \tilde{\boldsymbol{v}}_s\right] = -\frac{e}{m_0 s_s}\tilde{\boldsymbol{B}}_s, \\
&\frac{1}{n_{0s}}\frac{\partial \tilde{g}_s}{\partial t} + \left(\nabla \cdot \tilde{\boldsymbol{w}}_s\right) = 0,
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
&\nabla \cdot \tilde{\boldsymbol{E}}_s = -4\pi e \tilde{n}_s, \\
&\nabla \cdot \tilde{\boldsymbol{B}}_s = -4\pi e \tilde{g}_s, \\
&s_s \nabla \times \tilde{\boldsymbol{E}}_s = -\frac{\partial \tilde{\boldsymbol{B}}_s}{\partial t} + 4\pi e n_{0s}\tilde{\boldsymbol{w}}_s, \\
&s_s \nabla \times \tilde{\boldsymbol{B}}_s = \frac{\partial \tilde{\boldsymbol{E}}_s}{\partial t} - 4\pi e n_{0s}\tilde{\boldsymbol{v}}_s.
\end{aligned} \tag{14}
$$

From the systems (13) and (14) we have the following wave equations for the parameters of electron fluid

$$
\begin{aligned}
&\left(\frac{\partial^2}{\partial t^2} - s_s^2\Delta + \omega_{sp}^2\right)\tilde{n}_s = 0, \\
&\left(\frac{\partial^2}{\partial t^2} - s_s^2\Delta + \omega_{sp}^2\right)\tilde{\boldsymbol{v}}_s = 0, \\
&\left(\frac{\partial^2}{\partial t^2} - s_s^2\Delta + \omega_{sp}^2\right)\tilde{g}_s = 0, \\
&\left(\frac{\partial^2}{\partial t^2} - s_s^2\Delta + \omega_{sp}^2\right)\tilde{\boldsymbol{w}}_s = 0,
\end{aligned} \tag{15}
$$

and for the internal fields

$$
\begin{aligned}
&\left(\frac{\partial^2}{\partial t^2} - s_s^2\Delta + \omega_{sp}^2\right)\tilde{\boldsymbol{E}}_s = 0, \\
&\left(\frac{\partial^2}{\partial t^2} - s_s^2\Delta + \omega_{sp}^2\right)\tilde{\boldsymbol{B}}_s = 0.
\end{aligned} \tag{16}
$$

The parameter ω_{sp} is the plasma frequency of electron fluid

$$\omega_{sp}^2 = \frac{4\pi n_{0s} e^2}{m_0}. \tag{17}$$

The schematic draw of dispersion relation for equations (15) and (16) is represented in Fig. 1.

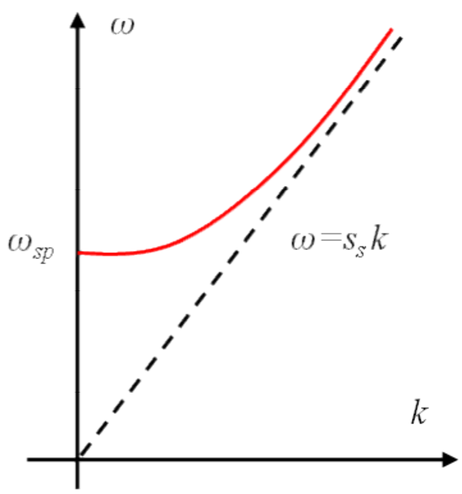

Fig. 1. The schematic draw of dispersion curve for sound waves in ideal electron fluid.

5. Relation to the Londons equations

Let us consider the flow of a superconducting condensate without taking into account the vortex motion. Then

$$\begin{aligned} g_s &= 0, \\ \boldsymbol{w}_s &= 0, \end{aligned} \tag{18}$$

and linearized equations (4) are transformed into

$$\frac{1}{s_s}\frac{\partial}{\partial t}\boldsymbol{v}_s + \frac{s_s}{n_{0s}}\nabla n_s = -\frac{e}{m_0 s_s}\boldsymbol{E}_s, \tag{19}$$

$$\nabla \times \boldsymbol{v}_s = \frac{e}{m_0 s_s}\boldsymbol{B}_s, \tag{20}$$

$$\frac{1}{n_{0s}}\frac{\partial}{\partial t}n_s + \nabla \cdot \boldsymbol{v}_s = 0, \tag{21}$$

Rewriting these equations in the usual terms of the density of charge and current (taking into account the sign of the electron charge), we have

$$\boldsymbol{E}_s = \frac{4\pi}{s_s^2}\lambda^2 \frac{\partial \boldsymbol{j}_s}{\partial t} + 4\pi\lambda^2 \nabla \rho_s, \tag{22}$$

$$\boldsymbol{B}_s = -\frac{4\pi}{s_s}\lambda^2 \nabla \times \boldsymbol{j}_s, \tag{23}$$

$$\frac{\partial \rho_s}{\partial t} + \nabla \cdot \boldsymbol{j}_s = 0, \tag{24}$$

where

$$\begin{aligned} \rho_s &= -en_s, \\ \boldsymbol{j}_s &= -en_{0s}\boldsymbol{v}_s, \\ \lambda^2 &= \frac{m_0 s_s^2}{4\pi e^2 n_{0s}}. \end{aligned} \tag{25}$$

Corresponding wave equations are

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - s_s^2\Delta + \omega_{ep}^2\right)\rho_s &= 0, \\ \left(\frac{\partial^2}{\partial t^2} - s_s^2\Delta + \omega_{ep}^2\right)\boldsymbol{j}_s &= 0, \end{aligned} \tag{26}$$

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - s_s^2\Delta + \omega_{ep}^2\right)\boldsymbol{E}_s &= 0, \\ \left(\frac{\partial^2}{\partial t^2} - s_s^2\Delta + \omega_{ep}^2\right)\boldsymbol{B}_s &= 0. \end{aligned} \tag{27}$$

Thus, equations (22)-(24), (26) and (27) have the form of the London equations [17,18]. However, the speed of propagation of perturbations in the electron fluid is equal to the speed of electronic sound, and not the speed of light, which seems to be more adequate from a physical point of view.

6. Hydrodynamic model of electron gas in a normal metal

In a normal metal electrons collide with a crystal lattice, which is accompanied by energy dissipation processes. Let us denote the frequency of collisions by ε. Then the linearized equations describing the motion of the electron fluid in a normal metal are written as follows:

$$\begin{aligned} \frac{1}{s_n}\left(\frac{\partial}{\partial t} + \varepsilon\right)\boldsymbol{v}_n + \nabla u_n + \nabla \times \boldsymbol{w}_n &= -\frac{e}{ms_n}\boldsymbol{E}_n, \\ \frac{1}{s_n}\left(\frac{\partial}{\partial t} + \varepsilon\right)u_n + \nabla \cdot \boldsymbol{v}_n &= 0, \\ \frac{1}{s_n}\left(\frac{\partial}{\partial t} + \varepsilon\right)\boldsymbol{w}_n + \nabla \xi_n - \nabla \times \boldsymbol{v}_n &= -\frac{e}{ms_n}\boldsymbol{B}_n, \\ \frac{1}{s_n}\left(\frac{\partial}{\partial t} + \varepsilon\right)\xi_n + \nabla \cdot \boldsymbol{w}_n &= 0, \end{aligned} \tag{28}$$

and

$$
\begin{aligned}
&\nabla\cdot\boldsymbol{E}_n=-4\pi e n_n,\\
&\nabla\cdot\boldsymbol{B}_n=-4\pi e g_n,\\
&\left(\frac{\partial}{\partial t}+\varepsilon\right)\boldsymbol{B}_n+s_n\nabla\times\boldsymbol{E}_n=4\pi e n_{0n}\boldsymbol{w}_n,\\
&\left(\frac{\partial}{\partial t}+\varepsilon\right)\boldsymbol{E}_n-s_n\nabla\times\boldsymbol{B}_n=4\pi e n_{0n}\boldsymbol{v}_n.
\end{aligned}
\tag{29}
$$

Here index n means that values correspond to electrons in normal metal. From equations (28) and (29) we have the following relations for energy and momentum:

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\left(\boldsymbol{v}_n^2+\boldsymbol{w}_n^2+u_n^2+\xi_n^2\right)+\varepsilon\left(\boldsymbol{v}_n^2+\boldsymbol{w}_n^2+u_n^2+\xi_n^2\right)+\\
&s_n\nabla\cdot\left(\left(\boldsymbol{v}_n\times\boldsymbol{w}_n\right)+u_n\boldsymbol{v}_n+\xi_n\boldsymbol{w}_n\right)=-\frac{e}{m}\left(\boldsymbol{v}_n\cdot\boldsymbol{E}_n+\boldsymbol{w}_n\cdot\boldsymbol{B}_n\right),
\end{aligned}
\tag{30}
$$

and

$$
\begin{aligned}
&\frac{1}{8\pi}\frac{\partial}{\partial t}\left(\boldsymbol{E}_n^2+\boldsymbol{B}_n^2\right)+\frac{\varepsilon}{4\pi}\left(\boldsymbol{E}_n^2+\boldsymbol{B}_n^2\right)+\\
&\frac{s_n}{4\pi}\nabla\cdot\left(\boldsymbol{E}_n\times\boldsymbol{B}_n\right)=e n_{0n}\left(\boldsymbol{v}_n\cdot\boldsymbol{E}_n+\boldsymbol{w}_n\cdot\boldsymbol{B}_n\right).
\end{aligned}
\tag{31}
$$

The expressions (30) and (31) are the analogs of Poynting theorem (known in electrodynamics) for electron fluid in normal metal.

7. Sound waves in normal metal

Let us consider small fluctuations of electron fluid parameters in normal metal near the equilibrium state

$$
\begin{aligned}
&n_n=n_{0n}+\tilde{n}_n,\\
&\boldsymbol{v}_n=\tilde{\boldsymbol{v}}_n,\\
&g_n=\tilde{g}_n,\\
&\boldsymbol{w}_n=\tilde{\boldsymbol{w}}_n.
\end{aligned}
\tag{32}
$$

The linearized equations are

$$
\begin{aligned}
&\frac{1}{s_n}\left(\frac{\partial}{\partial t}+\varepsilon\right)\tilde{\boldsymbol{v}}_n+\frac{s_n}{n_{0n}}\nabla\tilde{n}_n+\left[\nabla\times\tilde{\boldsymbol{w}}_n\right]=-\frac{e}{ms}\tilde{\boldsymbol{E}}_n,\\
&\frac{1}{n_{0n}}\left(\frac{\partial}{\partial t}+\varepsilon\right)\tilde{n}_n+\left(\nabla\cdot\tilde{\boldsymbol{v}}_n\right)=0,\\
&\frac{1}{s_n}\left(\frac{\partial}{\partial t}+\varepsilon\right)\tilde{\boldsymbol{w}}_n+\frac{s_n}{n_{0n}}\nabla\tilde{g}_n-\left[\nabla\times\tilde{\boldsymbol{v}}_n\right]=-\frac{e}{ms}\tilde{\boldsymbol{B}}_n,\\
&\frac{1}{n_{0n}}\left(\frac{\partial}{\partial t}+\varepsilon\right)\tilde{g}_n+\left(\nabla\cdot\tilde{\boldsymbol{w}}_n\right)=0.
\end{aligned}
\tag{33}
$$

and fields satisfy the equations (29). Then from (29) and (33) we obtain the following wave equations:

$$\left(\left(\frac{\partial}{\partial t}+\varepsilon\right)^2-s_n^2\Delta+\omega_{np}^2\right)\tilde{n}_n=0,$$
$$\left(\left(\frac{\partial}{\partial t}+\varepsilon\right)^2-s_n^2\Delta+\omega_{np}^2\right)\tilde{\boldsymbol{v}}_n=0,$$
$$\left(\left(\frac{\partial}{\partial t}+\varepsilon\right)^2-s_n^2\Delta+\omega_{np}^2\right)\tilde{g}_n=0, \tag{34}$$
$$\left(\left(\frac{\partial}{\partial t}+\varepsilon\right)^2-s_n^2\Delta+\omega_{np}^2\right)\tilde{\boldsymbol{w}}_n=0,$$

and

$$\left(\left(\frac{\partial}{\partial t}+\varepsilon\right)^2-s_n^2\Delta+\omega_{np}^2\right)\tilde{\boldsymbol{E}}_n=0,$$
$$\left(\left(\frac{\partial}{\partial t}+\varepsilon\right)^2-s_n^2\Delta+\omega_{np}^2\right)\tilde{\boldsymbol{B}}_n=0. \tag{35}$$

The dispersion relation for equations (34) and (35) is

$$\omega^2-2i\omega\varepsilon-\varepsilon^2-s_n^2k^2-\omega_{np}^2=0. \tag{36}$$

This equation has two roots

$$\omega=i\varepsilon\pm\sqrt{s_n^2k^2+\omega_{np}^2}. \tag{37}$$

We have damped waves with decrement ε. The real part of (37) is represented in Fig. 2. If $k\to\infty$ we have the asymptote

$$\omega=s_nk. \tag{38}$$

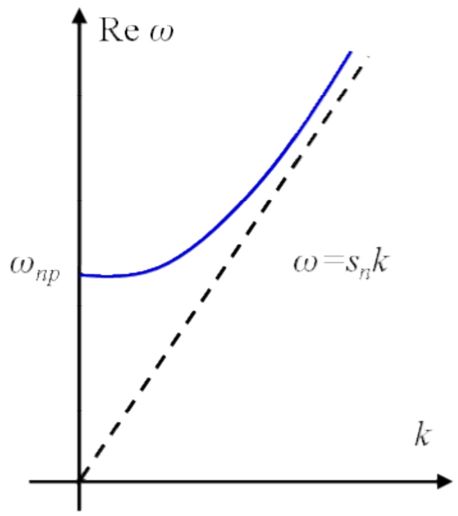

Fig. 2. The schematic draw of dispersion curve for electron sound waves in normal metal.

8. Two-fluid model of mixed state in superconductor

The system of equations describing the waves in mixture of normal and superconducting electrons is written as follows:

$$
\begin{aligned}
&\frac{1}{s_n}\left(\frac{\partial}{\partial t}+\varepsilon\right)\tilde{\boldsymbol{v}}_n+\frac{s_n}{n_{0n}}\nabla\tilde{n}_n+\left[\nabla\times\tilde{\boldsymbol{w}}_n\right]=-\frac{e}{ms_n}\tilde{\boldsymbol{E}}_n,\\
&\frac{1}{n_{0n}}\left(\frac{\partial}{\partial t}+\varepsilon\right)\tilde{n}_n+\left(\nabla\cdot\tilde{\boldsymbol{v}}_n\right)=0,\\
&\frac{1}{s_n}\left(\frac{\partial}{\partial t}+\varepsilon\right)\tilde{\boldsymbol{w}}_n+\frac{s_n}{n_{0n}}\nabla\tilde{g}_n-\left[\nabla\times\tilde{\boldsymbol{v}}_n\right]=-\frac{e}{ms_n}\tilde{\boldsymbol{B}}_n,\\
&\frac{1}{n_{0n}}\left(\frac{\partial}{\partial t}+\varepsilon\right)\tilde{g}_n+\left(\nabla\cdot\tilde{\boldsymbol{w}}_n\right)=0,
\end{aligned}
\tag{39}
$$

and

$$
\begin{aligned}
&\frac{1}{s_s}\frac{\partial\tilde{\boldsymbol{v}}_s}{\partial t}+\frac{s_s}{n_{0s}}\nabla\tilde{n}_s+\left[\nabla\times\tilde{\boldsymbol{w}}_s\right]=-\frac{e}{ms_s}\tilde{\boldsymbol{E}}_s,\\
&\frac{1}{n_{0s}}\frac{\partial\tilde{n}_s}{\partial t}+\left(\nabla\cdot\tilde{\boldsymbol{v}}_s\right)=0,\\
&\frac{1}{s_s}\frac{\partial\tilde{\boldsymbol{w}}_s}{\partial t}+\frac{s_s}{n_{0s}}\nabla\tilde{g}_s-\left[\nabla\times\tilde{\boldsymbol{v}}_s\right]=-\frac{e}{ms_s}\tilde{\boldsymbol{B}}_s,\\
&\frac{1}{n_{0s}}\frac{\partial\tilde{g}_s}{\partial t}+\left(\nabla\cdot\tilde{\boldsymbol{w}}_s\right)=0,
\end{aligned}
\tag{40}
$$

and

$$
\begin{aligned}
&\nabla\cdot\tilde{\boldsymbol{E}}_n=-4\pi e\left(\tilde{n}_n+\tilde{n}_s\right),\\
&\nabla\cdot\tilde{\boldsymbol{B}}_n=-4\pi e\left(\tilde{g}_n+\tilde{g}_s\right),\\
&\left(\frac{\partial}{\partial t}+\varepsilon\right)\tilde{\boldsymbol{B}}_n+s_n\nabla\times\tilde{\boldsymbol{E}}_n=4\pi e\left(n_{0n}\tilde{\boldsymbol{w}}_n+n_{0s}\tilde{\boldsymbol{w}}_s\right),\\
&\left(\frac{\partial}{\partial t}+\varepsilon\right)\tilde{\boldsymbol{E}}_n-s_n\nabla\times\tilde{\boldsymbol{B}}_n=4\pi e\left(n_{0n}\tilde{\boldsymbol{v}}_n+n_{0s}\tilde{\boldsymbol{v}}_s\right),
\end{aligned}
\tag{41}
$$

and

$$
\begin{aligned}
&\nabla\cdot\tilde{\boldsymbol{E}}_s=-4\pi e\left(\tilde{n}_n+\tilde{n}_s\right),\\
&\nabla\cdot\tilde{\boldsymbol{B}}_s=-4\pi e\left(\tilde{g}_n+\tilde{g}_s\right),\\
&\frac{\partial\tilde{\boldsymbol{B}}_s}{\partial t}+s_s\nabla\times\tilde{\boldsymbol{E}}_s=4\pi e\left(n_{0n}\tilde{\boldsymbol{w}}_n+n_{0s}\tilde{\boldsymbol{w}}_s\right),\\
&\frac{\partial\tilde{\boldsymbol{E}}_s}{\partial t}-s_s\nabla\times\tilde{\boldsymbol{B}}_s=4\pi e\left(n_{0n}\tilde{\boldsymbol{v}}_n+n_{0s}\tilde{\boldsymbol{v}}_s\right).
\end{aligned}
\tag{42}
$$

Wave equations for sound waves can be represented in the following form

$$\left\{\left(\left(\frac{\partial}{\partial t}+\varepsilon\right)^2 - s_n^2\Delta + \omega_{np}^2\right)\left(\frac{\partial^2}{\partial t^2} - s_s^2\Delta + \omega_{sp}^2\right) - \omega_{np}^2\omega_{sp}^2\right\}\tilde{\boldsymbol{P}} = 0, \tag{43}$$

where generalized parameter $\tilde{\boldsymbol{P}}$ takes the meanings

$$\tilde{\boldsymbol{P}} \in \left\{\tilde{n}_s, \tilde{g}_s, \tilde{\boldsymbol{v}}_s, \tilde{\boldsymbol{w}}_s, \tilde{\boldsymbol{E}}_s, \tilde{\boldsymbol{B}}_s, \tilde{n}_n, \tilde{g}_n, \tilde{\boldsymbol{v}}_n, \tilde{\boldsymbol{w}}_n, \tilde{\boldsymbol{E}}_n, \tilde{\boldsymbol{B}}_n\right\}. \tag{44}$$

The dispersion relation is

$$\left(\omega^2 - 2i\omega\varepsilon - \varepsilon^2 - s_n^2k^2 - \omega_{np}^2\right)\left(\omega^2 - s_s^2k^2 - \omega_{sp}^2\right) - \omega_{np}^2\omega_{sp}^2 = 0. \tag{45}$$

The schematic draw of dispersion relation (45) is represented in Fig. 3.

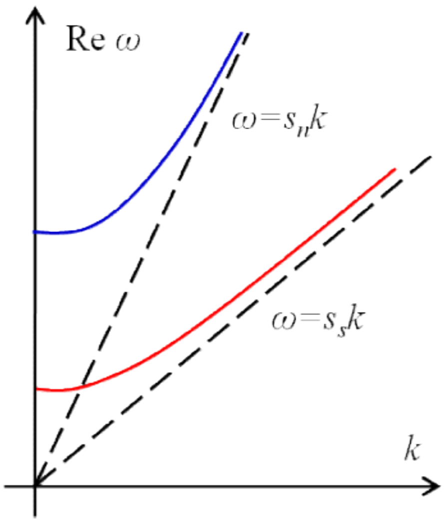

Fig. 3. The schematic plots of dispersion curves for electron sound waves in mixed state of superconductor.

9. Conclusion

Thus, we have proposed the self-consistent hydrodynamic model consisting of the equations for vortex flow of electron fluid and the equations for frozen-in electromagnetic field. The frozen-in fields satisfy modified Maxwell's equations, which show that the fields are incorporated in electron fluid and propagate at the speed of sound. This approach enables the description of superconducting condensate as the ideal electron fluid. In the case of vortex-less flow it leads us to the equations very close to London equations, but the speed of fluctuations propagation is equal to the sped of electron sound, not the speed of light. We believe that in simple model the damping processes are described by additional parameter, which means the collision frequency. It allows us to describe the electron fluid in normal metal and propose two-fluid model of mixed state in superconductor. The advantages of self-consistent equations are illustrated by derivation and analysis of wave equations for electron sound waves in metals.

Aknowlegements

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