Model P$(\varphi)_4$ Quantum Field Theory. A Nonstandard Approach Based on Nonstandard Pointwise-Defined Quantum Fields

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Abstract. A new non-Archimedean approach to interacted quantum fields is presented. In proposed approach, a field operator $\phi(x,t)$ no longer a standard tempered operator-valued distribution, but a non-classical operator-valued function. We prove using this novel approach that the quantum field theory with Hamiltonian $P(\varphi)_4$ exists and that the corresponding C^*- algebra of bounded observables satisfies all the Haag-Kastler axioms. In particular we prove that the $\lambda(\varphi^4)_4$ quantum field theory model is Lorentz covariant.

INTRODUCTION

Extending the real numbers $\mathbb{R}$ to include infinite and infinitesimal quantities originally enabled D. Laugwitz [1] to view the delta distribution $\delta(x)$ as a nonstandard point function. Independently A. Robinson [2] demonstrated that distributions could be viewed as generalized polynomials. Luxemburg [3] and Sloan [4] presented an alternate representative of distributions as internal functions within the context of canonical Robinson's theory of nonstandard analysis. For further information on nonstandard real analysis, we refer to [5]-[6].

Abbreviation 1.In this paper we adopt the following notations. For a standard set E we often write E_{st}. For a set E_{st} let ${}^{\sigma}E_{\text{st}}$ be a set ${}^{\sigma}E_{\text{st}} = \{{}^*x | x \in E_{\text{st}}\}$. We identify z with ${}^{\sigma}z$ i.e., $z \equiv {}^{\sigma}z$ for all $z \in \mathbb{C}$. Hence, ${}^{\sigma}E_{\mathbf{st}} = E_{\text{st}}$ if $E \subseteq \mathbb{C}$, e.g., ${}^{\sigma}\mathbb{C} = \mathbb{C}$, ${}^{\sigma}\mathbb{R} = \mathbb{R}$, ${}^{\sigma}P = P$, ${}^{\sigma}L_+^{\uparrow} = L_+^{\uparrow}$, etc. Let ${}^*\mathbb{R}_{\approx}$, ${}^*\mathbb{R}_{\approx+}$, ${}^*\mathbb{R}_{\text{fin}}$, ${}^*\mathbb{R}_{\infty}$, and ${}^*\mathbb{N}_{\infty}$ denote the sets of infinitesimal hyper-real numbers, positive infinitesimal hyper-real numbers, finite hyper-real numbers, infinite hyper-real numbers and infinite hyper natural numbers, respectively. Note that ${}^*\mathbb{R}_{\text{fin}} = {}^*\mathbb{R} \backslash {}^*\mathbb{R}_{\infty}$, ${}^*\mathbb{C} = {}^*\mathbb{R} + \mathrm{i}{}^*\mathbb{R}$, ${}^*\mathbb{C}_{\text{fin}} = {}^*\mathbb{R}_{\text{fin}} + \mathrm{i}{}^*\mathbb{R}_{\text{fin}}$.

Definition 1.[5]. Let $\{X, O\}$ be a standard topological space and let ${}^*X$ be the nonstandard extension of X. Let O_x denote the set of open neighbourhoods of point $x \in X$. The monad $mon_O(x)$ of x is the subset of ${}^*X$ defined by $mon_O(x) = \cap\{{}^*O | O \subset O_x\}$.The set of near standard points of ${}^*X$ is the subset of ${}^*X$ defined by $nst\,({}^*X) = \cup\{mon_O(x) | x \in X\}$. It is shown that $\{X, O\}$ is Hausdorff space if and only if $x \neq y$ implies $mon_O(x) \cap mon_O(y) = \emptyset$. Thus for any Hausdorff space $\{X, O\}$, we can define the equivalence relation $\approx_O$ on $nst\,({}^*X)$ so that $x \approx_O y$ if and only if $x \in mon_O(z)$ and $y \in mon_O(z)$ for some $z \in X$.

Definition 2.The standard Schwartz space of rapidly decreasing test functions on $\mathbb{R}^n, n \in \mathbb{N}$ is the standard function space is defined by $S(\mathbb{R}^n, \mathbb{C}) = \{f \in C^{\infty}(\mathbb{R}^n, \mathbb{C}) | \forall \alpha, \beta \in \mathbb{N}^n[\|f\|_{\alpha,\beta} < \infty]\}$, where

$$\|f\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n}\left|x^{\alpha}\left(D^{\beta}f(x)\right)\right|.$$

Remark 1.If f is a rapidly decreasing function, then for all $\alpha \in \mathbb{N}^n$ the integral of $\left|x^{\alpha}D^{\beta}f(x)\right|$ exists

$$\int_{\mathbb{R}^n} \left| x^\alpha D^\beta f(x) \right| d^n x < \infty.$$

Definition 3.The internal Schwartz space of rapidly decreasing test functions on ${}^*\mathbb{R}^n, n \in {}^*\mathbb{N}$ is the function space defined by ${}^*S({}^*\mathbb{R}^n, {}^*\mathbb{C}) = \{ {}^*f \in {}^*C^{{}^*\infty}({}^*\mathbb{R}^n, {}^*\mathbb{C}) | \forall \alpha, \beta \in {}^*\mathbb{N}^n [{}^*\| {}^*f \|_{\alpha,\beta} < {}^*\infty] \}$, where

$${}^*\| {}^*f \|_{\alpha,\beta} = \sup{}^* \left\{ x^\alpha \left(D^\beta f(x) \right) | x \in \mathbb{R}^n \right\}.$$

Remark 2.If f is a rapidly decreasing function, $f \in S(\mathbb{R}^n, \mathbb{C})$, then for all $\alpha, \beta \in {}^*\mathbb{N}^n$ the internal integral of $\left| {}^*x^\alpha D^\beta {}^*f(x) \right|$ exists

$${}^*\int_{{}^*\mathbb{R}^n} \left| {}^*x^\alpha D^\beta {}^*f(x) \right| d^n x < {}^*\infty.$$

Here $D^\beta {}^*f(x) = {}^*\left(D^\beta f(x) \right)$.

Definition 4.The Schwartz space of essentially rapidly decreasing test functions on ${}^*\mathbb{R}^n, n \in {}^*\mathbb{N}$ is the function space defined by

$${}^*S_{\text{fin}}({}^*\mathbb{R}^n, {}^*\mathbb{C}) = \left\{ {}^*f \in {}^*C^{{}^*\infty}({}^*\mathbb{R}^n, {}^*\mathbb{C}) | \forall(\alpha, \beta)(\alpha, \beta \in {}^*\mathbb{N}^n) \exists c_{\alpha\beta} \left(c_{\alpha\beta} \in {}^*\mathbb{R}_{\text{fin}} \right) \forall x (x \in {}^*\mathbb{R}^n) \left[\left| x^\alpha \left({}^*D^\beta {}^*f(x) \right) \right| < c_{\alpha\beta} \right] \right\}.$$

Remark 3.If ${}^*f \in {}^*S_{\text{fin}}({}^*\mathbb{R}^n, {}^*\mathbb{C})$, then for all $\alpha \in {}^*\mathbb{N}^n$ the internal integral of $\left| {}^*x^\alpha D^\beta {}^*f(x) \right|$ exists and finitely bounded above

$${}^*\int_{{}^*\mathbb{R}^n} \left| {}^*x^\alpha D^\beta {}^*f(x) \right| d^n x < d_{\alpha\beta}, d_{\alpha\beta} \in {}^*\mathbb{R}_{\text{fin}}.$$

Abbreviation 2.The standard Schwartz space of rapidly decreasing test functions on $\mathbb{R}^n$ we will be denote by $S(\mathbb{R}^n)$. Let ${}^*S({}^*\mathbb{R}^n), n \in {}^*\mathbb{N}$ denote the space of ${}^*\mathbb{C}$-valued rapidly decreasing internal test functions on ${}^*\mathbb{R}^n, n \in {}^*\mathbb{N}$ and let ${}^*S_{\text{fin}}({}^*\mathbb{R}^n), n \in {}^*\mathbb{N}$ denote the set of ${}^*\mathbb{C}_{\text{fin}}$ -valued essentially rapidly decreasing test functions on ${}^*\mathbb{R}^n, n \in {}^*\mathbb{N}$. If $h(\omega, x): \mathbb{R} \times \mathbb{R}^n$ and $f: \mathbb{R}^n \to \mathbb{C}$ are Lebesgue measurable on $\mathbb{R}^{4n}$ we shall write $\langle {}^*h, {}^*f \rangle$ for internal Lebesgue integral ${}^*\int_{{}^*\mathbb{R}^n} {}^*h {}^*f \, d^n x$ with ${}^*f \in {}^*S_{\text{fin}}({}^*\mathbb{R}^n)$. Certain internal functions ${}^*h(\omega, x): {}^*\mathbb{R} \times {}^*\mathbb{R}^n \to {}^*\mathbb{C}$ define classical distribution $\tau(f)$ by the rule [3],[4]:

$$\tau(f) = \text{st}(\langle {}^*h, {}^*f \rangle). \tag{1}$$

Here $\text{st}(a)$ is the standard part of a and $\text{st}(\langle {}^*h, {}^*f \rangle)$ exists.
Definition 5.We shall say that ${}^*h(\omega, x)$ with $\omega = \varpi \in {}^*\mathbb{R}_\infty$ is an internal representative to distribution $\tau(f)$ and we will write symbolically $\tau(x_1, \dots, x_n) \approx {}^*h(\omega, x_1, \dots, x_n)$ if the equation (1) holds.
Definition6. **[**6**].**We shall say that certain internal functions ${}^*h(\omega, x): {}^*\mathbb{R} \times {}^*\mathbb{R}^n \to {}^*\mathbb{C}$ is a finite tempered distribution if ${}^*f \in {}^*S_{\text{fin}}({}^*\mathbb{R}^n)$ implies $|{}^*h, {}^*f| \in {}^\sigma\mathbb{R} = \mathbb{R}$. A functions ${}^*h(\omega, x): {}^*\mathbb{R} \times {}^*\mathbb{R}^n \to {}^*\mathbb{C}$ is called infinitesimal tempered distribution if ${}^*f \in {}^*S_{\text{fin}}({}^*\mathbb{R}^n)$ implies $|{}^*h, {}^*f| \in {}^*\mathbb{R}_\approx$ .The space of infinitesimal tempered distribution is denoted by ${}^*S_\approx({}^*\mathbb{R}^n)$.
Definition 7.We shall say that certain internal functions ${}^*h(\omega, x): {}^*\mathbb{R} \times {}^*\mathbb{R}^{4n} \to {}^*\mathbb{C}$ is a Lorentz $\approx$ -invariant tempered distribution if ${}^*f \in {}^*S_{\text{fin}}({}^*\mathbb{R}^n)$ and $\Lambda \in {}^\sigma L_+^\uparrow$ implies $\langle {}^*h, {}^*f(\Lambda x_1, \dots, \Lambda x_n) \rangle \approx \langle {}^*h, {}^*f(x_1, \dots, x_n) \rangle$.
Example 1. Let us consider Lorentz invariant distribution

$$D(x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\boldsymbol{kr}} \frac{\sin \omega t}{\omega} d^3 k = \frac{1}{2\pi} \delta(r^2 - t^2) \text{sign}(t). \tag{2}$$

Here $\omega = |\boldsymbol{k}| = \sqrt{k_1^2 + k_2^2 + k_3^2}$ and $\boldsymbol{r} = (x_1, x_2, x_3)$, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. It easily verify that distribution $D(x)$ has the following internal representative

$$D(x, \varpi) = \frac{1}{(2\pi)^3} {}^*\!\!\int_{|\boldsymbol{k}| \leq \varpi} e^{i\boldsymbol{kr}} \frac{\sin \omega t}{\omega} d^3 k. \tag{3}$$

Here $\varpi \in {}^*\mathbb{R}_\infty$. By integrating in (3) over angle variables we get

$$D(x, \varpi) = \frac{1}{8\pi^2 r} {}^*\!\!\int_0^\varpi \left\{ e^{i\omega(r-t)} + e^{-i\omega(r-t)} - e^{i\omega(r+t)} - e^{-i\omega(r+t)} \right\} d\omega. \tag{4}$$

From (4) by canonical calculation finally we get

$$D(x, \varpi) \approx \frac{1}{4\pi^2 r} \left[\frac{\sin \varpi(r-t)}{r-t} - \frac{\sin \varpi(r+t)}{r+t} \right] \approx \frac{\delta(r-t) - \delta(r+t)}{4\pi^2 r} = \frac{1}{2\pi} \delta(r^2 - t^2) \mathrm{sign}(t). \tag{5}$$

Example2. We consider now the following Lorentz invariant distribution:

$$D_1(x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\boldsymbol{kr}} \frac{\cos \omega t}{\omega} d^3 k = \frac{1}{2\pi^2} \frac{1}{x^2}. \tag{6}$$

It easily verify that distribution $D(x)$ has the following internal representative

$$D_1(x, \varpi) = \frac{1}{(2\pi)^3} {}^*\!\!\int_{|\boldsymbol{k}| \leq \varpi} e^{i\boldsymbol{kr}} \frac{\cos \omega t}{\omega} d^3 k. \tag{7}$$

Here $\varpi \in {}^*\mathbb{R}_\infty$. By integrating in (7) over angle variables we get

$$D_1(x, \varpi) \approx -\frac{i}{8\pi^2 r} {}^*\!\!\int_0^\varpi \left\{ e^{i\omega(r-t)} - e^{-i\omega(r-t)} + e^{i\omega(r+t)} - e^{-i\omega(r+t)} \right\} d\omega. \tag{8}$$

From (8) finally we get

$$D_1(x, \varpi) \approx -\frac{i}{8\pi^2 r} \left[\frac{-2}{i(r-t)} + \frac{-2}{i(r+t)} + \frac{2\cos \varpi(r-t)}{i(r-t)} + \frac{2\cos \varpi(r+t)}{i(r+t)} \right] \approx \frac{1}{2\pi^2} \frac{1}{x^2}. \tag{9}$$

Example 3.We consider now the following Lorentz invariant distribution

$$\Delta_c(x) = \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} e^{i(\boldsymbol{kr} - \varepsilon(\boldsymbol{k})|t|)} \frac{d^3 k}{\varepsilon(\boldsymbol{k})} = -\frac{m}{8\pi} \frac{H_1^{(2)}\left(-im\sqrt{|x^2|}\right)}{m\sqrt{|x^2|}}. \tag{10}$$

Here $-x^2 < 0$, $\varepsilon(\boldsymbol{k}) = \sqrt{|\boldsymbol{k}^2| + m^2}$ and $H_1^{(2)}$ is a Hankel function of the second kind. It easily verify that distribution $\Delta_c(x)$ has the following internal representative

$$\Delta_c(x, \varpi) = \frac{1}{2(2\pi)^3} {}^*\!\!\int_{|\boldsymbol{k}| \leq \varpi} e^{i(\boldsymbol{kr} - \varepsilon(\boldsymbol{k})|t|)} \frac{d^3 k}{\varepsilon(\boldsymbol{k})} \tag{11}$$

From (10)-(11) it follows ${}^*\Delta_c(x) = \Delta_c(x, \varpi) + \breve{\Delta}_c(x)$ where

$$\breve{\Delta}_c(x) = \frac{1}{2(2\pi)^3} {}^*\!\!\int_{|\boldsymbol{k}| > \varpi} e^{i(\boldsymbol{kr} - \varepsilon(\boldsymbol{k})|t|)} \frac{d^3 k}{\varepsilon(\boldsymbol{k})}. \tag{12}$$

Note that for all $\Lambda \in {}^\sigma L_+^\uparrow$, $\breve{\Delta}_c(\Lambda x) \in {}^*S_\approx({}^*\mathbb{R}^n)$ and therefore for all $\Lambda \in {}^\sigma L_+^\uparrow$, $\Delta_c(\Lambda x, \varpi) \approx \Delta_c(x, \varpi)$, i.e., $\Delta_c(x, \varpi)$ is a Lorentz $\approx$ -invariant tempered distribution, see definition 4. Thus we can set $t = 0$ in (11). By integrating in (11) over angle variables and using substitution of variables $|\boldsymbol{k}| = m \sinh(u)$ we get

$$\Delta_c(x,\varpi) \approx \frac{m}{8\pi^2 ir} {}^*\int_{-\ln\varpi}^{\ln\varpi} \exp\big(imr\sinh(u)\big)du. \quad (13)$$

Note that

$$^*H_1^{(2)}(x) = \frac{\pi}{i}\int_{-^*\infty}^{^*\infty} \exp\big(imr\sinh(u)\big)du = \Delta_c(x,\varpi) + \Xi(x,\varpi), \quad (14)$$

$$\Xi(x,\varpi) = \frac{\pi}{i}\int_{-^*\mathbb{R}}^{-\ln\varpi} \exp\big(imr\sinh(u)\big)du + \int_{\ln\varpi}^{^*\mathbb{R}} \exp\big(imr\sinh(u)\big)du. \quad (15)$$

From (13)-(15) finally we obtain $\Delta_c(x,\varpi) \approx H_1^{(2)}(x)$, since $\Xi(x,\varpi) \in {}^*S_{\approx}({}^*\mathbb{R}^n)$.
Example 4. Let us consider Lorentz invariant distribution

$$\Delta(x-y) = \int\{\exp[-ip(x-y)] - \exp[ip(x-y)]\}\,\delta(p^2-m^2)\vartheta(p^0)d^4p. \quad (16)$$

From (16) one obtains $\Delta(x-y) = \Xi_1(x-y) - \Xi_2(x-y)$, where

$$\Xi_1(x-y) = \int\big\{\exp\{[i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})] - i\omega(\boldsymbol{p})(x^0-y^0)\}\big\}\frac{d^3p}{\sqrt{\boldsymbol{p}^2+m^2}}, \quad (17)$$

$$\Xi_2(x-y) = \int\big\{\exp\{[-i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})] + i\omega(\boldsymbol{p})(x^0-y^0)\}\big\}\frac{d^3p}{\sqrt{\boldsymbol{p}^2+m^2}}, \quad (18)$$

$\omega(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2+m^2}$. It easily verify that distribution (17) and (18) has the following internal representatives

$$\Xi_1(x-y,\varpi) = {}^*\int_{|\boldsymbol{k}|\le\varpi}\big\{\exp\{[i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})] - i\omega(\boldsymbol{p})(x^0-y^0)\}\big\}\frac{d^3p}{\sqrt{\boldsymbol{p}^2+m^2}}. \quad (19)$$

$$\Xi_2(x-y,\varpi) = {}^*\int_{|\boldsymbol{k}|\le\varpi}\big\{-\exp\big[[i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})] + i\omega(\boldsymbol{p})(x^0-y^0)\big]\big\}\frac{d^3p}{\sqrt{\boldsymbol{p}^2+m^2}}. \quad (20)$$

Note that ${}^*\Delta(x-y) = [\Xi_1(x-y,\varpi) + \Xi_2(x-y,\varpi)] + \big[\breve{\Xi}_1(x-y,\varpi) + \breve{\Xi}_2(x-y,\varpi)\big]$, where

$$\breve{\Xi}_1(x-y,\varpi) = {}^*\int_{|\boldsymbol{k}|>\varpi}\big\{\exp\{[i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})] - i\omega(\boldsymbol{p})(x^0-y^0)\}\big\}\frac{d^3p}{\sqrt{\boldsymbol{p}^2+m^2}}, \quad (21)$$

$$\breve{\Xi}_2(x-y,\varpi) = {}^*\int_{|\boldsymbol{k}|>\varpi}\big\{-\exp\big[[i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})] + i\omega(\boldsymbol{p})(x^0-y^0)\big]\big\}\frac{d^3p}{\sqrt{\boldsymbol{p}^2+m^2}}. \quad (22)$$

Note that for all $\Lambda \in {}^\sigma L_+^\uparrow$, $\breve{\Xi}_1(\Lambda(x-y),\varpi) + \breve{\Xi}_1(\Lambda(x-y),\varpi) \in {}^*S_{\approx}({}^*\mathbb{R}^n)$ and therefore for all $\Lambda \in {}^\sigma L_+^\uparrow$, ${}^*\Delta\big(\Lambda(x-y)\big) \approx \Delta(\Lambda(x-y),\varpi) = \Xi_1(\Lambda(x-y),\varpi) + \Xi_2(\Lambda(x-y),\varpi)$, i.e., $\Delta(x-y,\varpi)$ is a Lorentz $\approx$-invariant tempered distribution, see definition 4. From Eq.(20) by replacement $\boldsymbol{p} \to -\boldsymbol{p}$ we obtain

$$\Xi_1(x-y,\varpi) = -{}^*\int_{|\boldsymbol{k}|\le\varpi}\big\{\exp\{[i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})] + i\omega(\boldsymbol{p})(x^0-y^0)\}\big\}\frac{d^3p}{\sqrt{\boldsymbol{p}^2+m^2}}. \quad (23)$$

From (19) and (23) we get

$$\Delta(x-y,\varpi) = \Xi_1(x-y,\varpi) + \Xi_2(x-y,\varpi) = {}^*\int_{|\boldsymbol{k}|\le\varpi}\sin[\omega(\boldsymbol{p})(x^0-y^0)]\exp[i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})]\frac{d^3p}{\sqrt{\boldsymbol{p}^2+m^2}}. \quad (24)$$

Thus for any points x and y separated by spacelike interval from (24) we obtain that

$$\Delta(x-y,\varpi) \approx 0, \quad (25)$$

since $\Delta(x-y,\varpi)$ is a Lorentz $\approx$-invariant tempered distribution. From (25) for any points x and y separated by spacelike interval we obtain that: $\mathrm{st}\big(\Delta(x-y,\varpi)\big)\equiv 0$.
Definition 8. [8].For each $m>0$, let $H_m=\{p\in\mathbb{R}^4|p\cdot\tilde{p}=m^2,m>,p_0>0\}$, where $\tilde{p}=(p^0,-p^1,-p^2,-p^3)$. Here the sets H_m which are standard mass hyperboloids, are invariant under ${}^{\sigma}L_{+}^{\uparrow}$. Let j_m be the homeomorphism of H_m onto $\mathbb{R}^3$ given by $j_m:(p_0,p_1,p_2,p_3)\to(p_1,p_2,p_3)=\boldsymbol{p}$. Define a measure $\Omega_m(E)$ on H_m by

$$\Omega_m(E)=\int_{j_m(E)}\frac{d^3\boldsymbol{p}}{\sqrt{|\boldsymbol{p}|^2+m^2}}.$$

The measure $\Omega_m(E)$ is ${}^{\sigma}L_{+}^{\uparrow}$-invariant [8].
Theorem 1.[8].Let μ be a polynomially bounded #-measure with support in $\bar{V}_{+}$. If μ is ${}^{\sigma}L_{+}^{\uparrow}=L_{+}^{\uparrow}$- invariant, there exists a polynomially bounded measure ρ on $[0,\infty)$ and a constant c so that for any $f\in S(\mathbb{R}^4)$

$$\int_{{}^*\mathbb{R}^4}f\,d\,\mu=cf(0)+\int_0^{\infty}d\,\rho\,(m)\left(\int_{\mathbb{R}^3}\frac{f\left(\sqrt{|\boldsymbol{p}|^2+m^2},p_1,p_2,p_3\right)d^3\boldsymbol{p}}{\sqrt{|\boldsymbol{p}|^2+m^2}}\right).\tag{26}$$

Theorem 2.Let μ is a polynomially bounded $L_{+}^{\uparrow}$- invariant measure with support in $\bar{V}_{+}$. Let $\mathcal{F}(f)$ be a linear $*$-continuous functional $\mathcal{F}:{}^*S_{\mathrm{fin}}\,({}^*\mathbb{R}^4)\to{}^*\mathbb{R}_{\mathrm{fin}}$ defined by ${}^*\!\int_{{}^*\mathbb{R}^4}f\,d\,\mu$ and there exists a polynomially bounded measure ρ on $[0,\infty)$ such that $\int_0^{{}^*\infty}d\,{}^*\rho\,(m)\in{}^*\mathbb{R}_{\mathrm{fin}}$ and a constant $c\in{}^*\mathbb{R}_{\mathrm{fin}}$ so that (1) holds. Then for any $f\in{}^*S_{\ \mathrm{fin}}({}^*\mathbb{R}^4)$ and for any $\varkappa\in{}^*\mathbb{R}_{\infty}$ the following property holds

$$\mathcal{F}(f)\approx cf(0)+\int_0^{{}^*\infty}d\,{}^*\rho\,(m)\left(\int_{|p|\le\varkappa}\frac{f\left(\sqrt{|\boldsymbol{p}|^2+m^2},p_1,p_2,p_3\right)d^{\#3}\boldsymbol{p}}{\sqrt{|\boldsymbol{p}|^2+m^2}}\right)\tag{27}$$

Definition 9.Let $\chi(\varkappa,\boldsymbol{p})$ be a function such that: $\chi(\varkappa,\boldsymbol{p})\equiv 1$ if $|\boldsymbol{p}|\le\varkappa$, $\chi(\varkappa,\boldsymbol{p})\equiv 0$ if $|\boldsymbol{p}|>\varkappa$, $\varkappa\in{}^*\mathbb{R}_{\infty}$.Define internal measure $\Omega_{m,\varkappa}$ on ${}^*H_m$ by

$$\Omega_{m,\varkappa}(E)={}^*\!\int_{{}^*H_m}\frac{\chi(\varkappa,\boldsymbol{p})d^3\boldsymbol{p}}{\sqrt{|\boldsymbol{p}|^2+m^2}}.\tag{28}$$

Theorem 3.[8]: Let $W_2(x_1,x_2)$ be the two-point function of a field theory satisfying the Wightman axioms and the additional condition that $(\psi_0,\varphi(f)\psi_0)=0$ for all $f\in S(\mathbb{R}^4)$. Then there exists a polynomially bounded positive measure $\rho(m)$ on $[\![0,\infty)$ so that for all for all $f\in S(\mathbb{R}^4)$

$$W_2(f)=\left(\psi_0,\varphi(\bar{f})\varphi(f)\psi_0\right)=\int\bar{f}(x_1)f(x_2)\,W_2(x_1-x_2)d^4xd^4y=\int_0^{\infty}\left(\int_{H_m}\hat{f}d\Omega_m\right)d\rho(m).\tag{29}$$

Theorem 4: Let $W_2(x_1,x_2)$ be the two-point function of a field theory mentioned in Theorem3. Then for all $f\in S_{\mathrm{fin}}({}^*\mathbb{R}^4)$ and for any $\varkappa\in{}^*\mathbb{R}_{\infty}$ the following property holds

$${}^*W_2(f)\approx{}^*\!\int_0^{{}^*\infty}\left({}^*\!\int_{{}^*H_m}\hat{f}d\Omega_{m,\varkappa}\right)d^*\rho(m).\tag{30}$$

Definition 10.1) Let $L(H)$ be algebra of the all densely defined linear operators in standard Hilbert space H. Operator-valued distribution on $\mathbb{R}^n$, that is a map $\varphi:S(\mathbb{R}^n)\to L(H)$ such that there exists a dense subspace $D\subset H$ satisfying:
1. for each $f\in S(\mathbb{R}^n)$ the domain of φ contains D,
2. the induced map: $S\to End(D)$, $f\to\varphi\,(f)$, is linear,
3. for each $h_1\in D$ and $h_2\in H$ the assignment $f\to\langle h_2,\varphi(f)h_1\rangle$ is a tempered distribution.
2) Certain operator-valued internal function $\varphi\,({}^*f,\varpi):{}^*S\big({}^*\mathbb{R}^n\big)\to{}^*L({}^*H)$ is an internal representative for standard operator valued distribution $\varphi\,(f)$ if for each near standard vectors $\tilde{h}_1\in{}^*D$ and $\tilde{h}_2\in{}^*H$ the equality holds

$$\langle h_2, \varphi(f)h_1\rangle = \mathrm{st}\big({}^*\langle \tilde{h}_2, \varphi({}^*f, \varpi)\tilde{h}_1\rangle\big), \tag{31}$$

where $h_1 \approx \tilde{h}_1$ and $h_2 \approx \tilde{h}_2$.

Definition 11.[9]. Let H be a Hilbert space and denote by H^n the n-fold tensor product $H^n = H\otimes H\otimes \cdots \otimes H$. Set $H^0 = \mathbb{C}$ and define $\mathcal{F}(H) = H^n$. $\mathcal{F}(H)$ is called the Fock space over Hilbert space H. Notice $\mathcal{F}(H)$ will be separable if H is. We set now $H = L_2(\mathbb{R}^3)$ then an element $\psi \in \mathcal{F}(H)$ is a sequence of $\mathbb{C}$-valued functions $\psi = \{\psi_0, \psi_1(x_1), \psi_2(x_1,x_2), \quad \psi_2(x_1,x_2,x_3), \dots, \psi_n(x_1,\dots,x_n)\}$, $n \in \mathbb{N}$ and such that $|\psi_0|^2 + \sum_{n\in\mathbb{N}}(\int |\psi_n(x_1,\dots,x_n)|^2 d^{3n}x) < \infty$.

Definition 12.[8]. We define now external operator $a(p)$ on $\mathcal{F}_s$ with domain D_S by

$$(a(p)\psi)^{(n)} = \sqrt{n+1}\,\psi^{(n+1)}(p,k_1,\dots k_n). \tag{32}$$

The formal adjoint of the operator $a(p)$ reads

$$(a^\dagger(p)\psi)^{(n)} = \tfrac{1}{\sqrt{n}}\textstyle\sum_{l=1}^{n}\delta^{(3)}(p-k_l)\psi^{(n-1)}(k_1,\dots,k_{l-1},k_{l+1},\dots,k_n) \tag{33}$$

Definition 13.[8]. A vector $\{\psi^{(n)}\}_{n=1}^{\infty}$ for which $\psi^{(n)} = 0$ for all except finitely many n is called a finite particle vector. We will denote the set of finite particle vectors by F_0. The vector $\Omega_0 = \langle 1,0,0,\dots\rangle$ is called the vacuum.

Definition 14: We let now ${}^*D_{*S} = \{{}^*\psi | {}^*\psi \in {}^*F_0, {}^*\psi^{(n)} \in {}^*S({}^*\mathbb{R}^{3n}), n \in {}^*\mathbb{N}\}$ and for each $p \in {}^*\mathbb{R}^{3n}$ we define an internal operator ${}^*a(p)$ on ${}^*\mathcal{F}_s$ with domain ${}^*D_{*S}$ by

$$({}^*a(p)\psi)^{(n)} = \sqrt{n+1}\,{}^*\psi^{(n+1)}(p,k_1,\dots k_n). \tag{34}$$

The formal $*$-adjoint of the operator ${}^*a$ reads

$$({}^*a^\dagger(p)\psi)^{(n)} = \tfrac{1}{\sqrt{n}}\textstyle\sum_{l=1}^{n}{}^*\delta^{(3)}(p-k_l){}^*\psi^{(n-1)}(k_1,\dots,k_{l-1},k_{l+1},\dots,k_n). \tag{35}$$

We express the free internal scalar field and the time zero fields with hyperfinite momentum cutoff $\varkappa \in {}^*\mathbb{R}_\infty$ in terms of ${}^*a^\dagger(p)$ and ${}^*a(p)$ as quadratic forms on ${}^*D_{*S}$ by

$${}^*\Phi_{m,\varkappa}(x,t) =$$

$$(2\pi)^{-3/2}\,{}^*\!\int_{|p|\le\varkappa}\big\{\big(\exp(\mu(p)t - ipx)\big){}^*a^\dagger(p) + \big(\exp(\mu(p)t + ipx)\big){}^*a\,(p)\big\}\frac{d^3p}{\sqrt{2\mu(p)}}, \tag{36}$$

$${}^*\varphi_{m,\varkappa}(x,t) = (2\pi)^{-3/2}\,{}^*\!\int_{|p|\le\varkappa}\big\{\big(\exp(-ipx)\big){}^*a^\dagger(p) + \big(\exp(ipx)\big){}^*a\,(p)\big\}\frac{d^3p}{\sqrt{2\mu(p)}}, \tag{37}$$

$${}^*\pi_{m,\varkappa}(x,t) = (2\pi)^{-3/2}\,{}^*\!\int_{|p|\le\varkappa}\big\{\big(\exp(-ipx)\big){}^*a^\dagger(p) + \big(\exp(ipx)\big){}^*a\,(p)\big\}\frac{d^3p}{\sqrt{\mu(p)/2}}. \tag{38}$$

Theorem 5: Let $\Phi_m(x,t)$ and $\varphi_m(x,t)$, $\pi_m(x,t)$ be the free standard scalar field and the time zero fields respectively. Then for any $\varkappa \in {}^*\mathbb{R}_\infty$ the operator valued internal functions (35)-(37) gives internal representatives for standard operator valued distributions $\Phi_m(x,t)$ and $\varphi_m(x,t)$, $\pi_m(x,t)$ respectively.

Definition 15: Let $\{X, \|\cdot\|\}$ be a standard Banach space. For $x \in {}^*X$ and $\varepsilon > 0, \varepsilon \approx 0$ we define the open $\approx$-ball about x of radius ε to be the set $B_\varepsilon(x) = \{y \in {}^*X | {}^*\|x-y\| < \varepsilon\}$.

Definition 16.Let $\{\{X, \|\cdot\|\}$ be a standard Banach space, $Y \subset X$, thus ${}^*Y \subset {}^*X$ and let $x \in {}^*X$.Then x is an $*$-accumulotion point of ${}^*Y$ if for any $\varepsilon \in {}^*\mathbb{R}_{\approx+}$ there is a hyper infinite sequence $\{x_n\}_{n=1}^{*\infty}$in ${}^*Y$ such that $\{x_n\}_{n=1}^{*\infty} \cap (B_\varepsilon(x)\backslash\{x\} \neq \emptyset)$.

Definition 17: Let $\{\{X, \|\cdot\|\}$ be a standard Banach space, let${}^*Y \subseteq {}^*X$,${}^*Y$ is $*$-closed if any $*$-accumulation point of

${}^{*}Y$ is an element of ${}^{*}Y$.

Definition 18. Let $\{\{X,\|\cdot\|\}$ be a standard Banach space. We shall say that internal hyper infinite sequence $\{x_n\}_{n=1}^{*\infty}$ in ${}^{*}X$ is $*$-converges to $x\in{}^{*}X$ as $n\to{}^{*}\infty$ if for any $\varepsilon\in{}^{*}\mathbb{R}_{\approx+}$ there is $N\in{}^{*}\mathbb{N}$ such that for any $n>N$: ${}^{*}\|x-y\|<\varepsilon$.

Definition 19. Let $\{\{X,\|\cdot\|_X\}$, $\{\{Y,\|\cdot\|_Y\}$ be a standard Banach spaces. A linear internal operator $A\colon D(A)\subseteq{}^{*}X\to{}^{*}Y$ is $*$-closed if for every internal hyper infinite sequence $\{x_n\}_{n=1}^{*\infty}$ in $D(A)$ $*$-converging to $x\in{}^{*}X$ such that $Ax_n\to y\in{}^{*}Y$ as $n\to{}^{*}\infty$ one has $x\in D(A)$ and $Ax=y$. Equivalently , A is $*$-closed if its graph is $*$-closed in the direct sum ${}^{*}X\oplus{}^{*}Y$.

Definition 20. Let H be a standard Hilbert space. The graph of the internal linear transformation $T\colon{}^{*}H\to{}^{*}H$ is the set of pairs $\{\langle\varphi,T\varphi\rangle|\varphi\in D(T)\}$. The graph of T, denoted by $\Gamma(\mathrm{T})$, is thus a subset of ${}^{*}H\times{}^{*}H$ which is internal Hilbert space with inner product $(\langle\varphi_1,\psi_1\rangle,\langle\varphi_2,\psi_2\rangle)=(\varphi_1,\varphi_2)+(\psi_1,\psi_2)$.The operator T is called a $*$-closed operator if $\Gamma(\mathrm{T})$ is a $*$-closed subset of Cartesian product ${}^{*}H\times{}^{*}H$.

Definition 21. Let H be a standard Hilbert space. Let T_1 and T be internal operators on internal Hilbert space ${}^{*}H$. Note that if $\Gamma(T_1)\supset\Gamma(T)$, then T_1 is said to be an extension of T and we write $T_1\supset T$. Equivalently, $T_1\supset T$ if and only if $D(T_1)\supset D(T)$ and $T_1\varphi=T\varphi$ for all $\varphi\in D(T)$.

Definition 22. An internal operator T on ${}^{*}H$ is $*$-closable if it has a $*$-closed extension. Every $*$-closable internal operator T has a smallest $*$-closed extension, called its $*$-closure, which we denote by $*$-$\bar{T}$.

Definition 23. Let H be a standard Hilbert space. Let T be a $*$-densely defined internal linear operator on internal Hilbert space ${}^{*}H$. Let $D(T^{*})$ be the set of $\varphi\in{}^{*}H$ for which there is a vector $\xi\in{}^{*}H$ with $(T\psi,\varphi)=(\varphi,\xi)$ for all $\psi\in D(T)$, then for each $\varphi\in D(T^{*})$, we define $T^{*}\varphi=\xi$. T^{*} is called the $*$-adjoint of T. Note that $S\subset T$ implies $T^{*}\subset S^{*}$.

Definition 24. Let H be a standard Hilbert space. A $*$-densely defined internal linear operator T on internal Hilbert space ${}^{*}H$ is called symmetric (or Hermitian) if $T\subset T^{*}$. Equivalently, T is symmetric if and only $if\ (T\varphi,\psi)=(\varphi,T\psi)$ for all $\varphi,\psi\in D(T)$.

Definition 25. Let H be a standard Hilbert space. A symmetric internal linear operator T on internal Hilbert space ${}^{*}H$ is called essentially self- $*$-adjoint if its $*$-closure $*$-$\bar{T}$ is self- $*$-adjoint. If T is $*$-closed, a subset $D\subset D(T)$ is called a $*$-core for T if $*$- $\left(\overline{T\restriction D}\right)=T$. If T is essentially self- $*$-adjoint, then it has one and only one self -$*$-adjoint extension.

Theorem 6: Let $n_1,n_2\in\mathbb{N}$ and suppose that $W\left(k_1,\dots k_{n_1},p_1,\dots,p_{n_2}\right)\in{}^{*}L_2\left({}^{*}\mathbb{R}^{3(n_1+n_2)}\right)$ where $W\left(k_1,\dots k_{n_1},p_1,\dots,p_{n_2}\right)$ is a ${}^{*}\mathbb{C}$-valued internal function on ${}^{*}\mathbb{R}^{3(n_1+n_2)}$. Then there is a unique operator T_W on ${}^{*}\mathcal{F}({}^{*}L_2({}^{*}\mathbb{R}^3))$ so that ${}^{*}D_{*S}\subset D(T_W)$ is a $*$- core for T_W and

1) as ${}^{*}\mathbb{C}$-valued quadratic forms on ${}^{*}D_{*S}\times{}^{*}D_{*S}$

$$T_W={}^{*}\!\!\int_{{}^{*}\mathbb{R}^{3(n_1+n_2)}}W\left(k_1,\dots k_{n_1},p_1,\dots,p_{n_2}\right)\left(\prod_{i=1}^{n_1}{}^{*}a^{\dagger}(k_i)\right)\left(\prod_{i=1}^{n_2}{}^{*}a(p_i)\right)d^{n_1}k\,d^{n_2}p$$

2) As ${}^{*}\mathbb{C}$-valued quadratic forms on $D_{*S}\times D_{*S}$

$$T_W^{*}={}^{*}\!\!\int_{{}^{*}\mathbb{R}^{3(n_1+n_2)}}W\left(k_1,\dots k_{n_1},p_1,\dots,p_{n_2}\right)\left(\prod_{i=1}^{n_1}{}^{*}a^{\dagger}(k_i)\right)\left(\prod_{i=1}^{n_2}{}^{*}a(p_i)\right)d^{n_1}k\,d^{n_2}p$$

3) On vectors in ${}^{*}F_0$ the operators T_W and T_W^{*} are given by the explicit formulas

$$\left(T_W({}^{*}\psi)\right)^{(l-n_2+n_1)}=$$

$$K(l,n_1,n_2){}^{*}\mathbf{S}\left[{}^{*}\!\!\int_{|p_1|\leq\varpi}\dots{}^{*}\!\!\int_{|p_{n_2}|\leq\varpi}W\left(k_1,\dots k_{n_1},p_1,\dots,p_{n_2}\right){}^{*}\psi^{(l)}\left(p_1,\dots,p_{n_2},k_1,\dots k_{n_1}\right)d^{3n_2}p\right],\qquad(39)$$

$\left(T_W^{*}({}^{*}\psi)\right)^{n}=0$ if $n<n_1-n_2$,

$$\left(T_W^*({}^*\psi)\right)^{(l-n_1+n_2)} =$$

$$K(l,n_2,n_1)\,{}^*\mathbf{S}\left[{}^*\!\!\int_{|p_1|\le\varpi}\cdots{}^*\!\!\int_{|p_{n_2}|\le\varpi} W(k_1,\dots k_{n_1},p_1,\dots,p_{n_2})\,{}^*\psi^{(l)}(p_1,\dots,p_{n_2},k_1,\dots k_{n_1})d^{3n_1}k\right] \quad (40)$$

$\left(T_W^*({}^*\psi)\right)^n = 0$, if $n < n_2 - n_1$.

Here $\mathbf{S}$ is the symmetrization operator defined in [9] and $K(l,n_2,n_1) = \left[\frac{l!(l+n_1-n_2)!}{(l-n_2)^2}\right]^{1/2}$, $n_1, n_2 \in \mathbb{N}, l \in {}^*\mathbb{N}$.

Proof: For vectors ${}^*\psi \in D_{{}^*S}$ we define $T_W({}^*\psi)$ by the formula (39). By the Schwarz inequality and the fact that ${}^*\mathbf{S}$ is a projection we get

$$\left({}^*\left\|\left(T_W({}^*\psi)\right)^{(l-n_2+n_1)}\right\|\right)^2 \le K(l,n_1,n_2)\,{}^*\left\|\left({}^*\psi^{(l)}\right)\right\|^2 {}^*\|W\|^2. \quad (41)$$

Let us now define the operator $T_W^*({}^*\psi)$ on $D_{{}^*S}$ by the formula (39), then for all ${}^*\varphi, {}^*\psi \in D_{{}^*S}$, then one obtains directly ${}^*({}^*\varphi, T_W{}^*\psi) = {}^*(T_W^*{}^*\varphi, {}^*\psi)$. Thus, T_W is $*$-closable and T_W^* is the restriction of the $*$-adjoint of T_W on $D_{{}^*S}$. We will use T_W to denote $*$-$\bar{T}_W$ and T_W^* to denote the $*$-adjoint of T_W. By the definition of T_W, $D_{{}^*S}$ is a $*$-core and further, since T_W is bounded on the l-particle vectors in $D_{{}^*S}$ we get ${}^*F_0 \subset D(T_W)$. Since the right-hand side of (39) is also bounded on the l-particle vectors, equation (38) represents T_W on all l-particle vectors. The proof of the statement (2) about T_W^* is the same.

Definition 26.[8]. Define standard Q -space by $Q = \times_{k=1}^{\infty}\mathbb{R}$. Let σ be the σ-algebra generated by infinite products of measurable sets in $\mathbb{R}$ and set $\mu = \otimes_{k=1}^{\infty}\mu_k$ with $d\mu_k = \pi^{-1/2}\exp(-x_k^2/2)$. Denote the points of Q by $q = \langle q_1, q_2, \dots \rangle$. Then $\langle Q, \mu\rangle$ is a measure space and the set of the all functions of the form $P_n(q) = P(q_1, q_2, \dots, q_n)$, where $P_n(q)$ is a polynomial and $n \in \mathbb{N}$ is arbitrary, is dense in $L_2(Q, d\mu)$. Remind that there exists a unitary map $S\colon \mathcal{F}_s(H) \to L_2(Q, d\mu)$ of Fock space $\mathcal{F}_s(H)$ onto $L_2(Q, d\mu)$ so that $\mathrm{S}\varphi(f_k)\mathrm{S}^{-1} = q_k$ and $\mathrm{S}\Omega_0 = 1$. Here $\{f_k\}_{k=1}^{\infty}$ is an orthonormal basis for H. Then by transfer one obtains internal measure space ${}^*\langle Q, \mu\rangle = \langle {}^*Q, {}^*\mu\rangle$ and internal unitary map ${}^*\mathrm{S}\colon \mathcal{F}_s(H) \to {}^*L_2({}^*Q, d{}^*\mu)$ so that ${}^*\mathrm{S}\varphi(f_r){}^*\mathrm{S}^{-1} = q_r, r \in {}^*\mathbb{N}$ and ${}^*\mathrm{S}\Omega_0 = 1$. Here $\{f_r\}_{r=1}^{{}^*\infty}$ is an orthonormal basis for ${}^*H$.

Theorem 7. Let ${}^*\varphi_\varkappa(x,t)$ be internal free scalar boson field of mass m at time $t = 0$ with hyperfinite momentum cutoff $\varkappa$ in four-dimensional space-time. Let $g(x)$ be a real-valued internal function in ${}^*L_2({}^*\mathbb{R}^3) \cap {}^*L_1({}^*\mathbb{R}^3)$. Then the operator

$${}^*H_{I,\varkappa}(g) = \lambda(\varkappa)\,{}^*\!\!\int_{{}^*\mathbb{R}^3} g(x) : {}^*\varphi_\varkappa^4(x) : d^3x \quad (42)$$

is a well-defined internal symmetric operator on ${}^*D_{{}^*S_{\mathrm{fin}}}$. Here $:{}^*\varphi_\varkappa^4(x) \coloneqq {}^*\varphi_\varkappa^4(x) + d_2(\varkappa)\left({}^*\varphi_\varkappa^2(x)\right) + d_1(\varkappa)$. where the coefficients $d_2(\varkappa)$ and $d_1(\varkappa)$ are independent of x. Let S denote the unitary map of $\mathcal{F}_s(H)$ onto $L_2(Q, d\mu)$ considered in [8]. Then $V = {}^*\mathrm{S}{}^*H_{I,\varkappa}(g){}^*\mathrm{S}^{-1}$ is multiplication by internal function $V_{I,\varkappa}(q)$ which satisfies:
(a) $V_{I,\varkappa}(q) \in {}^*L_p({}^*Q, d{}^*\mu)$ for all $p \in {}^*\mathbb{N}$, (b) $\exp\left(-tV_{I,\varkappa}(q)\right) \in {}^*L_1({}^*Q, d{}^*\mu)$ for all $t \in [0, {}^*\infty)$.

Proof: Note that for each $x \in {}^*\mathbb{R}^3$, the operator ${}^*\mathrm{S}({}^*\varphi_\varkappa(x)){}^*\mathrm{S}^{-1}$ is just the operator on internal measurable space ${}^*L_2({}^*\mathrm{Q}, d{}^*\mu)$ on which this operator acts by multiplying by the function $\sum_{k=1}^{{}^*\infty} c_k(x,\varkappa)q_k$, where $c_k(x,\varkappa) = (2\pi)^{3/2}\left(f_k, (\mu(p))^{1/2}\exp(ipx)\right)$. Furthermore, $\sum_{k=1}^{{}^*\infty}|c_k(x,\varkappa)|^2 = (2\pi)^{3/2}\,{}^*\left\|\mu(p)^{1/2}\right\|_2^2$ so ${}^*\mathrm{S}\left({}^*\varphi_\varkappa^4(x)\right){}^*\mathrm{S}^{-1}$and ${}^*\mathrm{S}\left({}^*\varphi_\varkappa^2(x)\right){}^*\mathrm{S}^{-1}$ are in ${}^*L_2({}^*\mathrm{Q}, d{}^*\mu)$ and the corresponding ${}^*L_2({}^*\mathrm{Q}, d{}^*\mu)$-norms are uniformly bounded in x. Therefore, since $g \in {}^*L_1({}^*\mathbb{R}^3)$ the operator ${}^*\mathrm{S}\left({}^*H_{I,\varkappa}(g)\right){}^*\mathrm{S}^{-1}$ is just the operator on internal measurable space

${}^*L_2({}^*\Omega, d{}^*\mu)$ on which this operator acts by multiplying by the ${}^*L_2({}^*\mathrm{Q}, d{}^*\mu)$-function which we denote by $V_{\varkappa,\lambda}(q)$. Let us consider now the expression for ${}^*H_{I,\varkappa}(g){}^*\Omega$, obviously this is a vector $(0,0,0,0,\psi^4,0,\dots)$ with

$$\psi^4(p_1,p_2,p_3,p_4) = {}^*\!\!\int_{{}^*\mathbb{R}^3} \frac{\lambda(\varkappa)g(x)\prod_{i=1}^4[\chi(\varkappa,p_i)]\exp\left(-ix\sum_{i=1}^{i=4}p_i\right)d^3x}{(2\pi)^{3/2}\prod_{i=1}^4[2\mu(p_i)]^{1/2}}. \quad (43)$$

Here $\chi(\varkappa,p)\equiv 1$ if $|p|\le\varkappa$, $\chi(\varkappa,p)\equiv 0$ if $|p|>\varkappa$, $\varkappa\in{}^*\mathbb{R}_\infty$. We choose now the parameter $\lambda=\lambda(\varkappa)\approx 0$ such that ${}^*\|\psi^4\|_2^2\in\mathbb{R}$ and therefore we obtain ${}^*\left\|{}^*H_{I,\varkappa,\lambda(\varkappa)}(g)\Omega_0\right\|_2^2\in\mathbb{R}$, since ${}^*\left\|{}^*H_{I,\varkappa,\lambda(\varkappa)}(g)\Omega_0\right\|_2^2={}^*\|\psi^4\|_2^2$. But, since ${}^*\mathrm{S}{}^*\Omega_0=1$, we get the equalities

$${}^*\left\|{}^*H_{I,\varkappa,\lambda(\varkappa)}(g)\Omega_0\right\|_2 = \left\|{}^*\mathrm{S}\,H_{I,\varkappa,\lambda(\varkappa)}(g){}^*\mathrm{S}^{-1}\right\|_{{}^*L_2({}^*Q,d{}^*\mu)} = {}^*\left\|V_{I,\varkappa,\lambda(\varkappa)}(q)\right\|_{{}^*L_2({}^*Q,d{}^*\mu)}. \quad (44)$$

From (43) we get that ${}^*\left\|V_{I,\varkappa,\lambda(\varkappa)}(q)\right\|_{{}^*L_2({}^*Q,d{}^*\mu)}\in\mathbb{R}$ and it is easily verify, that each polynomial $P(q_1,q_2,\dots,q_n)$, is $n\in{}^*\mathbb{N}$ in the domain of the operator $V_{I,\varkappa,\lambda(\varkappa)}(q)$ and ${}^*\mathrm{S}\,{}^*H_{I,\varkappa,\lambda(\varkappa)}(g){}^*\mathrm{S}^{-1}\equiv V_{I,\varkappa,\lambda(\varkappa)}(q)$ on that domain. Since ${}^*\Omega_0$ is in the domain of ${}^*H^p{}_{I,\varkappa,\lambda(\varkappa)}(g), p\in{}^*\mathbb{N}$, 1 is in the domain of the operator $V^p{}_{I,\varkappa,\lambda(\varkappa)}(q)$ for all $p\in{}^*\mathbb{N}$. Thus, for all $p\in{}^*\mathbb{N}$ $V_{I,\varkappa,\lambda(\varkappa)}(q)\in{}^*L_{2p}({}^*Q,d{}^*\mu)$, since ${}^*\mu\,({}^*Q\,)$ is finite, we conclude that $V_{I,\varkappa,\lambda(\varkappa)}(q)\in{}^*L_p({}^*Q,d{}^*\mu)$ for all $p\in{}^*\mathbb{N}$.

(b) Remind Wick's theorem asserts that $:{}^*\varphi^j_{m,\varkappa}(x):=\sum_{i=0}^{[j/2]}(-1)^i\frac{j!}{(j-2i)!i!}c^i_\varkappa{}^*\varphi^{(j-2i)}_{m,\varkappa}(x)$ with $c_\varkappa={}^*\left\|{}^*\varphi_{m,\varkappa}(x){}^*\Omega_0\right\|_2^2$. For $j=4$ we get $-O(c_\varkappa^2)\le:\ {}^*\varphi^4_{m,\varkappa}(x):$ and therefore $-\left({}^*\!\int_{{}^*\mathbb{R}^3}g(x)\,d^3x\right)O(c_\varkappa^2)\le{}^*H_{I,\varkappa,\lambda(\varkappa)}(g)$. Finally we obtain ${}^*\!\int_{{}^*Q}\exp\left(-t(:\ {}^*\varphi^4_{m,\varkappa}(x):)\right)d\,{}^*\mu\ \le\exp\left(O(c_\varkappa^2)\right)$ and this inequality finalized the proof.

Theorem 8.[8]. Let $\langle M,\mu\rangle$ be a σ-measure standard space with $\mu(M)=1$and let H_0 be the generator of a hypercontractive semigroup on $L_2(M,d\mu)$. Let V be a $\mathbb{R}$-valued measurable function on $\langle M,\mu\rangle$ such that $V\in L_p(M,d\mu)$ for all $p\in[\![1,\infty)$ and $\exp(-tV)\in L_1(M,d\mu)$ for all $t>0$. Then H_0+V is essentially self-adjoint on $C^\infty(H_0\,)\cap D(V)$ and is bounded below. Here $C^\infty(H_0\,)=\bigcap_{p\in\mathbb{N}}D\left(H_0^p\right)$.

Theorem 9. Let $\langle M,\mu\rangle$ be a σ-measure space with $\mu(M)=1$and let H_0 be the generator of a hypercontractive semigroup on $L_2(M,d\mu)$. Let$V$ be a ${}^*\mathbb{R}$-valued internal measurable function on $\langle{}^*M,{}^*\mu\rangle$ such that $V\in{}^*L_p({}^*M,d{}^*\mu)$ for all $p\in[1,{}^*\infty)$ and ${}^*\exp(-tV)\in{}^*L_1({}^*M,d{}^*\mu)$ for all $t>0$. Assume that a set $C^{*\infty}({}^*H_0\,)\cap D(V)$ is internal. Then operator ${}^*H_0+V$ is essentially self-$*$ -adjoint internal operator on $C^{*\infty}({}^*H_0\,)\cap D(V)$ and it is hyper finitely bounded below. Here $C^{*\infty}({}^*H_0\,)=\bigcap_{p\in{}^*\mathbb{N}}D\left({}^*H_0^p\right)$.

Proof. It follows immediately by transfer from theorem 8.

Remark 4: Let $V_{I,\varkappa,\lambda}$ be operator on internal measurable space ${}^*L_2({}^*\Omega,d{}^*\mu)$ on which this operator acts by multiplying by the ${}^*L_2({}^*\mathrm{Q},d{}^*\mu)$-function$V_{I,\varkappa,\lambda}$, see proof to Theorem 7. Note that for this operator a set $C^{*\infty}({}^*H_0\,)\cap D\left(V_{I,\varkappa,\lambda}\right)$ is not internal and therefore Theorem9 no longer holds. But without this theorem we cannot conclude that operator ${}^*H_0+V_{I,\varkappa,\lambda}$ is essentially self-$*$ -adjoint internal operator on $C^{*\infty}({}^*H_0\,)\cap D\left(V_{I,\varkappa,\lambda}\right)$. Thus Robinson's transfer is of no help in the case corresponding to operator $V_{I,\varkappa,\lambda}$ considered above. In order to resolve this issue, we will use non conservative extension of the model theoretical nonstandard analysis, see [10]-[14].

NON CONSERVATIVE EXTENSION OF THE MODEL THEORETICAL NONSTANDARD ANALYSIS

Remind that Robinson nonstandard analysis (RNA) many developed using set theoretical objects called super-structures [2]-[7]. A superstructure $\mathrm{V(S)}$ over a set S is defined in the following way: $\mathrm{V_0(S) = S}$, $\mathrm{V}_{n+1}(\mathrm{S}) = \mathrm{V}_n(\mathrm{S}) \cup \mathrm{P}(\mathrm{V}_n(\mathrm{S}))$, $\mathrm{V(S)} = \bigcup_{n\in\mathbb{N}} \mathrm{V}_{n+1}(\mathrm{S})$. Making $\mathrm{S} = \mathbb{R}$ will suffice for virtually any construction necessary in analysis. Bounded formulas are formulas where all quantifiers occur in the form: $\forall x\ (x \in y \to \cdots)$, $\exists x\ (x \in y \to \cdots)$. A nonstandard embedding is a mapping $*\colon \mathrm{V}(X) \to \mathrm{V}(Y)$ from a superstructure $\mathrm{V}(X)$ called the standard universe, into another superstructure $\mathrm{V}(Y)$ called nonstandard universe, satisfying the following postulates:

1. $Y = {}^*X$
2. *Transfer Principle*. For every bounded formula $\Phi(x_1, \dots, x_n)$ and elements $a_1, \dots, a_n \in V(X)$ the property $\Phi(a_1, \dots, a_n)$ is true for $a_1, \dots, a_n$ in the standard universe if and only if it is true for ${}^*a_1, \dots, {}^*a_n$ in the nonstandard universe: $V(X) \models \Phi(x_1, \dots, x_n) \leftrightarrow V(Y) \models \Phi({}^*a_1, \dots, {}^*a_n)$.
3. *Non-triviality.* For every infinite set A in the standard universe, the set $\{{}^*a | a \in A\}$ is a proper subset of ${}^*A$.

Definition 27**.** A set x is internal if and only if x is an element of ${}^*A$ for some $A \in \mathrm{V}(\mathbb{R})$. Let X be a set and $A = \{A_i\}_{i\in I}$ a family of subsets of X .Then the collection A has the infinite intersection property, if any infinite sub collection $J \subset I$ has non-empty intersection. Nonstandard universe is σ -saturated if whenever $\{A_i\}_{i\in I}$ is a collection of internal sets with the infinite intersection property and the cardinality of I is less than or equal to σ.

Remark 5. For each standard universe $U = V(X)$ there exists canonical language L_U and for each nonstandard universe $W = V(Y)$ there exists corresponding canonical nonstandard language ${}^*L = L_W$ [5],[7]

4.*The restricted rules of conclusion.* If Let A and B well formed, closed formulas so that $A, B \in {}^*L$. If $W \vDash A$, then $\neg A \nvdash_{RMP} B$. Thus, if a statement A holds in nonstandard universe, we cannot obtain from formula $\neg A$ any formula B whatsoever.

Definition 28.[10]-[14]. A set $S \subset {}^*\mathbb{N}$ is a hyper inductive if the following statement holds in $V(Y)$:

$$\bigwedge_{\alpha\in{}^*\mathbb{N}}(\alpha \in S \to \alpha^+ \in S).$$

Here $\alpha^+ = \alpha + 1$.Obviously a set ${}^*\mathbb{N}$ is a hyper inductive.

5. *Axiom of hyper infinite induction*

$$\forall S(S \subset {}^*\mathbb{N})\{\forall\beta(\beta \subset {}^*\mathbb{N})[\![\bigwedge_{1\leq\alpha<\beta}(\alpha \in S \to \alpha^+ \in S)]\!] \to S = {}^*\mathbb{N}\}.$$

Example 5: Remind the proof of the following statement: structure $(\mathbb{N}, <, =)$ is a well-ordered set.

Proof: Let X be a nonempty subset of $\mathbb{N}$. Suppose X does not have a <-least element. Then consider the set $\mathbb{N}\backslash X$.

Case1. $\mathbb{N}\backslash X = \emptyset$. Then $X = \mathbb{N}$ and so 0 is a $<$ -least element but this is a contradiction.

Case2. $\mathbb{N}\backslash X \neq \emptyset$. Then $1 \in \mathbb{N}\backslash X$ otherwise 1 is a $<$ -least element but this is a contradiction. Assume now that there exists some $n \in \mathbb{N}\backslash X$ such that $n \neq 1$, but since we have supposed that X does not have a $<$ -least element, thus $n + 1 \notin X$. Thus we see that for all n the statement $n \in \mathbb{N}\backslash X$ implies that $n + 1 \in \mathbb{N}\backslash X$. We can conclude by axiom of induction that $n \in \mathbb{N}\backslash X$ for all $n \in \mathbb{N}$. Thus $\mathbb{N}\backslash X = \mathbb{N}$ implies $X = \emptyset$. This is a contradiction to X being a non-empty subset of $\mathbb{N}$. Remind that structure $({}^*\mathbb{N}, <, =)$ is not a well-ordered set [5]-[7]. We set now $X_1 = {}^*\mathbb{N}\backslash\mathbb{N}$ and thus${}^*\mathbb{N}\backslash X_1 = \mathbb{N}$. In contrast with a set X mentioned above the assumption $n \in {}^*\mathbb{N}\backslash X_1$ implies that $n + 1 \in {}^*\mathbb{N}\backslash X_1$ if and only if n is finite, since for any infinite $n \in {}^*\mathbb{N}\backslash\mathbb{N}$ the assumption $n \in {}^*\mathbb{N}\backslash X_1$ contradicts with a true statement $V(Y) \vDash n \notin {}^*\mathbb{N}\backslash X_1$=$\mathbb{N}$ and therefore in accordance with postulate 4 we cannot obtain from $n \in {}^*\mathbb{N}\backslash X_1$ any closed formula B whatsoever.

Theorem 10.[14]. (Generalized Recursion Theorem) Let S be a set, $c \in S$ and $g\colon S \times {}^*\mathbb{N} \to S$ is any function with $\mathrm{dom}(g) = S \times {}^*\mathbb{N}$ and $\mathrm{range}(g) \subseteq S$, then there exists a function $\mathcal{F}\colon {}^*\mathbb{N} \to S$ such that: 1) $\mathrm{dom}(\mathcal{F}) = {}^*\mathbb{N}$ and $\mathrm{range}(\mathcal{F}) \subseteq S$; 2) $\mathcal{F}(1) = c$; 3) for all $x \in {}^*\mathbb{N}, \mathcal{F}(n + 1) = g(\mathcal{F}(n), n)$.

Definition 29.[12]-[14]. (1) Suppose that S is a standard set on which a binary operations $(\cdot + \cdot)$ and $(\cdot\times\cdot)$ is defined and under which S is closed. Let $\{x_k\}_{k\in{}^*\mathbb{N}}$ be any hyper infinite sequence of terms of ${}^*S$. For every hyper natural $n \in {}^*\mathbb{N}$ we denote by $Ext\text{-}\sum_{k=1}^{n} x_k$ the element of ${}^*S$ uniquely determined by the following canonical conditions:

(a) $Ext\text{-}\sum_{k=1}^{1} x_k = x_1$; (b) $Ext\text{-}\sum_{k=1}^{n+1} x_k = Ext\text{-}\sum_{k=1}^{n} x_k + x_{n+1}$ for all $n \in {}^*\mathbb{N}$.
(2) For every hyper natural $n \in {}^*\mathbb{N}_\infty$ we denote by $Ext\text{-}\prod_{k=1}^{n} x_k$ the element of ${}^*S$ uniquely determined by the following canonical conditions: (a) $Ext\text{-}\prod_{k=1}^{1} x_k = x_1$; (b) $Ext\text{-}\prod_{k=1}^{n+1} x_k = (Ext\text{-}\prod_{k=1}^{n} x_k) \times x_{n+1}$ for all $n \in {}^*\mathbb{N}$.
Theorem11. [14]. (1) suppose that S is a standard set on which a binary operation $(\cdot + \cdot)$ is defined and under which S is closed and that $(\cdot + \cdot)$ is associative on S. Let $\{x_k\}_{k\in {}^*\mathbb{N}}$ be any hyper infinite sequence of terms of ${}^*S$. Then for any $n, m \in {}^*\mathbb{N}$ we have: $Ext\text{-}\sum_{k=1}^{n+m} x_k = Ext\text{-}\sum_{k=1}^{n} x_k + Ext\text{-}\sum_{k=1}^{m} x_k$;
(2) suppose that S is a standard set on which a binary operation $(\cdot\times\cdot)$ is defined and under which S is closed and that $(\cdot\times\cdot)$ is associative on S. Let $\{x_k\}_{k\in {}^*\mathbb{N}}$ be any hyper infinite sequence of terms of ${}^*S$. Then for any $n, m \in {}^*\mathbb{N}$ we have: $Ext\text{-}\prod_{k=1}^{n+m} x_k = (Ext\text{-}\prod_{k=1}^{n} x_k) \times (Ext\text{-}\prod_{k=1}^{m} x_k)$; (3) for any $z \in {}^*S$ and for any $n \in {}^*\mathbb{N}_\infty$ we have: $z \times (Ext\text{-}\sum_{k=1}^{n} x_k) = Ext\text{-}\sum_{k=1}^{n} z \times x_k$.

External non-Archimedean Field ${}^*\mathbb{R}_c^\#$ by Cauchy Completion of the Internal Non -Archimedean Field ${}^*\mathbb{R}$.

Definition 30. A hyper infinite sequence of hyperreal numbers from ${}^*\mathbb{R}$ is a function $a\colon {}^*\mathbb{N} \to {}^*\mathbb{R}$ from the hyper natural numbers ${}^*\mathbb{N}$ into the hyperreal numbers ${}^*\mathbb{R}$.We usually denote such a function by $n \mapsto a_n$, so the terms in the sequence are written as $\{a_1, a_2, \ldots, a_n, \ldots\}$.To refer to the whole hyper infinite sequence, we will write $\{a_n\}_{n=1}^{{}^*\infty}$ or $\{a_n\}_{n\in {}^*\mathbb{N}}$.
Abbreviation 3. For a standard set E we often write $E_{\rm st}$, let ${}^\sigma E_{\rm st} = \{{}^*x | x \in E_{\rm st}\}$.We identify z with ${}^\sigma z$ i.e., $z \equiv {}^\sigma z$ for all $z \in \mathbb{C}$. Hence, ${}^\sigma E_{\rm st} = E_{\rm st}$ if $E \subseteq \mathbb{C}$, e.g., ${}^\sigma\mathbb{C} = \mathbb{C}$, ${}^\sigma\mathbb{R} = \mathbb{R}$, etc.Let ${}^*\mathbb{R}_c^\#$, ${}^*\mathbb{R}_{c,\approx}^\#$, ${}^*\mathbb{R}_{c,\approx+}^\#$, ${}^*\mathbb{R}_{c,\rm fin}^\#$, ${}^*\mathbb{R}_{c,\infty}^\#$, ${}^*\mathbb{N}_\infty$denote the sets of Cauchy hyper-real numbers, Cauchy infinitesimal hyper-real numbers, Cauchy positive infinitesimal hyper-real numbers, Cauchy finite hyper-real numbers, Cauchy infinite hyper-real numbers and infinite hyper natural numbers, respectively. Note that ${}^*\mathbb{R}_{c,\rm fin}^\# = {}^*\mathbb{R}_c^\# \backslash {}^*\mathbb{R}_{c,\infty}^\#$.
Definition 31. Let $\{a_n\}_{n=1}^{{}^*\infty}$ be a hyper infinite ${}^*\mathbb{R}$- valued sequence mentioned above. We shall say that $\{a_n\}_{n=1}^{{}^*\infty}$ #-tends to 0 if, given any $\varepsilon \in {}^*\mathbb{R}_{\approx+}$, there is a hyper natural number $N \in {}^*\mathbb{N}$ such that for all $n > N$, $|a_n| \le \varepsilon$. We denote this symbolically by $a_n \to_\# 0$.
Definition 32. Let $\{a_n\}_{n=1}^{{}^*\infty}$ be a hyper infinite ${}^*\mathbb{R}$-valued sequence mentioned above. We shall say that $\{a_n\}_{n=1}^{{}^*\infty}$ #-tends to $q \in {}^*\mathbb{R}$ if, given any $\varepsilon \in {}^*\mathbb{R}_{\approx+}$, there is a hyper natural number $N \in {}^*\mathbb{N}$ such that for all $n > N$, $|a_n - q| \le \varepsilon$ and we denote this symbolically by $a_n \to_\# q$ or by $\#\text{-}\lim_{n\to {}^*\infty} a_n = q$.
Definition 33. Let $\{a_n\}_{n=1}^{{}^*\infty}$ be a hyper infinite ${}^*\mathbb{R}$-valued sequence mentioned above. We shall say that sequence $\{a_n\}_{n=1}^{{}^*\infty}$ is bounded if there is a hyperreal $M \in {}^*\mathbb{R}$ such that for any $n \in {}^*\mathbb{N}$, $|a_n| \le M$.
Definition 34. Let $\{a_n\}_{n=1}^{{}^*\infty}$ be a hyper infinite ${}^*\mathbb{R}$- valued sequence mentioned above. We shall say that $\{a_n\}_{n=1}^{{}^*\infty}$ is a Cauchy hyper infinite ${}^*\mathbb{R}$- valued sequence if , given any $\varepsilon \in {}^*\mathbb{R}_{\approx+}$, there is a hyper natural number $N(\varepsilon) \in {}^*\mathbb{N}$ such that for any $m, n > N$, $|a_n - a_m| < \varepsilon$.
Theorem 12. If $\{a_n\}_{n=1}^{{}^*\infty}$ is a #-convergent hyper infinite ${}^*\mathbb{R}$-valued sequence, i.e., that is, $a_n \to_\# q$ for some hyperreal number $q, q \in {}^*\mathbb{R}$ then $\{a_n\}_{n=1}^{{}^*\infty}$ is a Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequence.
Theorem 13. If $\{a_n\}_{n=1}^{{}^*\infty}$ is a Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequence, then it is finitely bounded or hyper finitely bounded; that is, there is some finite or hyperfinite $M \in {}^*\mathbb{R}_+$ such that $|a_n| \le M$ for all $n \in {}^*\mathbb{N}$.
Definition 35. Let S be a set, with an equivalence relation $(\cdot \sim \cdot)$ on pairs of elements. For$s \in S$, denote by $cl[s]$ the set of all elements in S that are related to s. Then for any $s, t \in S$, either $cl[s] = cl[t]$ or $cl[s]$ and $cl[t]$ are disjoint.
Remark 6.The hyperreal numbers ${}^*\mathbb{R}_c^\#$ will be constructed as equivalence classes of Cauchy hyper infinite ${}^*\mathbb{R}$-valued

sequences. Let $\mathcal{F}\{{}^*\mathbb{R}\}$ denote the set of all Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequences of hyperreal numbers. We define the equivalence relation on a set $\mathcal{F}\{{}^*\mathbb{R}\}$.

Definition 36. Let $\{a_n\}_{n=1}^{{}^*\infty}$ and $\{b_n\}_{n=1}^{{}^*\infty}$ be in $\mathcal{F}\{{}^*\mathbb{R}\}$. Say they are #-equivalent if $a_n - b_n \to_{\#} 0$ i.e., if and only if the hyper infinite ${}^*\mathbb{R}$-valued sequence $\{a_n - b_n\}_{n=1}^{{}^*\infty}$ #-tends to 0.

Theorem 14. Definition above yields an equivalence relation on a set $\mathcal{F}\{{}^*\mathbb{R}\}$.

Definition 37. The external hyperreal numbers ${}^*\mathbb{R}_c^{\#}$ are the equivalence classes $cl[\{a_n\}]$ of Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequences of hyperreal numbers, as per definition above. That is, each such equivalence class is an external hyperreal number.

Definition 38. Given any hyperreal number $q \in {}^*\mathbb{R}$**,** define a hyperreal number $q^{\#}$to be the equivalence class of the hyper infinite ${}^*\mathbb{R}$-valued sequence $\{a_n = q\}_{n=1}^{{}^*\infty}$consisting entirely of $q \in {}^*\mathbb{R}$. So we view ${}^*\mathbb{R}$ as being inside ${}^*\mathbb{R}_{\mathbf{c}}^{\#}$ by thinking of each hyperreal number $q \in {}^*\mathbb{R}$ as its associated equivalence class $q^{\#}$. It is standard to abuse this notation, and simply refer to the equivalence class as q as well.

Definition 39. Let $s, t \in {}^*\mathbb{R}_c^{\#}$, so there are Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequences $\{a_n\}_{n=1}^{{}^*\infty}, \{b_n\}_{n=1}^{{}^*\infty}$ of hyperreal numbers with $s = cl[\{a_n\}]$ and $t = cl[\{b_n\}]$.

(a) Define $s + t$ to be the equivalence class of the hyper infinite sequence $\{a_n + b_n\}_{n=1}^{{}^*\infty}$.

(b) Define $s \times t$ to be the equivalence class of the hyper infinite sequence $\{a_n + b_n\}_{n=1}^{{}^*\infty}$.

Theorem 15. The operations $+, \times$ in definition above by the requirements (a) and (b) are well-defined.

Theorem 16. Given any hyperreal number $s \in {}^*\mathbb{R}_c^{\#}, s \neq 0$ there is a hyperreal number $t \in {}^*\mathbb{R}_c^{\#}$ such that $s \times t = 1$.

Theorem 17. If $\{a_n\}_{n=1}^{{}^*\infty}$ is a Cauchy hyper infinite sequence which does not #-tend to 0, then there is some $N \in {}^*\mathbb{N}$ such that, for all $n > N, a_n \neq 0$.

Definition 40. Let $s \in {}^*\mathbb{R}_c^{\#}$. Say that s is positive if $s \neq 0$, and if $s = cl[\{a_n\}]$ for some Cauchy hyper infinite sequence of hyperreal numbers such that for some $N \in {}^*\mathbb{N}, a_n > 0$ for all $n > N$. Then for a given two hyperreal numbers s, t, say that $s > t$ if $s - t$ is positive.

Theorem 18. Let $s, t \in {}^*\mathbb{R}_c^{\#}$ be hyperreal numbers such that $s > t$, and let $r \in {}^*\mathbb{R}_c^{\#}$, then $s + r > t + r$.

Theorem 19. Let $s, t \in {}^*\mathbb{R}_c^{\#}$ be hyperreal numbers such that $s, t > 0$. Then there is $m \in {}^*\mathbb{N}$ such that $m \times s > t$.

Theorem 20. Given any hyperreal number $r \in {}^*\mathbb{R}_c^{\#}$, and any hyperreal number $\varepsilon > 0, \varepsilon \approx 0$, there is a hyperreal number $q \in {}^*\mathbb{R}_c^{\#}$ such that $|r - q| < \varepsilon$.

Definition 40. Let $S \subsetneq {}^*\mathbb{R}_c^{\#}$ be a non-empty set of hyperreal numbers. A hyperreal number $x \in {}^*\mathbb{R}_c^{\#}$ is called an upper bound for S if $x \geq s$ for all $s \in S$. A hyperreal number x is the least upper bound (or supremum: sup$S$) for S if x is an upper bound for S and $x \leq y$ for every upper bound y of S.

Remark 7. The order $\leq$ given by Definition above obviously is $\leq$-incomplete.

Definition 41. Let $S \subsetneq {}^*\mathbb{R}_c^{\#}$ be a non-empty set of hyperreal numbers. We will say that:

(1) S is $\leq$ -admissible above if the following conditions are satisfied:

(a) S is finitely bounded or hyper finitely bounded above;

(b) let $A(S)$ be a set such that $\forall x[x \in A(S) \Leftrightarrow x \geq S]$ then for any $\varepsilon > 0, \varepsilon \approx 0$ there are $\alpha \in S$ and $\beta \in A(S)$ such that $\beta - \alpha \leq \varepsilon \approx 0$. (2) S is $\leq$ -admissible belov if the following conditions are satisfied:

(a) S is finitely bounded or hyper finitely bounded below;

(b) let $L(S)$ be a set such that $\forall x[x \in L(S) \Leftrightarrow x \leq S]$ then for any $\varepsilon > 0, \varepsilon \approx 0$ there are $\alpha \in S$ and $\beta \in L(S)$ such that $\alpha - \beta \leq \varepsilon \approx 0$.

Theorem 21.[14].(a) Any $\leq$-admissible above subset $S \subset {}^*\mathbb{R}_c^{\#}$ has the least upper bound property. (b) Any $\leq$-admissible above subset $S \subset {}^*\mathbb{R}_c^{\#}$ has the greatest lower bound property.

Theorem 22.[14]. (Generalized Nested Intervals Theorem) Let $\{I_n\}_{n=1}^{{}^*\infty} = \{[a_n, b_n]\}_{n=1}^{{}^*\infty}, [a_n, b_n] \subset {}^*\mathbb{R}_c^{\#}$ be a hyper infinite sequence of #-closed intervals satisfying each of the following conditions: (a) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq I_n \supseteq \cdots$ (b) $b_n - a_n \to_{\#} 0$ as $n \to {}^*\infty$, Then $\bigcap_{n=1}^{{}^*\infty} I_n$consists of exactly one hyperreal number $x \in {}^*\mathbb{R}_c^{\#}$.

Theorem 23.[14]. (Generalized Squeeze Theorem) Let $\{a_n\}_{n=1}^{{}^*\infty}, \{c_n\}_{n=1}^{{}^*\infty}$ be two hyper infinite sequences

#-converging to L, and $\{b_n\}_{n=1}^{*\infty}$ a hyper infinite sequence. If $\forall n > K, K \in {}^*\mathbb{N}$ we have $a_n \leq b_n \leq c_n$, then b_n also #-converges to L.
Theorem 24.[14]. If #-$\lim_{n\to{}^*\infty,}|a_n| = 0$, then #-$\lim_{n\to{}^*\infty}, a_n = 0$.
Theorem 25.[14]. (Generalized Bolzano -Weierstrass Theorem) Any finitely or hyper finitely bounded hyper infinite ${}^*\mathbb{R}_c^\#$ -valued sequence has #-convergent hyper infinite subsequence.
Definition 42. Let $\{a_n\}_{n=1}^{*\infty}$ be ${}^*\mathbb{R}_c^\#$-valued sequence. Say that a sequence $\{a_n\}_{n=1}^{*\infty}$ #-tends to 0 if, given any $\varepsilon > 0, \varepsilon \approx 0$, there is a hyper natural number $N \in {}^*\mathbb{N}_\infty, N = N(\varepsilon)$ such that, for all $n > N$, $|a_n| \leq \varepsilon$.
Definition43. Let $\{a_n\}_{n=1}^{*\infty}$ be ${}^*\mathbb{R}_c^\#$-valued hyper infinite sequence. We call $\{a_n\}_{n=1}^{*\infty}$ a Cauchy hyper infinite sequence if given any hyperreal number $\varepsilon \in {}^*\mathbb{R}_{c,\approx+}^\#$, there is a hypernatural number $N = N(\varepsilon)$ such that for any $m, n > N$, $|a_n - a_m| < \varepsilon$.
Theorem 26. If $\{a_n\}_{n=1}^{*\infty}$is a #-convergent hyper infinite sequence i.e., $a_n \to_\# b$ for some hyperreal number $b \in {}^*\mathbb{R}_c^\#$, then $\{a_n\}_{n=1}^{*\infty}$ is a Cauchy hyper infinite sequence.
Theorem 27.If $\{a_n\}_{n=1}^{*\infty}$ is a Cauchy hyper infinite sequence, then it is bounded; that is, there is some $M \in {}^*\mathbb{R}_c^\#$ such that $|a_n| \leq M$ for all $n \in {}^*\mathbb{N}$.
Theorem 28.[14]. Any Cauchy hyper infinite sequence $\{a_n\}_{n=1}^{*\infty}$ has a #-limit in ${}^*\mathbb{R}_c^\#$; that is, there exists $b \in {}^*\mathbb{R}_c^\#$ such that $a_n \to_\# b$.
Remark 8. Note, that there exists canonical natural embedding ${}^*\mathbb{R} \hookrightarrow {}^*\mathbb{R}_c^\#$.
Remark 9.A nonempty set S of Cauchy hyperreal numbers ${}^*\mathbb{R}_c^\#$ is unbounded above if it has no hyperfinite upper bound, or unbounded below if it has no hyperfinite lower bound. It is convenient to adjoin to Cauchy hyperreal number system ${}^*\mathbb{R}_c^\#$ two points, $+\infty^\# = ({}^*{+\infty})^\#$ (which we also write more simply as $\infty^\#$) and $-\infty^\#$, and to define the order relationships between them and any Cauchy hyperreal number $x \in {}^*\mathbb{R}_c^\#$ by $-\infty^\# < x < \infty^\#$.
Definition 44. We will call $-\infty^\#$ and $\infty^\#$ are points at hyper infinity. If $S \subset {}^*\mathbb{R}_c^\#$ is a nonempty set of Cauchy hyper-reals, we write $\sup(S) = \infty^\#$ to indicate that S is unbounded above, and $\inf(S) = -\infty^\#$ to indicate that S is un-bounded below.
Definition 45.The (ε, δ) definition of the #-limit of a function $f: D \to {}^*\mathbb{R}_c^\#$ is as follows: let $f(x)$ is a ${}^*\mathbb{R}_c^\#$- valued function defined on a subset $D \subset {}^*\mathbb{R}_c^\#$ of the Cauchy hyperreal numbers. Let c be a #-limit point of D and let $L \in {}^*\mathbb{R}_c^\#$ be Cauchy hyperreal number. We say that #-$\lim_{x\to_\# c} f(x) = L$ if for every $\varepsilon \approx 0, \varepsilon > 0$ there exists a $\delta \approx 0, \delta > 0$ such that, for all $x \in D$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.
Definition 46.[13].The function $f: {}^*\mathbb{R}_c^\# \to {}^*\mathbb{R}_c^\#$ is a #-continuous (or micro continuous) at some point c of its domain if the #-limit of $f(x)$, as x #-approaches c through the domain of f, exists and is equal to $f(c)$: #-$\lim_{x\to_\# c} f(x) = f(c)$.
Theorem 29. [14]. Let $\{a_n\}_{n=1}^{*\infty}$ and $\{b_n\}_{n=1}^{*\infty}$ be ${}^*\mathbb{R}_c^\#$- valued hyper infinite sequences. Then the following equalities hold for any $n, k, l, j, m \in {}^*\mathbb{N}$:

$$b \times (Ext\text{-}\sum_{i=1}^{n} a_i) = Ext\text{-}\sum_{i=1}^{n} b \times a_i \tag{45}$$

$$Ext\text{-}\sum_{i=1}^{n} a_i \pm Ext\text{-}\sum_{i=1}^{n} b_i = Ext\text{-}\sum_{i=1}^{n}(a_i \pm b_i) \tag{46}$$

$$Ext\text{-}\sum_{i=k_0}^{k_1}\left(Ext\text{-}\sum_{j=l_0}^{l_1} a_{ij}\right) = Ext\text{-}\sum_{j=l_0}^{l_1}\left(Ext\text{-}\sum_{i=k_0}^{k_1} a_{ij}\right) \tag{47}$$

$$(Ext\text{-}\sum_{i=1}^{n} a_i) \times \left(Ext\text{-}\sum_{j=1}^{n} b_j\right) = Ext\text{-}\sum_{i=1}^{n}\left(Ext\text{-}\sum_{j=1}^{n} a_i \times b_j\right) \tag{48}$$

$$(Ext\text{-}\prod_{i=1}^{n} a_i) \times (Ext\text{-}\prod_{i=1}^{n} b_i) = Ext\text{-}\prod_{i=1}^{n} a_i \times b_i \tag{49}$$

$$(Ext\text{-}\prod_{i=1}^{n} a_i)^m = Ext\text{-}\prod_{i=1}^{n} a_i^m. \tag{50}$$

Theorem 30. [14]. Let $\{a_n\}_{i=1}^{n}$ and $\{b_n\}_{i=1}^{n}$ be ${}^*\mathbb{R}_c^{\#}$- valued hyperfinite sequences. Suppose that $a_i \le b_i, 1 \le i \le n$. Then the following equalities hold for any $n \in {}^*\mathbb{N}$:

$$Ext\text{-}\prod_{i=1}^{n} a_i \le Ext\text{-}\prod_{i=1}^{n} b_i. \quad (51)$$

Theorem 31. [14]. Let $\{a_n\}_{i=1}^{n}$ and $\{b_n\}_{i=1}^{n}$ be ${}^*\mathbb{R}_c^{\#}$- valued hyperfinite sequences. Then the following inequalities hold for any $n \in {}^*\mathbb{N}$:

$$(Ext\text{-}\prod_{i=1}^{n} a_i \times b_i)^2 \le (Ext\text{-}\prod_{i=1}^{n} a_i^2) \times (Ext\text{-}\prod_{i=1}^{n} b_i^2). \quad (52)$$

Definition 47.[13]. If $\{a_n\}_{n=1}^{*\infty}$ is a ${}^*\mathbb{R}_c^{\#}$- valued hyper infinite sequence, the symbol $Ext\text{-}\sum_{n=1}^{*\infty} a_n$ is a hyper infinite series, and a_n is the n-th term of the hyper infinite series.
Definition 48.[13]. We shall say that $Ext\text{-}\sum_{n=1}^{*\infty} a_n$ #-converges to the sum $A \in {}^*\mathbb{R}_c^{\#}$, and write $Ext\text{-}\sum_{n=1}^{*\infty} a_n = A$ if the hyper infinite sequence $\{A_n\}_{n=1}^{*\infty}$ defined by $A_m = Ext\text{-}\sum_{n=1}^{m} a_n$ #-converges to the sum A. The hyperfinite sum A_m is the n-th partial sum of $Ext\text{-}\sum_{n=1}^{*\infty} a_n$. If $\#\text{-}\lim_{m\to{}^*\infty,} A_m = \infty^{\#}$ or $-\infty^{\#}$, we say that $Ext\text{-}\sum_{n=1}^{*\infty} a_n$ #-diverges to $\infty^{\#}$ or to $-\infty^{\#}$.
Theorem 32.[13]. The sum $Ext\text{-}\sum_{n=1}^{*\infty} a_n$ of a #-convergent hyper infinite series is unique.

Hyper infinite sequences and series of ${}^*\mathbb{R}_c^{\#}$- valued functions

Definition 49.[13]. If $f_1, f_2, \dots, f_k, f_{k+1}, \dots, f_n, \dots n \in {}^*\mathbb{N}$ are ${}^*\mathbb{R}_c^{\#}$- valued functions on a subset $D \subset {}^*\mathbb{R}_c^{\#}$ we say that $\{f_n\}_{n=1}^{*\infty}$ is a hyper infinite sequence of ${}^*\mathbb{R}_c^{\#}$- valued functions on D.
Definition 50.[13]. Suppose that $\{f_n\}_{n=1}^{*\infty}$ is a hyper infinite sequence of ${}^*\mathbb{R}_c^{\#}$- valued functions on $D \subset {}^*\mathbb{R}_c^{\#}$ and the hyper infinite sequence of values $\{f_n(x)\}_{n=1}^{*\infty}$ #-converges for each x in some subset S of D. Then we say that $\{f_n(x)\}_{n=1}^{*\infty}$ #-converges pointwise on S to the #-limit function f, defined by $f(x) = \lim_{n\to{}^*\infty} f_n(x)$.
Definition 51.[13]. If $\{f_n(x)\}_{n=1}^{*\infty}$ is a hyper infinite sequence of ${}^*\mathbb{R}_c^{\#}$- valued functions on $D \subset {}^*\mathbb{R}_c^{\#}$, then

$$Ext\text{-}\sum_{n=1}^{*\infty} f_n(x) \quad (53)$$

is a hyper infinite series of functions on D. The partial sums of (1), are defined by $F_n(x) = Ext\text{-}\sum_{k=1}^{n} f_n(x)$. If hyper infinite sequence $\{F_n(x)\}_{n=1}^{*\infty}$#-converges pointwise to the #-limit function $F(x)$ on a subset $S \subset D$, we say that $\{F_n(x)\}_{n=1}^{*\infty}$#-converges pointwise to the sum $F(x)$ on S, and write $F(x) = Ext\text{-}\sum_{n=1}^{*\infty} f_n(x)$.
Definition 52.[13]. A hyper infinite series of the form $Ext\text{-}\sum_{n=1}^{*\infty}(x - x_0)^n, n \in {}^*\mathbb{N}$ is called a hyper infinite power series in $x - x_0$.

The #-Derivatives and Riemann #-Integral of ${}^*\mathbb{R}_c^{\#}$-Valued Functions $f: D \to {}^*\mathbb{R}_c^{\#n}$

Definition 53.[13] A function $f: D \to {}^*\mathbb{R}_c^{\#}$ #-differentiable at an #-interior point $x \in D$ of its domain $D \subset {}^*\mathbb{R}_c^{\#}$ if the difference quotient $f(x) - f(x_0)/x - x_0$ has a #-limit: $\#\text{-}\lim_{x\to_{\#} x_0}(f(x) - f(x_0)/x - x_0)$. In this case the #-limit is called the #-derivative of f at interior point x_0, and is denoted by $f^{\#\prime}(x_0)$ or by $d^{\#}f\,(x_0)/d^{\#}x$.
Definition 54. If f is defined on an #-open set $S \subset {}^*\mathbb{R}_c^{\#}$, we say that f is #-differentiable on S if f is #-differentiable at every point of S. If f is #-differentiable on S, then $f^{\#\prime}(x)$ is a function on S.We say that f is #-continuously #-differentiable on S if $f^{\#\prime}(x)$ is #-continuous on S.

Definition 55.If f is #-differentiable on a #-neighbourhood of x_0, it is reasonable to ask if $f^{\#\prime}(x)$ is #-differentiable at x_0. If so, we denote the #-derivative of $f^{\#\prime}(x)$ at x_0 by $f^{\#\prime\prime}(x_0)$ or by $f^{\#(2)}(x_0)$ and this is the second #-derivative of f at x_0. Continuing inductively by hyper infinite induction, if $f^{\#(n-1)}(x)$ is defined on a #-neighbourhood of x_0, then the n-th #-derivative of f at x_0 denoted by $f^{\#(n)}(x_0)$ or by $d^{\#(n)}f\ (x_0)/d^{\#}x^n$, where $n \in {}^*\mathbb{N}$.

Theorem 33.[13]. If f is #-differentiable at x_0 then f is #-continuous at x_0.

Theorem 34.[13]. If f and g are #-differentiable at x_0, then so are $f \pm g$ and $f \times g$ with:

(a) $(f \pm g)^{\#\prime}(x_0) = f^{\#\prime}(x_0) \pm g^{\#\prime}(x_0)$, (b) $(f \times g)^{\#\prime}(x_0) = f^{\#\prime}(x_0)g(x_0) + g^{\#\prime}(x_0)f(x_0)$.

(c) The quotient f/g is #-differentiable at x_0 if $g(x_0) \neq 0$ with $(f/g)^{\#\prime} = \frac{f^{\#\prime}(x_0)g(x_0) - g^{\#\prime}(x_0)f(x_0)}{g(x_0)^2}$.

(d) If $n \in {}^*\mathbb{N}$ and $f_i, 1 \leq i \leq n$ are #-differentiable at x_0, then so are $Ext\text{-}\sum_{i=1}^{n} f_i$ with:

$$(Ext\text{-}\textstyle\sum_{i=1}^{n} f_i)^{\#\prime}(x_0) = Ext\text{-}\sum_{i=1}^{n} f_i^{\#\prime}(x_0).$$

(e) If $n \in {}^*\mathbb{N}$ and $f^{\#(n)}(x_0), g^{\#(n)}(x_0)$ exist, then so does $(f \times g)^{\#(n)}(x_0)$ and

$$(f \times g)^{\#(n)}(x_0) = Ext\text{-}\textstyle\sum_{i=0}^{n} \binom{n}{i} f^{\#(i)}(x_0) g^{\#(n-i)}(x_0)$$

Theorem 35.[13]. (The Chain Rule). Suppose that g is #-differentiable at x_0 and f is #-differentiable at $g(x_0)$. Then the composite function $h = f \circ g$ defined by $h(x) = f(g(x))$ is #-differentiable at x_0 with $h^{\#\prime}(x_0) = f^{\#\prime}(g(x_0))g^{\#\prime}(x_0)$.

Theorem 36.[13]. (Generalized Taylor's Theorem) Suppose that $f^{\#(n)}(x), n \in {}^*\mathbb{N}$ exists on an #-open interval I about x_0, and let $x \in I$. Let $P_n(x, x_0)$ be the n-th Taylor hyper polynomial of f about x_0, $P_n(x, x_0) = Ext\text{-}\sum_{r=0}^{n} \frac{f^{\#(r)}(x_0)(x-x_0)^r}{r!}$ Then the remainder $R(x, x_0) = f(x) - P_n(x, x_0)$ can be written as

$$R(x, x_0) = \frac{f^{\#(n+1)}(c)(x-x_0)^n}{(n+1)!}. \tag{54}$$

Here c depends upon x and is between x and x_0.

Definition 56.[13] Let $[a, b] \subset {}^*\mathbb{R}_c^{\#}$. A hyperfinite partition of $[a, b]$ is a hyperfinite set of subintervals $[x_0, x_1], \ldots, [x_{n-1}, x_n]$, with $n \in {}^*\mathbb{N}_\infty$, where $a = x_0 < x_1 \ldots < x_n = b$. A set of these points $x_0, x_1, \ldots, x_n$ defines a hyperfinite partition P of $[a, b]$, which we denote by $P = \{x_i\}_{i=0}^{n}$. The points $x_0, x_1, \ldots, x_n$ are the partition points of P.The largest of the lengths of the subintervals $[x_{i-1}, x_i]$, $0 \leq i \leq n$ is the norm of $P = \{x_i\}_{i=0}^{n}$ denoted by $\|P\|$; thus, $\|P\| = \max_{1 \leq i \leq n}(x_i - x_{i-1})$.

Definition 57. Let P and P' are hyperfinite partitions of $[a, b]$, then P' is a refinement of P if every partition point of P is also a partition point of P'; that is, if P' is obtained by inserting additional points between those of P.

Definition 58. Let f be ${}^*\mathbb{R}_c^{\#}$- valued function $f: [a, b] \to {}^*\mathbb{R}_c^{\#}$, then we say that external hyperfinite sum σ^{Ext} defined by

$$\sigma^{Ext} = Ext\text{-}\textstyle\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}),\ x_{i-1} \leq c_i \leq x_i, \tag{55}$$

is a Riemann external hyperfinite sum of f over the hyperfinite partition $P = \{x_i\}_{i=0}^{n}$.

Definition 59.[13]. Let f be ${}^*\mathbb{R}_c^{\#}$- valued function $f: [a, b] \to {}^*\mathbb{R}_c^{\#}$, then we say that f is Riemann #-integrable on $[a, b]$ if there is a number $L \in {}^*\mathbb{R}_c^{\#}$ with the following property: for every $\varepsilon \approx 0, \varepsilon > 0$, there is a $\delta \approx 0, \delta > 0$ such that $|L - \sigma^{Ext}| < \delta$ if σ^{Ext} is any Riemann external hyperfinite sum of f over a partition P of $[a, b]$ such that $\|P\| < \delta$. In this case, we say that L is the Riemann #-integral of f over $[a, b]$, and we shall write

$$L = Ext\text{-}\textstyle\int_a^b f(x)d^{\#}x. \tag{56}$$

Thus the Riemann #-integral of ${}^*\mathbb{R}_c^\#$- valued function $f:[a,b]\to{}^*\mathbb{R}_c^\#$ over $[a,b]$ is defined as #-limit of the external hyperfinite sums (55) with respect to partitions of the interval $[a,b]$:

$$Ext\text{-}\int_a^b f(x)d^\#x = \#\text{-}\lim_{n\to{}^*\infty}\left(Ext\text{-}\sum_{i=1}^n f(c_i)\,(x_i - x_{i-1})\right). \tag{57}$$

Definition 60. A coordinate rectangle R in ${}^*\mathbb{R}_c^{\#n}, n\in{}^*\mathbb{N}$ is the external finite or hyperfinite Cartesian product of n #-closed intervals; that is, $R = Ext\text{-}\times_{i=1}^n [a_i,b_i]$. The content of R is $V(R) = Ext\text{-}\prod_{i=1}^n(b_i - a_i)$. The hyperreal numbers $b_i - a_i$, $1\le i\le n$ are the edge lengths of R. If they are equal, then R is finite or hyperfinite coordinate cube. If $a_l = b_l$ for some r, then $V(R)=0$ and we say that R is degenerate; otherwise, R is nondegenerate.
Definition54. If $R = Ext\text{-}\times_{i=1}^n [a_i,b_i]$ and $P_r = a_{r0} < a_{r1} <\cdots< a_{rm_r}$ is an external hyperfinite partition of $[a_r,b_r], 1\le r\le n$, then the set of all rectangles in ${}^*\mathbb{R}_c^{\#n}$ that can be written as $Ext\text{-}\times_{i=1}^n\left[a_{i,j_{i-1}},a_{i,j_i}\right]$, $1\le j_r\le m_r$, $1\le r\le n$ is a partition of R. We denote this partition by $P = Ext\text{-}\times_{r=1}^n P_r$ and define its norm to be the maximum of the norms of $P_i, 1\le i\le n$; thus, $\|P\| = \max_i\{P_i|1\le i\le n\}$.
Definition 61. If $P = Ext\text{-}\times_{i=1}^n P_i$ and $P' = Ext\text{-}\times_{i=1}^n P_i'$ are partitions of the same rectangle, then P' is a refinement of P if P_i' is a refinement of $P_i, 1\le i\le n$ as defined above.
Definition 62.Suppose that f is a ${}^*\mathbb{R}_c^\#$- valued function defined on a rectangle R in ${}^*\mathbb{R}_c^{\#n}, n\in{}^*\mathbb{N}$, $P=\{P_i\}_{i=1}^k$is a partition of R, and x_i is an arbitrary point in R_i, $1\le j\le k$. Then a Riemann external hyperfinite sum σ^{Ext} of f over the partition P is defined by

$$\sigma^{Ext} = Ext\text{-}\sum_{i=1}^k f(x_i)\,V(R_i) \tag{58}$$

Definition 63. Let f be a ${}^*\mathbb{R}_c^\#$- valued function defined on a rectangle R in ${}^*\mathbb{R}_c^{\#n}, n\in{}^*\mathbb{N}$. We say that f is Riemann #-integrable on R if there is a number L with the following property: for every $\varepsilon\approx 0, \varepsilon>0$, there is a $\delta\approx 0,\delta>0$ such that $|L-\sigma^{Ext}|<\delta$ if σ^{Ext} is any Riemann external hyperfinite sum of f over a partition P of R such that $\|P\|<\delta$. In this case, we say that L is the Riemann #-integral of f over R, and write

$$L = Ext\text{-}\int_R f(x)d^{\#n}x. \tag{59}$$

Thus the Riemann #-integral of ${}^*\mathbb{R}_c^\#$- valued function f defined on a rectangle R in ${}^*\mathbb{R}_c^{\#n}$ is defined as #-limit of the external hyperfinite sums (58) with respect to partitions of the rectangle R:

$$Ext\text{-}\int_R f(x)d^{\#n}x = \#\text{-}\lim_{n\to{}^*\infty}\left(Ext\text{-}\sum_{i=1}^k f(x_i)\,V(R_i)\right). \tag{60}$$

The ${}^*\mathbb{R}_c^\#$-Valued #-Exponential Function Ext-$\exp(x)$ and ${}^*\mathbb{R}_c^\#$-Valued Trigonometric Functions Ext-$\sin(x)$, Ext-$\cos(x)$

We define the #-exponential function Ext-$\exp(x)$ as the solution of the #-differential equation

$$f^{\#\prime}(x) = f(x), f(0)=1. \tag{61}$$

We solve it by setting $f(x) = Ext\text{-}\sum_{n=0}^{{}^*\infty} x^n$, $f^{\#\prime}(x) = Ext\text{-}\sum_{n=0}^{{}^*\infty} nx^n$. Therefore

$$Ext\text{-}\exp(x) = Ext\text{-}\sum_{n=0}^{{}^*\infty}\frac{x^n}{n!}. \tag{62}$$

From (1) we get $\left(Ext\text{-}\exp(x)\right)\left(Ext\text{-}\exp(y)\right) = Ext\text{-}\exp(x+y)$ for any $x,y\in{}^*\mathbb{R}_c^\#$.

We define the #- trigonometric functions $Ext\text{-}\sin x$ and $Ext\text{-}\cos x$ by

$$Ext\text{-}\sin x = Ext\text{-}\sum_{n=0}^{^*\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}, Ext\text{-}\cos x = Ext\text{-}\sum_{n=0}^{^*\infty}(-1)^n \frac{x^{2n}}{(2n)!}. \quad (63)$$

It can be shown that the series (1) #-converges for all $x \in {}^*\mathbb{R}_c^\#$ #-differentiating yields

$$(Ext\text{-}\sin x)^{\#\prime} = Ext\text{-}\cos x, (Ext\text{-}\cos x)^{\#\prime} = -(Ext\text{-}\sin x). \quad (64)$$

${}^*\mathbb{R}_{\mathbf{c}}^\#$ -Valued Schwartz Distributions

Definition 64.[13]. Let U be an #- open subset of ${}^*\mathbb{R}_c^{\#n}$ and $f: U \to {}^*\mathbb{R}_c^\#$. The partial derivative of f at the point $x = (x_1, x_2, \dots, x_i, \dots, x_n)$ with respect to the i-th variable x_i is defined as

$$\frac{\partial^\# f}{\partial^\# x_i} = \#\text{-}\lim_{h \to_\# 0} \frac{f(x_1, x_2, \dots, x_i + h, \dots, x_n) - f(x_1, x_2, \dots, x_i, \dots, x_n)}{h}.$$

Definition 65.A multi-index of size $n \in {}^*\mathbb{N}$ is an element in ${}^*\mathbb{N}^n$, the length of a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in {}^*\mathbb{N}^n$ is defined as $Ext\text{-}\sum_{i=1}^n \alpha_i$ and denoted by $|\alpha|$. We introduce the following notations for a given multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in {}^*\mathbb{N}^n$: $x^\alpha = Ext\text{-}\prod_{i=1}^n x_i^{\alpha_i}$; $\partial^{\#\alpha} = Ext\text{-}\prod_{i=1}^n \frac{\partial^{\#\alpha_i}}{\partial^\# x_i^{\alpha_i}}$ or symbolically $\partial^{\#\alpha} = Ext\text{-}\frac{\partial^{\#\alpha}}{\partial^\# x_1^{\alpha_1} \dots \partial^\# x_n^{\alpha_n}}$..

Definition 66.The Schwartz space of rapidly decreasing ${}^*\mathbb{C}_c^\#$- valued test functions on ${}^*\mathbb{R}_c^{\#n}$, $n \in {}^*\mathbb{N}$ is the function space defined by

$$\mathrm{S}^\#({}^*\mathbb{R}_c^{\#n}, {}^*\mathbb{C}_c^\#) = \{f \in C^{^*\infty}({}^*\mathbb{R}_c^{\#n}, {}^*\mathbb{C}_c^\#) | \forall(\alpha, \beta)(\alpha, \beta \in {}^*\mathbb{N}^n) \forall x(x \in {}^*\mathbb{R}_c^{\#n})[|x^\alpha D^{\#\beta} f(x)| < \infty^\#]\}.$$

Remark 10.Note that if $f \in S^\#({}^*\mathbb{R}_c^{\#n}, {}^*\mathbb{C}_c^\#)$ the integral of $x^\alpha |D^{\#\beta} f(x)|$ exists

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} |x^\alpha D^{\#\beta} f(x)| d^{\#n} < \infty^\#.$$

Definition 67.The Schwartz space of essentially rapidly decreasing ${}^*\mathbb{C}_c^\#$- valued test functions on ${}^*\mathbb{R}_c^{\#n}$, $n \in {}^*\mathbb{N}$ is the function space defined by

$$\mathrm{S}^\#({}^*\mathbb{R}_c^{\#n}, {}^*\mathbb{C}_c^\#) = \{f \in C^{^*\infty}({}^*\mathbb{R}_c^{\#n}, {}^*\mathbb{C}_c^\#) | \forall\alpha(\alpha \in \mathbb{N}^n) \forall\beta(\beta \in {}^*\mathbb{N}^n) \forall x(x \in {}^*\mathbb{R}_c^{\#n})[|x^\alpha D^{\#\beta} f(x)| < \infty]\}.$$

Remark 11.Note that if $f \in S^\#({}^*\mathbb{R}_c^{\#n}, {}^*\mathbb{C}_c^\#)$ the integral of $x^\alpha |D^{\#\beta} f(x)|, \alpha \in \mathbb{N}^n, \beta \in {}^*\mathbb{N}^n$ exists and

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} |x^\alpha D^{\#\beta} f(x)| d^{\#n} < \infty.$$

Definition 67.The Schwartz space of rapidly decreasing ${}^*\mathbb{C}_c^\#$- valued test functions on ${}^*\mathbb{R}_{c,\text{fin}}^{\#n}$, $n \in {}^*\mathbb{N}$ is the function space defined by

$$\breve{\mathrm{S}}^\#({}^*\mathbb{R}_{c,\text{fin}}^{\#n}, {}^*\mathbb{C}_c^\#) = \{f \in C^{^*\infty}({}^*\mathbb{R}_{c,\text{fin}}^{\#n}, {}^*\mathbb{C}_c^\#) | \forall(\alpha, \beta)(\alpha, \beta \in {}^*\mathbb{N}^n) \forall x(x \in {}^*\mathbb{R}_{c,\text{fin}}^{\#n})[|x^\alpha D^{\#\beta} f(x)| < \infty^\#]\}.$$

Remark 12.Note that if $f \in \breve{\mathrm{S}}^\#({}^*\mathbb{R}_{c,\text{fin}}^{\#n}, {}^*\mathbb{C}_c^\#)$ the integral of $x^\alpha |D^{\#\beta} f(x)|, \alpha \in {}^*\mathbb{N}^n, \beta \in {}^*\mathbb{N}^n$ exists and

$$Ext\text{-}\int_{{}^*\mathbb{R}^{\#n}_{c,\text{fin}}} \left| x^{\alpha} D^{\#\beta} f(x) \right| d^{\#n} < \infty^{\#}.$$

Definition 68.The Schwartz space of essentially rapidly decreasing ${}^*\mathbb{C}^{\#}_{c}$- valued test functions on ${}^*\mathbb{R}^{\#n}_{c,\text{fin}}$, $n \in {}^*\mathbb{N}$ is the function space defined by

$$\check{S}^{\#}_{\text{fin}}\left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}, {}^*\mathbb{C}^{\#}_{c}\right) =$$
$$\left\{ f \in C^{{}^*\infty}\left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}, {}^*\mathbb{C}^{\#}_{c}\right) | \forall(\alpha,\beta)(\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^n) \exists c_{\alpha\beta}\left(c_{\alpha\beta} \in {}^*\mathbb{R}^{\#}_{c,\text{fin}}\right) \forall x\left(x \in {}^*\mathbb{R}^{\#n}_{c,\text{fin}}\right)\left[\left|x^{\alpha}\left(D^{\#\beta} f(x)\right)\right| < c_{\alpha\beta}\right]\right\}.$$

Remark 13.Note that if $f \in S^{\#}_{\text{fin}}({}^*\mathbb{R}^{\#n}_{c}, {}^*\mathbb{C}^{\#}_{c})$ the integral of $\left| x^{\alpha} D^{\#\beta} f(x) \right|$ exists and finitely bounded above

$$Ext\text{-}\int_{{}^*\mathbb{R}^{\#n}_{c,\text{fin}}} \left| x^{\alpha} D^{\#\beta} f(x) \right| d^{\#n} < d_{\alpha\beta}, d_{\alpha\beta} \in {}^*\mathbb{R}^{\#}_{c,\text{fin}}.$$

Abbreviation 4. 1) The Schwartz space of rapidly decreasing test functions on ${}^*\mathbb{R}^{\#n}_{c}$ we will be denoting by $S^{\#}({}^*\mathbb{R}^{\#n}_{c})$ and let $S^{\#}_{\text{fin}}({}^*\mathbb{R}^{\#n}_{c})$ denote the set of ${}^*\mathbb{C}^{\#}_{c}$-valued essentially rapidly decreasing test functions on ${}^*\mathbb{R}^{\#n}_{c}$.
2) The Schwartz space of rapidly decreasing ${}^*\mathbb{C}^{\#}_{c}$- valued test functions on ${}^*\mathbb{R}^{\#n}_{c,\text{fin}}$ we will be denoting by $\check{S}^{\#}\left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}\right)$ and let $\check{S}^{\#}_{\text{fin}}\left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}\right)$ denote the set of ${}^*\mathbb{C}^{\#}_{c}$-valued essentially rapidly decreasing test functions on ${}^*\mathbb{R}^{\#n}_{c,\text{fin}}$.

Definition 69. A linear functional $u: S^{\#}({}^*\mathbb{R}^{\#n}_{c}) \to {}^*\mathbb{C}^{\#}_{c}$ is a #-continuous if there exist $C, k \in {}^*\mathbb{N}$ and constants $c_{\alpha\beta}$ such that $|u(\varphi)| \leq C\left(Ext\text{-}\sum_{|\alpha|\leq k, |\beta|\leq k} c_{\alpha\beta}\right)$. Here $\forall x(x \in {}^*\mathbb{R}^{\#n}_{c})\left[\left|x^{\alpha}\left(D^{\#\beta} \varphi(x)\right)\right| < c_{\alpha\beta}\right]$.

Definition 70. A linear functional $u: S^{\#}\left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}\right) \to {}^*\mathbb{C}^{\#}_{c}$ is a strongly #-continuous if there exist $C, k \in {}^*\mathbb{N}$ and constants $c_{\alpha\beta}$ such that $|u(\varphi)| \leq C\left(Ext\text{-}\sum_{|\alpha|\leq k, |\beta|\leq k} c_{\alpha\beta}\right) \in {}^*\mathbb{R}^{\#}_{c,\text{fin}}$.

Definition 71. A generalized function $u \in S^{\#\prime}({}^*\mathbb{R}^{\#n}_{c})$ is defined as a #-continuous linear functional on vector space $S^{\#}({}^*\mathbb{R}^{\#n}_{c})$, symbolically it written as $u: \varphi \to (u, \varphi)$. Thus space $S^{\#\prime}({}^*\mathbb{R}^{\#n}_{c})$ of generalized functions is the space dual to $S^{\#}({}^*\mathbb{R}^{\#n}_{c})$.

Definition 72. A generalized function $u \in S^{\#\prime}\left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}\right)$ is defined as a strongly #-continuous linear functional on vector space $S^{\#}\left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}\right)$, symbolically it written as $u: \varphi \to (u, \varphi)$. Thus space $S^{\#\prime}\left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}\right)$ of generalized functions is the space dual to $S^{\#}\left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}\right)$.

Definition 73. Convergence of a hyper infinite sequence $\{u_n\}_{n=1}^{{}^*\infty}$ of generalized functions in $S^{\#\prime}({}^*\mathbb{R}^{\#n}_{c})$ is defined as weak #-convergence of functionals in $S^{\#\prime}({}^*\mathbb{R}^{\#n}_{c})$ that is: $u_n \to_{\#} 0$, as $n \to {}^*\infty$, in $S^{\#\prime}({}^*\mathbb{R}^{\#n}_{c})$ means that $(u_n, \varphi) \to_{\#} 0$, as $n \to {}^*\infty$, for all $\varphi \in S^{\#}({}^*\mathbb{R}^{\#n}_{c})$.

Definition 74. Convergence of a hyper infinite sequence $\{u_n\}_{n=1}^{{}^*\infty}$ of generalized functions in $S^{\#\prime}\left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}\right)$ is defined as weak #-convergence of functionals in $S^{\#\prime}\left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}\right)$ that is: $u_n \to_{\#} 0$, as $n \to {}^*\infty$, in $S^{\#\prime}\left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}\right)$ means that $(u_n, \varphi) \to_{\#} 0$, as $n \to {}^*\infty$, for all $\varphi \in S^{\#}\left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}\right)$.

Definition 75. 1) Let $u \in S^{\#'}({}^*\mathbb{R}^{\#n}_{c})$ and let $x = Ay + b$ be a linear transformation of ${}^*\mathbb{R}^{\#n}_{c}$ onto ${}^*\mathbb{R}^{\#n}_{c}$. The generalized function $u(Ay + b) \in S^{\#'}({}^*\mathbb{R}^{\#n}_{c})$ is defined by

$$(u(Ay + b), \varphi) = \left(u, \frac{\varphi\left[A^{-1}(x - b)\right]}{|\det A|}\right). \qquad (65)$$

Formula (1) enables one to define generalized functions that are translation invariant, spherically symmetric, centrally symmetric, homogeneous, periodic, Lorentz invariant, etc.
2) Let the function $\alpha(x) \in C^{\#1}({}^*\mathbb{R}^{\#}_{c})$ have only simple zeros $x_k \in {}^*\mathbb{R}^{\#}_{c}, k \in {}^*\mathbb{N}$, the function $\delta(\alpha(x))$ is defined by

$$\delta(\alpha(x)) = Ext\text{-}\sum_{k=1}^{{}^*\infty} \frac{\delta(x - x_k)}{|\alpha^{\#\prime}(x_k)|}. \qquad (66)$$

3) Let $u \in S^{\#'}({}^*\mathbb{R}_c^{\#n})$, the generalized (weak) #-derivative $\partial^{\#\alpha}u$ of u of order α is defined as

$$(\partial^{\#\alpha}u, \varphi) = (-1)^{|\alpha|}(u, \partial^{\#\alpha}\varphi). \tag{67}$$

4) Let $u \in S^{\#'}({}^*\mathbb{R}_c^{\#n})$ and $g(x) \in C^{\#^*\infty}({}^*\mathbb{R}_c^{\#n})$, The product $gu = ug$ is defined by

$$(gu, \varphi) = (u, g\varphi). \tag{68}$$

5) Let $u_1 \in S^{\#'}({}^*\mathbb{R}_c^{\#n})$ and $u_2 \in S^{\#'}({}^*\mathbb{R}_c^{\#m})$ then their direct product is defined by the formula

$$(u_1 \times u_2, \varphi) = \big(u_1(x)(u_2(y), \varphi)\big),\ \ \varphi(x, y) \in S^{\#}\,({}^*\mathbb{R}_c^{\#n} \times {}^*\mathbb{R}_c^{\#m}). \tag{69}$$

6) The Fourier transform $\mathcal{F}[u]$ of a generalized function $u \in S^{\#'}({}^*\mathbb{R}_c^{\#n})$ is defined by the formula

$$(\mathcal{F}[u], \varphi) = (u, \mathcal{F}[\varphi]), \tag{70}$$

$$\mathcal{F}[\varphi] = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} \varphi(x)(Ext\text{-}\exp[i(\xi, x)])d^{\#n}x. \tag{71}$$

Since the operation $\varphi(x) \to \mathcal{F}[\varphi](\xi)$ is an isomorphism of $\mathrm{S}^{\#}({}^*\mathbb{R}_c^{\#n})$ onto $\mathrm{S}^{\#}({}^*\mathbb{R}_c^{\#n})$, the operation $u \to \mathcal{F}[u]$ is an isomorphism of $S^{\#'}({}^*\mathbb{R}_c^{\#n})$ onto $S^{\#'}({}^*\mathbb{R}_c^{\#n})$ and the inverse of $\mathcal{F}[u]$ is given by: $\mathcal{F}^{-1}[u] = (2\pi)^{-n}\mathcal{F}[u(-\xi)]$. The following formulas hold for $u \in S^{\#'}({}^*\mathbb{R}_c^{\#n})$: (a) $\partial^{\#\alpha}\,\mathcal{F}[u] = \mathcal{F}[(ix)^{\alpha}u]$, (b) $\mathcal{F}[\,\partial^{\#\alpha}u] = (i\xi)^{\alpha}\mathcal{F}[u]$,(c) if the generalized function $u_1 \in S^{\#'}({}^*\mathbb{R}_c^{\#n})$ has #-com-pact support, then $\mathcal{F}[u_1 * u_2] = \mathcal{F}[u_1]\mathcal{F}[u_2]$.
7) If the generalized function u is periodic with n-period $T = (T_1, \dots, T_n)$, then $u \in S^{\#'}({}^*\mathbb{R}_c^{\#n})$, and it can be expanded in a hyper infinite trigonometric series

$$u(x) = Ext\text{-}\sum_{|k|=0}^{{}^*\infty} c_k(u)(Ext\text{-}\exp[i(k\omega, x)]), |c_k(u)| \le A(1 + |k|)^m\,. \tag{72}$$

The series (1) #-converges to $u(x)$ in $S^{\#'}({}^*\mathbb{R}_c^{\#n})$, here $\omega = \left(\frac{2\pi}{T_1}, \dots, \frac{2\pi}{T_n}\right)$ and $k\omega = \left(\frac{2\pi k_1}{T_1}, \dots, \frac{2\pi k_n}{T_n}\right)$.

A NON-ARCHIMEDEAN METRIC SPACES ENDOWED WITH ${}^*\mathbb{R}_{\mathbf{c}}^{\#}$ -VALUED METRIC

Definition 76. A non-Archimedean metric space is an ordered pair $(M, d^{\#})$ where M is a set, and $d^{\#}$ is a #-metric on M i.e., ${}^*\mathbb{R}_{\mathbf{c}+}^{\#}$- valued function $d^{\#}: M \times M \to {}^*\mathbb{R}_{\mathbf{c}+}^{\#}$such that for any triplet $x, y, z \in M$, the following holds:
1. $d^{\#}(x, y) = 0 \Longrightarrow x = y$. 2. $d^{\#}(x, y) = d^{\#}(y, x)$. 3. $d^{\#}(x, z) \le d^{\#}(x, y) + d^{\#}(y, z)$.
Definition 77. A hyper infinite sequence $\{x_n\}_{n=1}^{{}^*\infty}$ of points in M is called #-Cauchy in $(M, d^{\#})$ if for every hyperreal $\varepsilon \in {}^*\mathbb{R}_{c+}^{\#}$ there exists some $N \in {}^*\mathbb{N}$ such that $d^{\#}(x_n, x_m) < \varepsilon$ if $n, m > N$.
Definition 78. A point x of the non-Archimedean metric space $(M, d^{\#})$ is the #-limit of the hyper infinite sequence$\{x_n\}_{n=1}^{{}^*\infty}$ if for all $\varepsilon \in {}^*\mathbb{R}_{c+}^{\#}$, there exists some $N \in {}^*\mathbb{N}$ such that $d^{\#}(x_n, x) < \varepsilon$ if $n > N$.
Definition 79. A non-Archimedean metric space is #-complete if any of the following equivalent conditions are satisfied: 1.Every hyper infinite #-Cauchy sequence $\{x_n\}_{n=1}^{{}^*\infty}$ of points in M has a #-limit that is also in M.
2.Every hyper infinite #-Cauchy sequence in M, #-converges in M that is, to some point of M.

For any non-Archimedean metric space $(M, d^{\#})$ one can construct a #-complete non-Archimedean metric space $(M', d^{\#})$ which is also denoted as $(\#\text{-}\overline{M}, d^{\#})$ and which contains M a #-dense subspace.
It has the following universal property: if K is any #-complete non-Archimedean metric space and $f: M \to K$ is any uniformly #-continuous function from M to K, then there exists a unique uniformly #-continuous function $f': M' \to K$ that extends f.The space #-$\overline{M}$ is determined up to #-isometry by this property (among all #-complete metric spaces #- isometrically containing non-Archimedean metric space $(\#\text{-}\overline{M}, d^{\#})$, and is called the #-completion of $(M, d^{\#})$.
The #-completion of M can be constructed as a set of equivalence classes of Cauchy hyper infinite sequences in M. For any two hyper infinite Cauchy sequences $\{x_n\}_{n=1}^{^*\infty}$ and $\{y_n\}_{n=1}^{^*\infty}$ in M, we may define their distance as $d^{\#\prime} = \#\text{-}\lim_{n\to\infty^{\#}} d^{\#}(x_n, y_n)$. This #-limit exists because the hyperreal numbers ${}^*\mathbb{R}_c^{\#}$ are #-complete. This is only a pseudo metric, not yet a metric, since two different hyper infinite Cauchy sequences may have the distance 0. But having distance 0 is an equivalence relation on the set of all hyper infinite Cauchy sequences, and the set of equivalence classes is a metric space, the #-completion of M. The original space is embedded in this space via the identification of an element x of M' with the equivalence class of hyper infinite sequences in M #-converging to x i.e., the equivalence class containing a hyper infinite sequence with constant value x. This defines a #-isometry onto a #-dense subspace, as required.
Example 6. Both ${}^*\mathbb{R}$ and ${}^*\mathbb{C}$ are internal metric spaces when endowed with the distance function $d(x, y) = |x - y|$.
Definition1. About any point $x \in M$ we define the #-open ball of radius $r \in {}^*\mathbb{R}_{c+}^{\#}$ about x as the set $B_r(x) = \{y \in M | d^{\#}(x, y) < r\}$. These #-open balls form the base for a topology on M.
Definition 80. A non-Archimedean metric space $(M, d^{\#})$ is called hyper finitely bounded if there exists some $r \in {}^*\mathbb{R}_{c,\mathrm{fin}+}$ such that $d^{\#}(x, y) < r$ for all $x, y \in M$.
Definition 81. A non-Archimedean metric space $(M, d^{\#})$ is called finitely bounded if there exists some $r \in {}^*\mathbb{R}_{c,\infty+}$ such that $d^{\#}(x, y) < r$ for all $x, y \in M$.
Definition 82. A non-Archimedean metric space $(M, d^{\#})$ is called hyper finitely bounded if there exists some $r \in {}^*\mathbb{R}_{c,\infty+}$ such that $d^{\#}(x, y) < r$ for all $x, y \in M$.
Definition 83. Let $(M, d^{\#})$ be a non-Archimedean metric space. A set $A \subset X$ is called finitely bounded if there exists some $r \in {}^*\mathbb{R}_{c,\mathrm{fin}+}$ such that $A \subset B_r(a)$, $a \in X$.
Definition 84. A non-Archimedean metric space $(M, d^{\#})$ is called #-compact if every hyper infinite sequence $\{x_n\}_{n=1}^{^*\infty}$ in M has a hyper infinite subsequence that #-converges to a point in M. This sort of compactness is known as hyper sequential compactness and, in a non-Archimedean metric spaces is equivalent to the topological notions of hyper countable #-compactness.
Definition 85. A topological space X is called hyper countably #-compact if it satisfies any of the following equivalent conditions: (a) every hyper countable open cover U of X (i.e., $card(U) = card({}^*\mathbb{N})$) has a finite or hyperfinite sub-cover.
For a function $f: M_1 \to M_2$ with a non-Archimedean metric spaces $(M_1, d_1^{\#})$ and $(M_2, d_2^{\#})$ the following definitions of uniform #-continuity and (ordinary) #-continuity hold.
Definition 86. A function f is called uniformly #-continuous if for every $\varepsilon \in {}^*\mathbb{R}_{c\approx+}^{\#}$ there exists $\delta \in {}^*\mathbb{R}_{c\approx+}$ such that for every $x, y \in M_1$ with $d_1^{\#}(x, y) < \delta$ we get $d_2^{\#}(f(x), f(y)) < \varepsilon$.
Definition 87. A function f is called #-continuous at $x \in M_1$ if for every $\varepsilon \in {}^*\mathbb{R}_{c\approx+}^{\#}$ there exists $\delta \in {}^*\mathbb{R}_{c\approx+}^{\#}$ such that for every $y \in M_1$ with $d_1^{\#}(x, y) < \delta$ we get $d_2^{\#}(f(x), f(y)) < \varepsilon$.

LEBESGUE #-INTEGRATION OF ${}^*\mathbb{R}_c^{\#}$ -VALUED FUNCTIONS

Let $C_0^{\#}({}^*\mathbb{R}_c^{\#n})$ be the space of all ${}^*\mathbb{R}_c^{\#}$-valued #-compactly supported #-continuous functions of ${}^*\mathbb{R}_c^{\#n}$. Define a #-norm on $C_0^{\#}$ by the Riemann #-integral [13]:

$$\|f\|_{\#} = Ext\text{-}\int |f(x)| d^{\#n}x, \quad (73)$$

Note that the Riemann #-integral exists for any #-continuous function $f: {}^*\mathbb{R}_c^{\#n} \to {}^*\mathbb{R}_c^{\#}$, see [13]. Then $C_0^{\#}({}^*\mathbb{R}_c^{\#n})$ is a #-normed vector space and thus in particular, it is a non-Archimedean metric space. All non-Archimedean metric space, have a non-Archimedean #-completion ($\#\text{-}\bar{M}, d^{\#}$). Let $L_1^{\#}$ be this #-completion. This space $L_1^{\#}$ is isomorphic to the space of Lebesgue #-integrable functions modulo the subspace of functions with #-integral zero. Furthermore, the Riemann integral (1) is a uniformly #-continuous linear functional with respect to the #-norm on $C_0^{\#}({}^*\mathbb{R}_c^{\#n})$ which is #-dense in $L_1^{\#}$. Hence the Riemann #- integral $Ext\text{-}\int f(x) d^{\#n}x$ has a unique extension to all of $L_1^{\#}$. This integral is precisely the Lebesgue #-integral.

Definition 88. Suppose that $1 \leq p < {}^*\infty$, and $[a,b]$ is an interval in ${}^*\mathbb{R}_c^{\#}$. We denote by $L_p^{\#}([a,b])$ the set of the all functions $f: [a,b] \to {}^*\mathbb{R}_c^{\#}$ such that $Ext\text{-}\int_a^b |f(x)|^p d^{\#}x < {}^*\infty$. We define the $L_p^{\#}$ -#-norm of f by

$$\|f\|_{\#p} = \left(Ext\text{-}\int_a^b |f(x)|^p d^{\#}x\right)^{1/p}. \quad (74)$$

More generally, if E is a subset of ${}^*\mathbb{R}_c^{\#n}$, which could be equal to ${}^*\mathbb{R}_c^{\#n}$ itself, then $L_p^{\#}(E)$ is the set of Lebesgue #-measurable functions $f: E \to {}^*\mathbb{R}_c^{\#}$ whose p-th power is Lebesgue #-integrable, with the #-norm

$$\|f\|_{\#p} = \left(Ext\text{-}\int_E |f(x)|^p d^{\#n}x\right)^{1/p}. \quad (75)$$

Definition 89. A set $X \subset {}^*\mathbb{R}_c^{\#n}$ is #-measurable if there exists $Ext\text{-}\int 1_X \, d^{\#n}x$, where 1_X is the indicator function.

Definition 90. A ${}^*\mathbb{R}_c^{\#}$ -valued function f on ${}^*\mathbb{R}_c^{\#n}$ is a #-measurable if a set $\{x | f(x) > t\}$ is a #-measurable set for all $t \in {}^*\mathbb{R}_c^{\#n}$.

Remark 14.To assign a value to the Lebesgue #-integral of the indicator function 1_X of a #-measurable set X consistent with the given #-measure $\mu^{\#}$, the only reasonable choice is to set: $Ext\text{-}\int 1_X d\,\mu^{\#} = \mu^{\#}(X)$.

Definition 91. A hyperfinite linear combination of indicator functions $f = Ext\text{-}\sum_{k=1}^{n} \alpha_k \, 1_{X_k}$ where the coefficients $\alpha_k \in {}^*\mathbb{R}_c^{\#}$ and X_k are disjoint #-measurable sets, is called a #-measurable simple function.

Definition 92. When the coefficients α_k are positive, we set $Ext\text{-}\int f d\,\mu^{\#} = Ext\text{-}\sum_{k=1}^{n} \alpha_k \, \mu^{\#}(X_k)$. For a non-negative #-measurable function f, let $\{f_n(x)\}_{n=1}^{{}^*\infty}$be a hyper infinite sequence of the simple functions $f_n(x)$ whose values is $\frac{k}{2^n}$ whenever $\frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n}$ for k a non-negative hyperinteger less than 4^n. Then we set

$$Ext\text{-}\int f d\,\mu^{\#} = \#\text{-}\lim_{n \to {}^*\infty} (Ext\text{-}\int f_n d\,\mu^{\#}).$$

Definition 93. If f is a #-measurable function of the set E to the reals including $\pm\infty^{\#}$, then we can write $f = f^+ - f^-$, where: 1) $f^+(x) = f(x)$ if $f(x) > 0$ and $f^+(x) = 0$ if $f(x) \leq 0$; 2) $f^-(x) = f(x)$ if $f(x) < 0$ and $f^-(x) = 0$ if $f(x) \geq 0$. Note that both f^+ and f^- are non-negative #-measurable functions and $|f| = f^+ + f^-$.

Definition 94.We say that the Lebesgue #-integral of the #-measurable function f exists, or is defined if at least one of $Ext\text{-}\int f^+ d\,\mu^{\#}$ and $Ext\text{-}\int f^- d\,\mu^{\#}$ is finite or hyperfinite. In this case we define

$$Ext\text{-}\int f d\,\mu^{\#} = (Ext\text{-}\int f^+ d\,\mu^{\#}) + (Ext\text{-}\int f^- d\,\mu^{\#}).$$

Theorem 37. Assuming that f is #-measurable and non-negative, the function $\breve{f}(x) = \{x \in E | f(x) > t\}$ is monotonically non-increasing. The Lebesgue #-integral may then be defined as the improper Riemann #-integral of $\breve{f}(x)$: $Ext\text{-}\int_E f d\,\mu^{\#} = Ext\text{-}\int_0^{{}^*\infty} \breve{f}(x) d^{\#}x$.

Definition 95. Let X be any set. We denote by 2^X the set of all subsets of X.A family $\mathcal{F} \subset 2^X$ is called a #-σ-algebra on X (or $\sigma^{\#}$-algebra on X) if: 1) $\emptyset \in \mathcal{F}$. 2) A family $\mathcal{F}$ is closed under complements, i.e. $A \in \mathcal{F}$ implies $X \backslash A \in \mathcal{F}$. 3) A family $\mathcal{F}$ is closed under hyper infinite unions, i.e. if $\{A_n\}_{n \in {}^*\mathbb{N}}$ is a hyper infinite sequence in $\mathcal{F}$ then

$\bigcup_{n\in {}^*\mathbb{N}} A_n \in \mathcal{F}$.

Theorem 38. If $\mathcal{F}$ is a #-σ-algebra on X then: 1) $\mathcal{F}$ is closed under hyper infinite intersections, i.e., if $\{A_n\}_{n\in {}^*\mathbb{N}}$ is a hyper infinite sequence in $\mathcal{F}$ then $\bigcap_{n\in {}^*\mathbb{N}} A_n \in \mathcal{F}$. 2) $X \in \mathcal{F}$.3) $\mathcal{F}$ is closed under hyperfinite unions and hyperfinite intersections.4) $\mathcal{F}$ is closed under set differences.5) $\mathcal{F}$ is closed under symmetric differences.

Theorem 39. If $\{A_\alpha\}_{\alpha\in I}$ is a collection of $\sigma^\#$-algebras on a set X, then $\bigcap_{\alpha\in I} A_\alpha$, is also an $\sigma^\#$-algebras on a set X.

Theorem 40. If $K \subset L$ then $\sigma^\#(K) \subset \sigma^\#(L)$.

Definition 96. (Borel $\sigma^\#$-algebra) Given a topological space X, the Borel $\sigma^\#$-algebra is the $\sigma^\#$-algebra generated by the #-open sets. It is denoted by $\mathcal{B}^\#(X)$. We call sets in $\mathcal{B}^\#(X)$ a Borel set. Specifically in the case $X = {}^*\mathbb{R}_c^{\#n}$ we have that $\mathcal{B}^\#({}^*\mathbb{R}_c^{\#n}) = \{U | U\ is\ \#\text{-open set}\}$. Note that the Borel $\sigma^\#$-algebra also contains all #-closed sets and is the smallest $\sigma^\#$-algebra with this property.

Definition 97. (#- Measures) A pair $(X, \mathcal{F})$ where $\mathcal{F}$ is an $\sigma^\#$-algebra on X is call a #- measurable space. Elements of $\mathcal{F}$ are called a #-measurable sets. Given a #-measurable space $(X, \mathcal{F})$, a function $\mu^\#: \mathcal{F} \to [0, {}^*\infty]$ is called a #-mea-sure on $(X, \mathcal{F})$ if: 1) $\mu^\#(\emptyset) = 0$. 2) For all hyper infinite sequences $\{A_n\}_{n\in {}^*\mathbb{N}}$ of pairwise disjoint sets in $\mathcal{F}$

$$\mu^\#\left(\bigcup_{n=1}^{{}^*\infty} A_n\right) = Ext\text{-}\sum_{n=1}^{{}^*\infty} \mu^\#(A_n). \tag{76}$$

A NON-ARCHIMEDEAN BANACH SPACES ENDOWED WITH ${}^*\mathbb{R}_{\mathbf{c}}^\#$ -VALUED NORM

A non-Archimedean normed space with ${}^*\mathbb{R}_c^\#$ -valued norm (#-norm) is a pair $(X, \|\cdot\|_\#)$ consisting of a vector space X over a non-Archimedean scalar field ${}^*\mathbb{R}_c^\#$ or complex field ${}^*\mathbb{C}_c^\# = {}^*\mathbb{R}_c^\# + i{}^*\mathbb{R}_c^\#$ together with a norm $\|\cdot\|_\#: X \to {}^*\mathbb{R}_c^\#$. Like any norms, this norm induces a translation invariant distance function, called the norm induced non-Archimedean ${}^*\mathbb{R}_c^\#$ -valued metric $d^\#(x, y)$ for all vectors $x, y \in X$, defined by $d^\#(x, y) = \|x - y\|_\# = \|y - x\|_\#$. Thus $d^\#(x, y)$ makes X into a non-Archimedean metric space $(X, d^\#)$.

Definition 98. A hyper infinite sequence $\{x_n\}_{n=1}^{{}^*\infty}$ in X is called $d^\#$ - Cauchy or Cauchy in $(X, d^\#)$ or $\|\cdot\|_\#$ -Cauchy if for every hyperreal $\varepsilon \in {}^*\mathbb{R}_{c+}^\#$ there exists some $N \in {}^*\mathbb{N}$ such that $d^\#(x_n, y_m) = \|x_n - y_n\|_\# < \varepsilon$ if $n, m > N$.

Definition 99.The metric $d^\#$ is called a #-complete metric if the pair $(X, d^\#)$ is a #-complete metric space, which by definition means for every $d^\#$- Cauchy sequence $\{x_n\}_{n=1}^{{}^*\infty}$ in $(X, d^\#)$, there exists some $x \in X$ such that $\#\text{-}\lim_{n\to{}^*\infty} \|x_n - x\|_\# = 0$.

Semigroups on non-Archimedean Banach spaces and their generators

Definition 100. A family of bounded operators $\{T(t) | 0 < t < {}^*\infty\}$ on external hyper infinite dimensional non-Archimedean Banach space X endowed with ${}^*\mathbb{R}_c^\#$ -valued #-norm $\|\cdot\|_\#$ is called a strongly #-continuous semigroup if: (a) $T(0) = I$, (b) $T(s)T(t) = T(s + t)$ for all $s, t \in {}^*\mathbb{R}_{c,+}^\#$, (c) For each $\phi \in X, t \mapsto T(t)$ is #-continuous mapping.

Definition101. A family $\{T(t) | 0 < t < {}^*\infty\}$ of bounded or hyper bounded operators on external hyper infinite dimensional Banach space X is called a contraction semigroup if it is a strongly #-continuous semigroup and moreover $\|T(t)\|_\# < 1$ for all $t \in [0, {}^*\infty)$.

Theorem 41. Let $T(t)$ is a strongly #-continuous semigroup on a non-Archimedean Banach space X, let $A\varphi = \#\text{-}\lim_{r\to_\# 0} A_r\varphi$ where $A_r = r^{-1}(I - T(r))$ and let $D(A) = \{\varphi | \exists (\#\text{-}\lim_{r\to_\# 0} A_r\varphi)\}$, then the operator A is #-closed and #-densely defined. Operator A is called the infinitesimal generator of the semigroup $T(t)$.

Definition 102. We will also say that A generates the semigroup $T(t)$ and write $T(t) = Ext\text{-}\exp(-tA)$.

Theorem 42. (Generalized Hille-Yosida theorem) A necessary and sufficient condition that #-closed linear operator

A on a non-Archimedean Banach space X generate a contraction semigroup is that: (a) $(-{}^*\infty, 0) \subset \rho(A)$, (b) $\|(\lambda + A)^{-1}\|_\# \leq \lambda^{-1}$ for all $\lambda > 0$.

Definition 103. Let X be a non-Archimedean Banach space, $\varphi \in X$.An element $l \in X^*$ that satisfies $\|l\|_\# = \|\varphi\|_\#$, and $l(\varphi) = \|\varphi\|_\#^2$ is called a normalized tangent functional to φ. By the generalized Hahn-Banach theorem, each $\varphi \in X$ has at least one normalized tangent functional.

Definition 104. A #-densely defined operator A on a non-Archimedean Banach space X is called accretive if for each $\varphi \in D(A)$, $\mathrm{Re}(l(A\varphi)) \geq 0$ for some normalized tangent functional to φ. Operator A is called maximal accretive if A is accretive and A has no proper accretive extension.

Remark 15. We remark that any accretive operator is #-closable. The #-closure of an accretive operator is again accretive, so every accretive operator has a smallest #-closed accretive extension.

Theorem 43. A #-closed operator A on a non-Archimedean Banach space X is the generator of a contraction semigroup if and only if A is accretive and $\mathrm{Ran}(\lambda_0 + A) = X$ for some $\lambda_0 > 0$.

Theorem 44. Let A be a #-closed operator on a non-Archimedean Banach space X. Then, if both A and it adjoint A^* are accretive, A generates a contraction semigroup.

Theorem 45. Let A be the generator of a contraction semigroup on a non-Archimedean Banach space X. Let D be a #-dense set, $D \subset D(A)$, so that $Ext\text{-}\exp(-tA): D \to D$. Then D is a #-core for A, i.e.,#-$\overline{A \restriction D} = A$.

Hypercontractive semigroups

In the previous section we discussed $L^p_\#$-contractive semigroups. In this section we give a self-adjointness theorem for operators of the form $A + V$ where V is a multiplication operator and A generates an $L^p_\#$-contractive semigroup that satisfies a strong additional property.

Definition 105. Let $\langle M. \mu^\# \rangle$ be a #-measure space with $\mu^\#(M) = 1$ and suppose that $A$is a positive self-adjoint operator on $L^2_\#(M, d^\#\mu^\#)$. We say that $Ext\text{-}\exp(-tA)$ is a hyper contractive semigroup if: (a) $Ext\text{-}\exp(-tA)$ is $L^p_\#$-contractive; (b) for some $b > 2$ and some constant C_b, there is a $T > 0$ so that $\|[Ext\text{-}\exp(-tA)]\varphi\|_{\#b} \leq \|\varphi\|_{\#2}$ for all $\varphi \in L^2_\#(M, d^\#\mu^\#)$

Note that the condition (a) implies that $Ext\text{-}\exp(-tA)$ is a strongly #-continuous contraction semigroup for all $p < {}^*\infty$. Holder's inequality shows that $\|\cdot\|_{\#q} \leq \|\cdot\|_{\#p}$ if $p \geq q$. Thus the $L^p_\#$-spaces are a nested family of spaces which get smaller as p gets larger; this suggests that (b) is a very strong condition. The following proposition shows that constant b plays no special role.

Theorem 46. Let $Ext\text{-}\exp(-tA)$ be a hypercontractive semigroup on $L^2_\#(M, d^\#\mu^\#)$. Then for all $p, q \in (1, {}^*\infty)$ there is a constant $C_{p,q}$ and a $t_{p,q} > 0$ so that if $> t_{p,q}$, then $\|Ext\text{-}\exp(-tA)\varphi\|_{\#p} < C_{p,q}\|\varphi\|_{\#q}$, for all $\varphi \in L^\#_q$.

Theorem 47. Let $\langle M, \mu^\# \rangle$ be a $\sigma^\#$-measure space with $\mu^\#(M) = 1$and let H_0 be the generator of a hypercontractive semi-group on $L_2(M, d^\#\mu^\#)$. Let V be a ${}^*\mathbb{R}^\#_c$ -valued measurable function on $\langle M, \mu^\# \rangle$ such that $V \in L^\#_p(M, d^\#\mu^\#)$ for all $p \in [\![1, {}^*\infty)$ and $Ext\text{-}\exp(-tV) \in L^\#_1(M, d^\#\mu^\#)$ for all $t > 0$. Then $H_0 + V$ is essentially self#-adjoint on $C^{^*\infty}(H_0) \cap D(V)$ and is bounded below. Here $C^{^*\infty}(H_0) = \bigcap_{p \in {}^*\mathbb{N}} D(H_0^p)$.

A NON-ARCHIMEDEAN HILBERT SPACES ENDOWED WITH ${}^*\mathbb{C}^\#_c$ -VALUED INNER PRODUCT

Definition 106. Let H be external hyper infinite dimensional vector space over complex field ${}^*\mathbb{C}^\#_c = {}^*\mathbb{R}^\#_c + \mathrm{i}{}^*\mathbb{R}^\#_c$. An inner product on H is a${}^*\mathbb{C}^\#_c$-valued function, $\langle \cdot,\cdot \rangle: H \times H \to {}^*\mathbb{C}^\#_c$, such that: (1) $\langle ax + by, z \rangle = \langle ax, z \rangle + \langle by, z \rangle$, (2) $\overline{\langle x, y \rangle} = \langle y, x \rangle$. (3) $\|x\|^2 \equiv \langle x, x \rangle \geq 0$ with equality $\langle x, x \rangle = 0$ if and only if $x = 0$.

Theorem 48. (Generalized Schwarz Inequality) Let $\{H, \langle \cdot,\cdot \rangle\}$be an inner product space, then for all $x, y \in H$:

$|\langle x,y\rangle| \leq \|x\|\|y\|$ and equality holds if and only if x and y are linearly dependent.
Theorem 49. Let $\{H,\langle\cdot,\cdot\rangle\}$be an inner product space, and $\|x\|_{\#} = \sqrt{\langle x,x\rangle}$. Then $\|\cdot\|_{\#}$ is a ${}^{*}\mathbb{R}_c^{\#}$ -valued #-norm on a space H. Moreover $\langle x,x\rangle$ is #-continuous on Cartesian product $H \times H$, where H is viewed as the #-normed space $\{H,\|\cdot\|_{\#}\}$.
Definition 107. A non-Archimedean Hilbert space is a #-complete inner product space.
Example 7. The standard inner product on ${}^{*}\mathbb{C}_c^{\#n}, n \in {}^{*}\mathbb{N}_{\infty}$ is given by external hyperfinite sum

$$\langle x,y\rangle = Ext\text{-}\sum_{i=1}^{n} \overline{x_i}\, y_i. \tag{77}$$

Here $x = \{x_i\}_{i=1}^{n}, y = \{y_i\}_{i=1}^{n}$, with $x_i, y_i \in {}^{*}\mathbb{C}_c^{\#}, 1 \leq i \leq n$, see [14].
Example 8. The sequence space $l_2^{\#}$ consists of all hyper infinite sequences $z = \{z_i\}_{i=1}^{{}^{*}\infty}$ of complex numbers in ${}^{*}\mathbb{C}_c^{\#}$ such that the hyper infinite series Ext-$\sum_{i=1}^{n}|z_i|^2$ #-converges. The inner product on $l_2^{\#}$ is defined by

$$\langle z,w\rangle = Ext\text{-}\sum_{i=1}^{{}^{*}\infty} \overline{z_i}\, w_i. \tag{78}$$

Here $z = \{z_i\}_{i=1}^{{}^{*}\infty}, w = \{w_i\}_{i=1}^{{}^{*}\infty}$ and the latter hyper infinite series #-converging as a consequence of the generalized Schwarz inequality and the #-convergence of the previous hyper infinite series.
Example 9. Let $C^{\#}[a,b]$ be the space of the ${}^{*}\mathbb{C}_c^{\#}$- valued #-continuous functions defined on the interval $[a,b] \subset {}^{*}\mathbb{R}_c^{\#}$, see [14]. We define an inner product on the space $C^{\#}[a,b]$ by the formula

$$\langle f,g\rangle = Ext\text{-}\int_a^b \overline{f(x)}\,g(x)\, d^{\#}x. \tag{79}$$

This space is not #-complete, so it is not a non-Archimedean Hilbert space. The #-complettion of $C^{\#}[a,b]$ with respect to the #-norm

$$\|f\|_{\#} = \left(Ext\text{-}\int_a^b |f(x)|^2\, d^{\#}x\right)^{1/2}, \tag{80}$$

is denoted by $L_2^{\#}[a,b]$.
Example 10. Let $C^{\#(k)}[a,b]$be the space of the ${}^{*}\mathbb{C}_c^{\#}$- valued functions with $k \in {}^{*}\mathbb{N}$ #-continuous #-derivatives on $[a,b] \subset {}^{*}\mathbb{R}_c^{\#}$, see [14].We define an inner product on the space $C^{\#(k)}[a,b]$ by the formula

$$\langle f,g\rangle = Ext\text{-}\sum_{i=0}^{k} \left(Ext\text{-}\int_a^b \overline{f^{\#(i)}(x)}\,g^{\#(i)}(x)\, d^{\#}x\right). \tag{81}$$

Here $f^{\#(i)}$ and $g^{\#(i)}$ denotes the i-th #-derivatives of f and g respectively.The corresponding #-norm is

$$\|f\|_{\#} = \left(Ext\text{-}\sum_{i=1}^{k}\left(Ext\text{-}\int_a^b \left|f^{\#(i)}(x)\right|^2 d^{\#}x\right)\right)^{1/2}. \tag{82}$$

This space is not #-complete, so it is not a non-Archimedean Hilbert space. The non-Archimedean Hilbert space obtained by #-complettion of $C^{\#(k)}[a,b]$ with respect to the #-norm (1) is non-Archimedean Sobolev space, denoted by $H^{\#k}[a,b]$.
Definition 108. The graph of the linear transformation $T: H \to H$ is the set of pairs $\{\langle\phi, T\phi\rangle | (\phi \in D(T))\}$. The graph of the operator T, denoted by $\Gamma(\mathrm{T})$, is thus a subset of $H \times H$ which is a non-Archimedean Hilbert space with the following inner product $(\langle\phi_1,\psi_1\rangle,\langle\phi_2,\psi_2\rangle)$. Operator T is called a #-closed operator if $\Gamma(\mathrm{T})$ is a #-closed subset of $H \times H$.
Definition 109. Let T_1 and T be operators on H. If $\Gamma(T_1) \supset \Gamma(\mathrm{T})$, then T_1 is said to be an extension of T and we write $T_1 \supset T$. Equivalently, $T_1 \supset T$ if and only if $D(T_1) \supset D(T)$ and $T_1\phi = T\phi$ for all $\phi \in D(T)$.
Definition 110. An operator T is #-closable if it has a #-closed extension. Every #-closable operator has a smallest

#-closed extension, called its #-closure, which we denote by #-T.
Theorem 50. If T is #-closable, then $\Gamma(\#\text{-}\overline{T}) = \#\text{-}\overline{\Gamma(T)}$.
Definition 111. Let $D(T^*)$ be the set of $\varphi \in H$ for which there is an $\xi \in H$ with $(T\psi, \varphi) = (\psi, \xi)$ for all $\psi \in D(T)$. For each $\varphi \in D(T^*)$, we define $T^*\varphi = \xi$.The operator T^* is called the adjoint of T. Note that $\varphi \in D(T^*)$ if and only if $|(T\psi, \varphi)| \le C\|\psi\|_\#$ for all $\psi \in D(T)$. Note that $S \subset T$ implies $T^* \subset S$.
Remark 16. Note that for ξ to be uniquely determined by the condition $(T\psi, \varphi) = (\psi, \xi)$ one needs the fact that $D(T)$ is #-dense in H. If the domain $D(T^*)$ is #-dense in H, then we can define $T^{**} = (T^*)^*$.
Theorem 51. Let T be a #-densely defined operator on a non-Archimedean Hilbert space H. Then: (a) T^* is #-closed. (b) The operator T is #-closabie if and only if $D(T^*)$ is -dense in which case $T = T^{**}$. (c) If T is #-closabie, then $(\#\text{-}\overline{T})^* = T^*$.
Definition 112. Let T be a #-closed operator on a non-Archimedean Hilbert space H. A complex number $\lambda \in {}^*\mathbb{C}_c^\#$ is in the resolvent set $\rho(T)$, if $\lambda I - T$ is a bijection of $D(T)$ onto H with a finitely or hyper finitely bounded inverse. If complex number $\lambda \in \rho(T)$, $R_\lambda = (\lambda I - T)^{-1}$ is called the resolvent of T at λ.
Definition 113. A #-densely defined operator T on a non-Archimedean Hilbert space is called symmetric or Hermitian if $T \subset T^*$, that is, $D(T) \subset D(T^*)$ and $T\varphi = T^*\varphi$ for all $\varphi \in D(T)$ and equivalently, T is symmetric if and only if $(T\varphi, \psi) = (\varphi, T\psi)$ for all $\varphi, \psi \in D(T)$.
Definition 114. A #-densely defined operator T is called self-#-adjoint if $T = T^*$, that is, if and only if T is symmetric and $D(T) = D(T^*)$.
Remark 17. A symmetric operator T is always #-closable, since $D(T)$ #-dense in H. If T is symmetric, T^* is a #-closed extension of T so the smallest #-closed extension T^{**} of T must be contained in T^*. Thus for symmetric operators, we have $T \subset T^{**} \subset T^*$, for #-closed symmetric operators we have $T = T^{**} \subset T^*$ and, for self-#-adjoint operators we have $T = T^{**} = T^*$. Thus a #-closed symmetric operator T is self-#-adjoint if and only if T^* is symmetric. Definition1. A symmetric operator T is called essentially self-adjoint if its #-closure #-$\overline{T}$ is self-#-adjoint. If T is #-closed, a subset $D \subset D(T)$ is called a core for T if #-$\overline{T \restriction D} = T$.
Remark 18. If T is essentially self-#-adjoint, then it has one and only one self-#-adjoint extension.
Definition 115. Let A be an operator on a non-Archimedean Hilbert Hilbert space $H^\#$.The set $C^{^*\infty}(A) = \bigcap_{n=1}^{^*\infty} D(A^n)$ is called the $C^{^*\infty}$-vectors for A. A vector $\varphi \in C^{^*\infty}(A)$ is called an #-analytic vector for A if

GENERALIZED TROTTER PRODUCT FORMULA

Theorem 52. Let A and B be self-adjoint operators on non-Archimedean Hilbert space $H^\#$. Suppose that the operator $A + B$ is self-#-adjoint on $D = D(A) \cap D(B)$, then the following equality holds

$$\text{s-\#-}\lim_{n\to{}^*\infty}\left[\left(Ext\text{-}\exp\left(\frac{itA}{n}\right)\right)\left(Ext\text{-}\exp\left(\frac{itB}{n}\right)\right)\right]^n = Ext\text{-}\exp[it(A+B)]. \quad (83)$$

Theorem 53. Let A and B be self-adjoint operators on non-Archimedean Hilbert space $H^\#$. Suppose that the operator $A + B$ is essentially self-#-adjoint on $D = D(A) \cap D(B)$, then the following equality holds

$$\text{s-\#-}\lim_{n\to{}^*\infty}\left[\left(Ext\text{-}\exp\left(\frac{itA}{n}\right)\right)\left(Ext\text{-}\exp\left(\frac{itB}{n}\right)\right)\right]^n = Ext\text{-}\exp[it(A+B)]. \quad (84)$$

Theorem 54. Let A and B be the generators of contraction semigroups on non-Archimedean Banach space $B^\#$.Suppose that the #-closure of $(A + B) \restriction D(A) \cap D(B)$ generates a contraction semigroup on $B^\#$. Then the following equality holds

$$\text{s-\#-}\lim_{n\to{}^*\infty}\left[\left(Ext\text{-}\exp\left(-\frac{tA}{n}\right)\right)\left(Ext\text{-}\exp\left(-\frac{tB}{n}\right)\right)\right]^n = Ext\text{-}\exp[-t(\#\text{-}\overline{A+B})]. \quad (85)$$

FOCK SPACE OVER NONARCHIMEDEAN HILBERT SPACE

Definition 116.Let $H^{\#}$ be a complex hyper infinite-dimensional non-Archimedean Hilbert space over field ${}^{*}\mathbb{C}_c^{\#}$ and denote by $H^{\#(n)}$ the n-fold tensor product: $H^{\#(n)} = Ext\text{-}\otimes_{k=1}^{n} H^{\#}, n \in {}^{*}\mathbb{N}$. Set $H^{\#(0)} = {}^{*}\mathbb{C}_c^{\#}$ and define $\mathcal{F}(H^{\#}) = Ext\text{-}\oplus_{n\in{}^{*}\mathbb{N}}\left(H^{\#(n)}\right)$. $\mathcal{F}(H^{\#})$ is called the Fock space over non-Archimedean Hilbert space $H^{\#}$. Set $H^{\#} = L_2^{\#}\left({}^{*}\mathbb{R}_c^{\#3}\right)$, then an element $\psi \in \mathcal{F}(H^{\#})$ is a hyper infinite sequence of ${}^{*}\mathbb{C}_c^{\#}$-valued functions $\psi = \{\psi_0, \psi_1(x_1), \psi_2(x_1, x_2),$ $\psi_2(x_1, x_2, x_3), \dots, \psi_n(x_1, \dots, x_n)\}, n \in {}^{*}\mathbb{N}$ and such that $|\psi_0|^2 + Ext\text{-}\sum_{n\in{}^{*}\mathbb{N}}(Ext\text{-}\int |\psi_n(x_1, \dots, x_n)|^2 d^{\#3n}x) < {}^{*}\infty$. Actually, it is not $\mathcal{F}(H^{\#})$ itself, but two of its subspaces which are used in quantum field theory. These two hyper infinite-dimensional subspaces are constructed as follows: Let P_n be the permutation group on $n \in {}^{*}\mathbb{N}$ elements and let $\{\varphi_k\}_{k=1}^{{}^{*}\infty}$be a basis for a space $H^{\#}$. For each $\sigma \in P_n$ we define an operator (which we also denote by σ) on basis elements of $H^{\#(n)}$ by $\sigma\left(Ext\text{-}\otimes_{i=1}^{n}\varphi_{k_i}\right) = Ext\text{-}\otimes_{i=1}^{n}\varphi_{k_{\sigma(i)}}$. The operator extends by linearity to a bounded operator (of #-norm one) on $H^{\#}$ and we can define $\breve{\mathbf{S}}_n^{\#} = \left(\frac{1}{n!}\right)\left(Ext\text{-}\sum_{\sigma\in P_n}\sigma\right)$. It is easily to show by definitions that $\breve{\mathbf{S}}_n^{\#2} = \breve{\mathbf{S}}_n^{\#}$ and $\breve{\mathbf{S}}_n^{\#*} = \breve{\mathbf{S}}_n^{\#}$ so $\breve{\mathbf{S}}_n^{\#}$ is an orthogonal projection. The range of $\breve{\mathbf{S}}_n^{\#}$ is called the n-fold symmetric tensor product of $H^{\#}$. We now define $\mathcal{F}_s^{\#}(H^{\#}) = Ext\text{-}\oplus_{n\in{}^{*}\mathbb{N}}\breve{\mathbf{S}}_n^{\#}H^{\#(n)}$. Non-Archimedean Hilbert space $\mathcal{F}_s^{\#}(H^{\#})$ is called the symmetric Fock space over non-Archimedean Hilbert space $H^{\#}$ or the Boson Fock space over non-Archimedean Hilbert space $H^{\#}$.

SEGAL QUANTIZATION OVER NONARCHIMEDEAN HILBERT SPACE

Let $H^{\#}$ be a complex non-Archimedean Hilbert space over field ${}^{*}\mathbb{C}_c^{\#}$ and let $\mathcal{F}(H^{\#}) = Ext\text{-}\oplus_{n\in{}^{*}\mathbb{N}}\left(H^{\#(n)}\right)$, where $H^{\#(n)} = Ext\text{-}\otimes_{k=1}^{n}H^{\#}$ be the Fock space over $H^{\#}$and let $\mathcal{F}_s(H^{\#})$ be the Boson subspace of $\mathcal{F}(H^{\#})$. Let $f \in H^{\#}$ be fixed. For vectors in $H^{\#(n)}$ of the form $\eta = Ext\text{-}\otimes_{i=1}^{n}\psi_i, n \in {}^{*}\mathbb{N}$ we define a map $b^{-}(f): H^{\#(n)} \to H^{\#(n-1)}$ by $b^{-}(f)\eta = (f, \psi_1)(Ext\text{-}\otimes_{i=2}^{n}\psi_i)$ and $b^{-}(f)$ extends by linearity to finite and hyperfinite linear combinations of such η, the extension is well defined, and $\|b^{-}(f)\eta\|_{\#} \leq \|f\|_{\#}\|\eta\|_{\#}$. Thus $b^{-}(f)$ extends to a bounded map (of #-norm $\|f\|_{\#}$) of $H^{\#(n)}$ into $H^{\#(n-1)}$. Since this holds for each $n \in {}^{*}\mathbb{N}$ (except for $n = 0$ in which case we define $b^{-}(f): H^{\#(0)} \to \{0\}$), $b^{-}(f)$ is a bounded operator of #-norm $\|f\|_{\#}$ from $\mathcal{F}(H^{\#})$ to $\mathcal{F}(H^{\#})$. It is easy to check that operator $b^{+}(f) = \left(b^{-}(f)\right)^{*}$takes each subspace $H^{\#(n)}$into $H^{\#(n+1)}$with the action $b^{+}(f)\eta = f \otimes Ext\text{-}\otimes_{i=1}^{n}\psi_i$ on product vectors. Note that the map $f \to b^{+}(f)$ is linear and the map $f \to b^{-}(f)$ is antilinear. Let S_n be the symmetrization operators introduced in previous section and then the operator $\breve{\mathbf{S}}^{\#} = Ext\text{-}\oplus_{n\in{}^{*}\mathbb{N}}\breve{\mathbf{S}}_n^{\#}$ is the projection onto the symmetric Fock space $\mathcal{F}_s(H^{\#}) = Ext\text{-}\oplus_{n\in{}^{*}\mathbb{N}}\breve{\mathbf{S}}_n^{\#}H^{\#(n)}$, we will write $\breve{\mathbf{S}}_n^{\#}H^{\#(n)} = H_s^{\#(n)}$and call $H_s^{\#(n)}$the n-particle subspace of $\mathcal{F}_s(H^{\#})$. Note that operator $b^{-}(f)$ takes space $\mathcal{F}_s(H^{\#})$ into itself, but the operator $b^{+}(f)$ does not. A vector $\psi = \left\{\psi^{(n)}\right\}_{n=1}^{{}^{*}\infty}$ with $\psi^{(n)} = 0$ for all except finite or hyperfinite set of number n is called a finite or hyperfinite particle vector correspondingly. We will denote the set of hyperfinite particle vectors by F_0. The vector $\Omega_0 = \langle 1,0,0,\dots\rangle$ is called the vacuum vector. Let A be any self-adjoint operator on $H^{\#}$ with domain of essential self-#-adjointness $D = D(A)$. Let $D_A = \left\{\psi \in F_0 | \psi^{(n)} \in Ext\text{-}\otimes_{i=1}^{n}D, n \in {}^{*}\mathbb{N}\right\}$ and define operator $d\Gamma^{\#}(A)$ on $D_A \cap H_s^{\#(n)}$ by $d\Gamma^{\#}(A)$=$A \otimes I \cdots \otimes I + I \otimes A \otimes \cdots \otimes I + \cdots + \otimes I \cdots \otimes I \otimes A$. Note that $d\Gamma^{\#}(A)$is essentially self-#-adjoint on D_A . Operator $d\Gamma^{\#}(A)$ is called the second quantization of the operator A. For example, let $A = I$, then its second quantization $N^{\#} = d\Gamma^{\#}(I)$ is essentially self-#-adjoint on F_0 and for $\psi \in H_s^{\#(n)}, N^{\#}\psi = n\psi$. $N^{\#}$ is called the number operator. If U is a unitary operator on space $H^{\#}$, we define $d\Gamma^{\#}(U)$ to be the unitary operator on $\mathcal{F}_s(H^{\#})$ which equals $Ext\text{-}\otimes_{i=1}^{n}U$ when restricted to $H_s^{\#(n)}$for $n > 0$, and which equals the identity on $H_s^{\#(0)}$. If Ext-$\exp(itA)$ is a

#-continuous unitary group on $H^{\#}$, then $\Gamma^{\#}(Ext\text{-}\exp(itA))$ is the group generated by $d\Gamma^{\#}(A)$, i.e., that expressed by the formula $\Gamma^{\#}(Ext\text{-}\exp(itA)) = Ext\text{-}\exp(itd\Gamma^{\#}(A))$.

Definition 117. We define the annihilation operator $a^{-}(f)$ on $\mathcal{F}_s(H^{\#})$ with domain F_0 by the formula

$$a^{-}(f) = \sqrt{N+1}b^{-}(f). \tag{86}$$

Operator $a^{-}(f)$ is called an annihilation operator because it takes each $(n+1)$-particle subspace into the n-particle subspace. For each ψ and η in F_0, $(\sqrt{N+1}b^{-}(f)\psi, \eta) = (\psi, S^{\#}b^{+}(f)\sqrt{N+1})$, then we get

$$(a^{-}(f))^{*} \upharpoonright F_0 = S^{\#}b^{+}(f)\sqrt{N+1}\,. \tag{87}$$

The operator $(a^{-}(f))^{*}$ is called a creation operator. Both $a^{-}(f)$ and $(a^{-}(f))^{*}$ #-closable; we denote their #-closures by $a^{-}(f)$ and $(a^{-}(f))^{*}$ also. The equation (1) implies that the Segal field operator $\Phi_S^{\#}(f)$ on F_0 defined by $\Phi_S^{\#}(f) = \frac{1}{\sqrt{2}}\left[a^{-}(f) + (a^{-}(f))^{*}\right]$ is symmetric and essentially self-#-adjoint. The mapping from $H^{\#}$ to the self-#-adjoint operators on $\mathcal{F}_s(H^{\#})$ given by $f \to \Phi_S^{\#}(f)$ is called the Segal quantization over $H^{\#}$. Note that the Segal quantization is a real linear map.

Theorem 55. Let $H^{\#}$ be hyper infinite dimensional Hilbert space over complex field ${}^{*}\mathbb{C}_c^{\#} = {}^{*}\mathbb{R}_c^{\#} + \mathrm{i}\,{}^{*}\mathbb{R}_c^{\#}$ and $\Phi_S^{\#}(f)$ the corresponding Segal quantization. Then:

(a) (self-#-adjointness) for each $f \in H^{\#}$ the operator $\Phi_S^{\#}(f)$ is essentially self-#-adjoint on $F_{\,0}$, the hyperfinite particle vectors;

(b) (cyclicity of the vacuum) the vector Ω_0 is in the domain of all hyperfinite products $Ext\text{-}\prod_{i=1}^{n}\Phi_S^{\#}(f_i)\,, n \in {}^{*}\mathbb{N}$ and the set $\{Ext\text{-}\prod_{i=1}^{n}\Phi_S^{\#}(f_i)\,|f_i \in H^{\#}, n \in {}^{*}\mathbb{N}\}$ is #-total in $\mathcal{F}_s(H^{\#})$;

(c) (commutation relations) for each $\psi \in F_0$ and $f, g \in H^{\#}$: $[\Phi_S^{\#}(f)\Phi_S^{\#}(g) - \Phi_S^{\#}(g)\Phi_S^{\#}(f)]\psi = i\mathrm{Im}(f,g)_{H^{\#}}\psi$;

(c') (generalized commutation relations) assuming that $(f,g)_{H^{\#}} \approx 0$ and $\psi \in F$ is a near standard vector we get $[\Phi_S^{\#}(f)\Phi_S^{\#}(g) - \Phi_S^{\#}(g)\Phi_S^{\#}(f)]\psi \approx 0$ and therefore $\mathrm{st}([\Phi_S^{\#}(f)\Phi_S^{\#}(g) - \Phi_S^{\#}(g)\Phi_S^{\#}(f)]\psi) = 0$;

(d) let $W(f)$ denotes the external unitary operator $Ext\text{-}\exp\left(i\Phi_S^{\#}(f)\right)$ then
$W(f+g) = \left[Ext\text{-}\exp\left(-\frac{i}{2}\mathrm{Im}(f,g)_{H^{\#}}\right)\right]W(f)W(g)$;

(e) (#-continuity) if $\{f_n\}_{n=1}^{^{*}\infty}$ is hyper infinite sequence such as $\#\text{-}\lim_{n\to{}^{*}\infty} f_n = f$ in $H^{\#}$ then:

1) $\#\text{-}\lim_{n\to{}^{*}\infty} W(f_n)\psi$ exists for all $\psi \in \mathcal{F}_s(H^{\#})$ and $\#\text{-}\lim_{n\to{}^{*}\infty} W(f_n)\psi = W(f)\psi$

2) $\#\text{-}\lim_{n\to{}^{*}\infty} \Phi_S^{\#}(f_n)\psi$ exists for all $\psi \in F_0$ and $\#\text{-}\lim_{n\to{}^{*}\infty} \Phi_S^{\#}(f_n)\psi = \Phi_S^{\#}(f)\psi$

(e) For every unitary operator U on $H^{\#}$, $\Gamma^{\#}(U): D\left(\#\text{-}\overline{\Phi_S^{\#}(f)}\right) \to D\left(\#\text{-}\overline{\Phi_S^{\#}(Uf)}\right)$ and for all $\psi \in D\left(\#\text{-}\overline{\Phi_S^{\#}(Uf)}\right)$,
$\Gamma^{\#}(U)\left(\#\text{-}\overline{\Phi_S^{\#}(Uf)}\right)\Gamma^{\#-1}(U)\psi = \#\text{-}\overline{\Phi_S^{\#}(Uf)}\psi$ for all $f \in H^{\#}$.

Remark 18. Henceforth we use $\Phi_S^{\#}(f)$ to denote the #-closure $\#\text{-}\overline{\Phi_S^{\#}(f)}$ of $\Phi_S^{\#}(f)$.

Definition 118. For each $m > 0, m \in \mathbb{R}$ let $H_m^{\#} = \{p \in {}^{*}\mathbb{R}_c^{\#4} | p \cdot \tilde{p} = m^2, p_0 > 0\}$, where $\tilde{p} = (p^0, -p^1, -p^2, -p^3)$, the sets $H_m^{\#}$, are called mass hyperboloids, are invariant under canonical Lorentz group ${}^{\sigma}L_{+}^{\uparrow}$. Let j_m be the #-homeomorphism of $H_m^{\#}$ onto ${}^{*}\mathbb{R}_c^{\#3}$ given by $j_m: \langle p_0, p_1, p_2, p_3\rangle \to \langle p_1, p_2, p_3\rangle = \boldsymbol{p}$. Define a #-measure $\Omega_m^{\#}$ on $H_m^{\#}$ for any #-measurable set $E \subset H_m^{\#}$ by

$$\Omega_m^{\#}(E) = Ext\text{-}\int_{j_m(E)} \frac{d^{\#3}\boldsymbol{p}}{\sqrt{|\boldsymbol{p}|^2 + m^2}}\,. \tag{88}$$

Theorem 56. Let $\mu^{\#}$ be a polynomially bounded #-measure with support in $\#\text{-}\bar{V}_{+}$. If $\mu^{\#}$ is ${}^{\sigma}L_{+}^{\uparrow} = L_{+}^{\uparrow}$- invariant, there exists a polynomially bounded #-measure $\rho^{\#}$ on $[0, \infty^{\#})$ and a constant c so that for any $f \in S^{\#}({}^{*}\mathbb{R}_c^{\#4})$

$$Ext\text{-}\int_{{}^{*}\mathbb{R}_c^{\#4}} f\, d^{\#}\mu^{\#} = cf(0) + Ext\text{-}\int_0^{^{*}\infty} d^{\#}\rho^{\#}(m)\left(Ext\text{-}\int_{{}^{*}\mathbb{R}_c^{\#3}} \frac{f\left(\sqrt{|\boldsymbol{p}|^2+m^2}, p_1, p_2, p_3\right) d^{\#3}\boldsymbol{p}}{\sqrt{|\boldsymbol{p}|^2+m^2}}\right). \tag{89}$$

Definition119. Let $\boldsymbol{\mathcal{F}}(f)$ be a linear #-continuous functional $\mathcal{F}: S^{\#}_{\text{fin}}({}^*\mathbb{R}^{\#4}_c) \to {}^*\mathbb{R}^{\#}_c$. Functional $\mathcal{F}$ is $L^{\uparrow}_{+}$- $\approx$ -invariant if for any $\Lambda \in L^{\uparrow}_{+}$ the following property holds $\mathcal{F}(f(\Lambda \mathrm{x})) \approx \mathcal{F}(f)$ for all $f \in S^{\#}_{\text{fin}}({}^*\mathbb{R}^{\#4}_c)$.
Theorem 57. Let $\mu^{\#}$ be a polynomially bounded $L^{\uparrow}_{+}$- invariant #-measure with support in #-$\bar{V}_{+}$. Let $\mathcal{F}(f)$ be a linear #-continuous functional $\mathcal{F}: S^{\#}_{\text{fin}}({}^*\mathbb{R}^{\#4}_c) \to {}^*\mathbb{R}^{\#}_{c,\text{fin}}$ defined by $Ext\text{-}\int_{{}^*\mathbb{R}^{\#4}_c} f\, d^{\#}\mu^{\#}$ and there exists a polynomially bounded #-measure $\rho^{\#}$ on $[0,\infty^{\#})$ such that $\int_0^{{}^*\infty} d^{\#}\rho^{\#}(m) \in {}^*\mathbb{R}^{\#}_{c,\text{fin}}$ and a constant $c \in {}^*\mathbb{R}^{\#}_{c,\text{fin}}$ so that (1) holds. Then for any $f \in S^{\#}_{\text{fin}}({}^*\mathbb{R}^{\#4}_c)$ and for any $\varkappa \in {}^*\mathbb{R}^{\#}_{c,\infty}$ the following property holds

$$\mathcal{F}(f) \approx cf(0) + Ext\text{-}\int_0^{{}^*\infty} d^{\#}\rho^{\#}(m)\left(Ext\text{-}\int_{|\boldsymbol{p}|\leq\varkappa} \frac{f\left(\sqrt{|\boldsymbol{p}|^2+m^2}, p_1, p_2, p_3\right)d^{\#3}\boldsymbol{p}}{\sqrt{|\boldsymbol{p}|^2+m^2}}\right). \quad (90)$$

Definition 120. Let $\chi(\varkappa, \boldsymbol{p})$ be a function such that: $\chi(\varkappa, \boldsymbol{p}) \equiv 1$ if $|\boldsymbol{p}| \leq \varkappa$, $\chi(\varkappa, \boldsymbol{p}) \equiv 0$ if $|\boldsymbol{p}| > \varkappa$. Define a #-measure $\Omega^{\#}_{m,\varkappa}$ on $H^{\#}_m$ by

$$\Omega^{\#}_{m,\varkappa}(E) = Ext\text{-}\int_{j_m(E)} \frac{\chi(\varkappa, \boldsymbol{p})d^{\#3}\boldsymbol{p}}{\sqrt{|\boldsymbol{p}|^2+m^2}}. \quad (91)$$

We use the Segal quantization to define the free Hermitian scalar field of mass m. We take $H^{\#} = L^{\#}_2(H^{\#}_m, d^{\#}\Omega^{\#}_{m,\varkappa})$. For each $f \in S^{\#}({}^*\mathbb{R}^{\#4}_c)$ we define $Ef \in H^{\#}$ by $Ef = 2\pi(Ext\text{-}\hat{f}) \upharpoonright H^{\#}_m$ where the Fourier transform is defined in terms of the Lorentz invariant inner product $p \cdot \tilde{x}$: $Ext\text{-}\hat{f} = \frac{1}{4\pi^2}\left(Ext\text{-}\int_{{}^*\mathbb{R}^{\#4}_c} Ext\text{-}\exp[i(p \cdot \tilde{x})]d^{\#4}x\right)$. If $\Phi^{\#}_{S,\varkappa}(\cdot)$ is the Segal quantization over $L^{\#}_2(H^{\#}_m, d^{\#}\Omega^{\#}_{m,\varkappa})$, we define for each ${}^*\mathbb{R}^{\#}_c$- valued $f \in S^{\#}({}^*\mathbb{R}^{\#4}_c)$: $\Phi^{\#}_{m,\varkappa}(f) = \Phi^{\#}_{S,\varkappa}(Ef)$ and for each ${}^*\mathbb{C}^{\#}_c$- valued $f \in S^{\#}({}^*\mathbb{R}^{\#4}_c)$ we define $\Phi^{\#}_{m,\varkappa}(f) = \Phi^{\#}_{m,\varkappa}(\mathrm{Re}f) + \mathrm{i}\Phi^{\#}_{m,\varkappa}(\mathrm{Im}f)$.
Definition 121. The mapping $f \to \Phi^{\#}_{m,\varkappa}(f)$ is called the free non-Archimedean Hermitian scalar field of mass m.
Definition 122. On $L^{\#}_2(H^{\#}_m, d^{\#}\Omega^{\#}_{m,\varkappa})$ we define the following unitary representation of the restricted Poincare group $L^{\uparrow}_{+}$: $(U_m(a,\Lambda)\psi)(p) = (Ext\text{-}\exp[i(p \cdot \tilde{a})])\psi(\Lambda^{-1}p)$ where we are using Λ to denote both an element of the abstract restricted Lorentz group and the corresponding element in the standard representation on $\mathbb{R}^4$.
Definition 123. The #-conjugation on a non-Archimedean Hilbert space $H^{\#}$ is an antilinear #-isometry $\mathbf{C}^{\#}$ so that the following equality holds $\mathbf{C}^{\#2} = I$.
Definition 124. Let $H^{\#}$ be a non-Archimedean Hilbert space over field ${}^*\mathbb{C}^{\#}_c$, $\Phi^{\#}_S(\cdot)$ the associated Segal quantization. Let $H^{\#}_{\mathbf{C}^{\#}} = \{f | \mathbf{C}^{\#}f = f\}$. For each $f \in H^{\#}_{\mathbf{C}^{\#}}$ we define $\varphi^{\#}(f) = \Phi^{\#}_S(f)$ and $\pi^{\#}(f) = \Phi^{\#}_S(if)$, the map $f \to \varphi^{\#}(f)$ is called the canonical free field over the doublet $\langle \boldsymbol{H}^{\#}, \mathbf{C}^{\#}\rangle$ and the map $f \to \pi^{\#}(f)$ is called the canonical conjugate momentum.
Theorem 58. Let $H^{\#}$ be a non-Archimedean Hilbert space over field ${}^*\mathbb{C}^{\#}_c$ with #-conjugation $\mathbf{C}^{\#}$. Let $\varphi^{\#}(\cdot)$ and $\pi^{\#}(\cdot)$ be the corresponding canonical fields. Then: (a) For each $f \in H^{\#}_{\mathbf{C}^{\#}}$, $\varphi^{\#}(f)$ is essentially self-#-adjoint on F_0.
(b) $\{\varphi^{\#}(f) | f \in H^{\#}_{\mathbf{C}^{\#}}\}$ is a commuting family of self-#-adjoint operators. (c) Ω_0 is a #-cyclic vector for the family $\{\varphi^{\#}(f) | f \in H^{\#}_{\mathbf{C}^{\#}}\}$. (d) If $\{f_n\}_{n=1}^{{}^*\infty}$ is hyper infinite sequence such as #-$\lim_{n\to{}^*\infty} f_n = f$ in $H^{\#}_{\mathbf{C}^{\#}}$, then #-$\lim_{n\to{}^*\infty} \varphi^{\#}(f_n)\psi$ exists for all $\psi \in F_0$ and #-$\lim_{n\to{}^*\infty} \varphi^{\#}(f_n)\psi = \varphi^{\#}(f)\psi$.
(e) #-$\lim_{n\to{}^*\infty}(Ext\text{-}\exp[i\varphi^{\#}(f_n)]\psi) = Ext\text{-}\exp[i\varphi^{\#}(f)]\psi$ for all $\psi \in \mathcal{F}_s(H^{\#})$. (f) Properties (a)-(e) hold with $\varphi^{\#}(f)$ replaced by $\pi^{\#}(f)$. (g) If $f, g \in H^{\#}_{\mathbf{C}^{\#}}$, then $[\varphi^{\#}(f)\varphi^{\#}(g) - \varphi^{\#}(g)\varphi^{\#}(f)]\psi = i(f,g)$ for all $\psi \in \mathcal{F}_s(H^{\#})$ and $(Ext\text{-}\exp[i\varphi^{\#}(f)])(Ext\text{-}\exp[i\pi^{\#}(f)]) = (Ext\text{-}\exp[i(f,g)])(Ext\text{-}\exp[i\pi^{\#}(f)])(Ext\text{-}\exp[i\varphi^{\#}(f)])$.
Definition1. We write now $f \in L^{\#}_2(H^{\#}_m, d^{\#}\Omega^{\#}_{m,\varkappa})$ as $f(p_0, \boldsymbol{p})$ and define now the #-conjugation $\mathbf{C}^{\#}$ by $\mathbf{C}^{\#}(f)(p_0, \boldsymbol{p}) = \overline{f(p_0, -\boldsymbol{p})}$. Note that $\mathbf{C}^{\#}$ is well-defined on $f \in L^{\#}_2(H^{\#}_m, d^{\#}\Omega^{\#}_{m,\varkappa})$ since $\langle p_0, -\boldsymbol{p}\rangle \in H^{\#}_m$ if and only if $\langle p_0, \boldsymbol{p}\rangle \in H^{\#}_m$.
Definition 125. We denote the canonical fields corresponding to $\mathbf{C}^{\#}$ by $\varphi^{\#}(\cdot)$ and $\pi^{\#}(\cdot)$ and define $\varphi^{\#}_{m,\varkappa}(f) = \varphi^{\#}(Ef)$ and $\pi^{\#}_{m,\varkappa}(f) = \pi^{\#}(\mu(\boldsymbol{p})Ef)$, $\mu(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2 + m^2}$ for ${}^*\mathbb{R}^{\#}_c$- valued $f \in L^{\#}_2({}^*\mathbb{R}^{\#4}_c)$, extending to all of

$L_2^{\#}({}^*\mathbb{R}_c^{\#4})$ by linearity. We let now $D_{S_{\text{fin}}^{\#}} = \{\psi | \psi \in F_0, \psi^{(n)} \in S_{\text{fin}}^{\#}({}^*\mathbb{R}_{c,\text{fin}}^{\#3n})\}$ and for each $p \in {}^*\mathbb{R}_c^{\#3}$ we define the operator $a(p)$ on $\mathcal{F}_s\left(L_2^{\#}({}^*\mathbb{R}_c^{\#3})\right)$ with domain $D_{S_{\text{fin}}^{\#}}$ by $(a(p)\psi)^{(n)} = \sqrt{n+1}\,\psi^{(n+1)}(p, k_1, \dots k_n)$, thus the formal #-adjoint of the operator $a(p)$ reads $(a^{\dagger}(p)\psi)^{(n)} = \frac{1}{\sqrt{n}}\sum_{l=1}^{n}\delta^{(3)}(p-k_l)\psi^{(n-1)}(k_1, \dots, k_{l-1}, k_{l+1}, \dots, k_n)$. Note that the formulas

$$a(g) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} a(p)g(-p)d^{\#3}p, \tag{92}$$

$$a^{\dagger}(g) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} a^{\dagger}(p)g(p)d^{\#3}p \tag{93}$$

hold for all $g \in S_{\text{fin}}^{\#}({}^*\mathbb{R}_c^{\#3})$ if the equalities (1)-(1) are understood in the sense of quadratic forms. That is, (1) means that for $\psi_1, \psi_2 \in D_{S_{\text{fin}}^{\#}} : (\psi_1, a(g)\psi_2) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}(\psi_1, a(p)\psi_2)g(-p)d^{\#3}p$ and similarly (1) means that for $\psi_1, \psi_2 \in D_{S_{\text{fin}}^{\#}} : (\psi_1, a(g)\psi_2) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}(\psi_1, a^{\dagger}(p)\psi_2)g(p)d^{\#3}p$. The particles number operator reads

$$N^{\#} = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} a^{\dagger}(p)a(p)\,d^{\#3}p. \tag{94}$$

The generator of time translations in the free scalar field theory of mass m is given by

$$H_0 = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \mu(p)a^{\dagger}(p)a(p)\,d^{\#3}p. \tag{95}$$

We express the free scalar field and the time zero fields in terms of $a^{\dagger}(p)$ and $a(p)$ as quadratic forms on $D_{S_{\text{fin}}^{\#}}$.

$$\Phi_{m,\varkappa}^{\#}(x,t) = (2\pi)^{-3/2}Ext\text{-}\int_{|p|\le\varkappa}\left\{\left(Ext\text{-}\exp(\mu(p)t - ipx)\right)a^{\dagger}(p) + \left(Ext\text{-}\exp(\mu(p)t + ipx)\right)a\,(p)\right\}\frac{d^{\#3}p}{\sqrt{2\mu(p)}}, \tag{96}$$

$$\varphi_{m,\varkappa}^{\#}(x) = (2\pi)^{-3/2}Ext\text{-}\int_{|p|\le\varkappa}\left\{\left(Ext\text{-}\exp(-ipx)\right)a^{\dagger}(p) + \left(Ext\text{-}\exp(ipx)\right)a\,(p)\right\}\frac{d^{\#3}p}{\sqrt{2\mu(p)}}, \tag{97}$$

$$\pi_{m,\varkappa}^{\#}(x) = (2\pi)^{-3/2}Ext\text{-}\int_{|p|\le\varkappa}\left\{\left(Ext\text{-}\exp(-ipx)\right)a^{\dagger}(p) + \left(Ext\text{-}\exp(ipx)\right)a\,(p)\right\}\frac{d^{\#3}p}{\sqrt{\mu(p)/2}}. \tag{98}$$

Theorem 59. Let $n_1, n_2 \in \mathbb{N}$ and suppose that $W(k_1, \dots k_{n_1}, p_1, \dots, p_{n_2}) \in L_2^{\#}\left({}^*\mathbb{R}_c^{\#3(n_1+n_2)}\right)$ where $W(k_1, \dots k_{n_1}, p_1, \dots, p_{n_2})$ is a ${}^*\mathbb{C}_{\text{c,fin}}^{\#}$-valued function on ${}^*\mathbb{R}_c^{\#3(n_1+n_2)}$. Then there is a unique operator T_W on $\mathcal{F}_s\left(L_2^{\#}({}^*\mathbb{R}_c^{\#3})\right)$ so that $D_{S_{\text{fin}}^{\#}} \subset D(T_W)$ is a #- core for T_W.

1) As ${}^*\mathbb{C}_c^{\#}$-valued quadratic forms on $D_{S_{\text{fin}}^{\#}} \times D_{S_{\text{fin}}^{\#}}$

$$T_W = Ext\text{-}\int_{{}^*\mathbb{R}^{3(n_1+n_2)}} W(k_1, \dots k_{n_1}, p_1, \dots, p_{n_2})\left(\prod_{i=1}^{n_1} a^{\dagger}(k_i, \varepsilon)\right)\left(\prod_{i=1}^{n_2} a(p_i, \varepsilon)\right)d^{\#3n_1}k d^{\#3n_2}p.$$

2) As ${}^*\mathbb{C}_c^{\#}$-valued quadratic forms on $D_{S_{\text{fin}}^{\#}} \times D_{S_{\text{fin}}^{\#}}$

$$T_W^* = Ext\text{-}\int_{{}^*\mathbb{R}^{3(n_1+n_2)}} \overline{W(k_1, \dots k_{n_1}, p_1, \dots, p_{n_2})}\left(\prod_{i=1}^{n_1} a^{\dagger}(k_i, \varepsilon)\right)\left(\prod_{i=1}^{n_2} a(p_i, \varepsilon)\right)d^{\#3n_1}k d^{\#3n_2}p.$$

3) If m_1 and m_2 are nonnegative integers so that $m_1 + m_2 = n_1 + n_2$, then

$$(1+N^{\#})^{-m_1/2}T_W(1+N^{\#})^{-m_2/2} \le C(m_1, m_2)\|W\|_{L_2^{\#}}.$$

4) On vectors in F_0 the operators T_W and T_W^* are given by the explicit formulas

$$\left(T_W(\psi)\right)^{l-n_2+n_1} =$$

$$K(l,n_1,n_2)\breve{\mathbf{S}}\left[Ext\text{-}\int_{|p_1|\leq\varkappa} \ldots Ext\text{-}\int_{|p_{n_2}|\leq\varkappa} W\left(k_1,\ldots k_{n_1},p_1,\ldots,p_{n_2}\right)\psi^{(l)}\left(p_1,\ldots,p_{n_2},k_1,\ldots k_{n_1}\right)d^{\#3n_2}p\right],$$

$\left(T_W(\psi)\right)^n = 0$ if $n < n_1 - n_2$,

$$\left(T_W^*(\psi)\right)^{l-n_1+n_2} =$$

$$K(l,n_2,n_1)\breve{\mathbf{S}}\left[Ext\text{-}\int_{|p_1|\leq\varkappa} \ldots\ Ext\text{-}\int_{|p_{n_2}|\leq\varkappa} \overline{W\left(k_1,\ldots k_{n_1},p_1,\ldots,p_{n_2}\right)}\,\psi^{(l)}\left(p_1,\ldots,p_{n_2},k_1,\ldots k_{n_1}\right)d^{\#3n_1}k\right]$$

$\left(T_W^*(\psi)\right)^n = 0$ if $n < n_2 - n_1$. Here $\breve{\mathbf{S}}$ is the symmetrization operator.

$Q^{\#}$-SPACE REPRESENTATION OF THE FOCK SPACE STRUCTURES

In this section the construction of a non-Archimedean $Q^{\#}$-space and $L_2^{\#}(Q^{\#},d^{\#}\mu^{\#})$, another representation of the Fock space structures are presented. In analogy with the one degree of freedom case where $\mathcal{F}^{\#}({}^*\mathbb{R}_c^{\#})$ is isomorphic to $L_2^{\#}({}^*\mathbb{R}_c^{\#},d^{\#}x)$ in such a way that $\Phi_S^{\#}(1)$ becomes multiplication by$x$, we will construct a $\sigma^{\#}$-measure space $\langle Q^{\#},\mu^{\#}\rangle$, with $\mu^{\#}(Q^{\#}) = 1$, and a unitary map $S^{\#}:\mathcal{F}_s^{\#}(H^{\#}) \to L_2^{\#}(Q^{\#},d^{\#}\mu^{\#})$ so that for each $f \in H_C^{\#}$, $S^{\#}\phi_\varkappa^{\#}(f)$ $S^{\#-1}$ acts on $L_2^{\#}(Q^{\#},d^{\#}\mu^{\#})$ by multiplication by a $\mu^{\#}$-measurable function. We can then show that in the case of the free scalar field of mass m in 4-dimensional space-time $M_4^{\#}$, $V = S^{\#}H_{I,\varkappa}^{\#}(g)S^{\#-1}$ is just multiplication by a function $V(q)$ which is in $L_2^{\#}(Q^{\#},d^{\#}\mu^{\#})$ for each $p \in {}^*\mathbb{N}$. Let $\{g_n\}_{n=1}^{^*\infty}$ be an orthonormal basis for $H^{\#}$ so that each $g \in H_C^{\#}$ and let $\{g_n\}_{n=1}^{N}, N \in {}^*\mathbb{N}$ be a finite or hyperfinite subcollection of the set $\{f_n\}_{n=1}^{^*\infty}$ .Let P_N be a set of the all external finite and hyperfinite polynomials $Ext\text{-}P[u_1,\ldots,u_N]$ and $\mathcal{F}_N^{\#}$ be the #-closure of the set $\{Ext\text{-}P[\varphi_\varkappa^{\#}(g_1),\ldots,\varphi_\varkappa^{\#}(g_N)]|P \in P_N\}$ in $\mathcal{F}_s^{\#}(H^{\#})$ and define a set $F_0^N = \mathcal{F}_N^{\#} \cap F_0$. From Theorem 55 it follows that $\varphi_\varkappa^{\#}(g_k)$ and $\pi_\varkappa^{\#}(g_k)$, for all $1 \leq k,l \leq N$ are essentially self-#-adjoint on F_0^N and that

$$(Ext\text{-}\exp[it\varphi_\varkappa^{\#}(g_k)])(Ext\text{-}\exp[it\pi_\varkappa^{\#}(g_l)]) =$$

$$\left(Ext\text{-}\exp\left[-ist\delta_{kl}\right]\right)(Ext\text{-}\exp[it\pi_\varkappa^{\#}(g_l)])(Ext\text{-}\exp[it\varphi_\varkappa^{\#}(g_k)])\ .$$

Therefore we have a representation of the generalized Weyl relations in which the vector Ω_0 satisfies the equality $([\varphi_\varkappa^{\#}(g_k)]^2 + [\pi_\varkappa^{\#}(g_l)]^2 - 1)\Omega_0 = 0$ and is cyclic for the operators $\{\varphi_\varkappa^{\#}(g_k)\}_{k=1}^{N}$. Therefore there is a unitary map $S^{\#(N)}:\mathcal{F}_N^{\#} \to L_2^{\#}({}^*\mathbb{R}_c^{\#N})$ such that: 1) $S^{\#(N)}\varphi_\varkappa^{\#}(g_k)\left(S^{\#(N)}\right)^{-1} = x_k$, 2) $S^{\#(N)}\pi_\varkappa^{\#}(g_k)\left(S^{\#(N)}\right)^{-1} = -\frac{1}{i}\frac{d^{\#}}{d^{\#}x_k}$ and 3) $S^{\#(N)}\Omega_0 = \pi^{-N/4}\left[Ext\text{-}\exp\left(-Ext\text{-}\sum_{k=1}^{N}\frac{x_k^2}{2}\right)\right]$. It is convenient to use the non-Archimedean Hilbert space $L_2^{\#}\left({}^*\mathbb{R}_c^{\#N},\pi^{-N/4}\left(Ext\text{-}\exp\left(-Ext\text{-}\sum_{k=1}^{N}\frac{x_k^2}{2}\right)\right)\right)d^{\#N}x$ instead of $L_2^{\#}({}^*\mathbb{R}_c^{\#N})$ so we let $d^{\#}\mu_k^{\#}= Ext\text{-}\exp\left(-\frac{x_k^2}{2}\right)d^{\#}x_k$ and define the operator $(Tf)(x) = \pi^{N/4}\left(Ext\text{-}\exp\left(Ext\text{-}\sum_{k=1}^{N}\frac{x_k^2}{2}\right)\right)$, Then T is a unitary map of $L_2^{\#}({}^*\mathbb{R}_c^{\#N})$ onto $L_2^{\#}\left({}^*\mathbb{R}_c^{\#N},Ext\text{-}\prod_{k=1}^{N}d^{\#}\mu_k^{\#}\right)$ and if we let $S_1^{\#(N)} = TS^{\#(N)}$ we get: 1) $S_1^{\#(N)}:\mathscr{F}_N^{\#} \to L_2^{\#}\left({}^*\mathbb{R}_c^{\#N},Ext\text{-}\prod_{k=1}^{N}d^{\#}\mu_k^{\#}\right)$,

2) $S_1^{\#(N)}\varphi_{\varkappa}^{\#}(g_k)\left(S_1^{\#(N)}\right)^{-1} = x_k$, 3) $S_1^{\#(N)}\pi_{\varkappa}^{\#}(g_k)\left(S_1^{\#(N)}\right)^{-1} = -\frac{x_k}{i} + \frac{1}{i}\frac{d^{\#}}{d^{\#}x_k}$ and 4) $S_1^{\#(N)}\Omega_0 = 1$, where 1 is the function identically one. Note that each #- measure $\mu_k^{\#}$ has mass one, which implies that

$$\langle\Omega_0, \left(Ext\text{-}\prod_{k=1}^{N} P_k\left(\varphi_{\varkappa}^{\#}(g_k)\right)\right)\Omega_0\rangle = \int_{{}^*\mathbb{R}_c^{\#N}} \left(Ext\text{-}\prod_{k=1}^{N} P_k(x_k)\right)\left(Ext\text{-}\prod_{k=1}^{N} d^{\#}\mu_k^{\#}\right) = \tag{99}$$

$$= Ext\text{-}\prod_{k=1}^{N}\int_{{}^*\mathbb{R}_c^{\#N}} P_k(x_k)\, d^{\#}\mu_k^{\#} = Ext\text{-}\prod_{k=1}^{N}\int_{{}^*\mathbb{R}_c^{\#N}} \langle\Omega_0, P_k(\varphi_{\varkappa}^{\#}(g_k)\Omega_0)\rangle.$$

Here $P_1, \therefore, P_N$ are external finite and hyperfinite polynomials. Now we can to construct directly the $\sigma^{\#}$-measure space $\langle Q^{\#}, \mu^{\#}\rangle$. We define a space $Q^{\#} = \times_{k=1}^{{}^*\infty}\, {}^*\mathbb{R}_c^{\#}$. Take the $\sigma^{\#}$-algebra generated by hyper infinite products of #-measurable sets in ${}^*\mathbb{R}_c^{\#}$ and set $\mu^{\#} = \otimes_{k=1}^{{}^*\infty}\mu_k^{\#}$. We denote the points of $Q^{\#}$ symbolically by $q = \langle q_1, q_2, \dots\rangle$, then $\langle Q^{\#}, \mu^{\#}\rangle$ is a $\sigma^{\#}$- measure space and the set of functions of the form $P(q_1, q_2, \dots)$, where P is a polynomial and $n \in {}^*\mathbb{N}$ is arbitrary, is #-dense in $L_2^{\#}(Q^{\#}, d^{\#}\mu^{\#})$. Let P be a polynomial in $N \in {}^*\mathbb{N}$ variables $P(x_1, x_2, \dots, x_N) = Ext\text{-}\sum_{l_1,\dots,l_N} c_{l_1,\dots,l_N} x_{k_1}^{l_1} \cdots x_{k_N}^{l_N}$ and define $\mathbf{S}^{\#}: P\left(\varphi_{\varkappa}^{\#}(g_{k_1}), \dots, \varphi_{\varkappa}^{\#}(g_{k_N})\right)\Omega_0 \to P\left(q_{k_1}, q_{k_2}, \dots, q_{k_N}\right)$. Then we get

$$\left(\varphi_{\varkappa}^{\#}(g_{k_1}), \dots, \varphi_{\varkappa}^{\#}(g_{k_N})\right)\Omega_0 = Ext\text{-}\sum_{\boldsymbol{l},\boldsymbol{m}} c_{\boldsymbol{l}}\bar{c}_{\boldsymbol{m}}\left(\Omega_0, \varphi_{\varkappa}^{\#}(g_{k_1})^{l_1+m_1}, \dots, \varphi_{\varkappa}^{\#}(g_{k_N})^{l_N+m_N}\Omega_0\right) =$$

$$Ext\text{-}\sum_{\boldsymbol{l},\boldsymbol{m}} c_{\boldsymbol{l}}\bar{c}_{\boldsymbol{m}}\int_{{}^*\mathbb{R}_c^{\#N}} q_{k_1}^{l_1+m_1} \times \dots \times q_N^{l_N+m_N}\left(Ext\text{-}\prod_{i=1}^{N} d^{\#}\mu_{k_i}^{\#}\right) = Ext\text{-}\int_{Q^{\#}}\left|P\left(x_{k_1}, x_{k_2}, \dots, x_{k_N}\right)\right|^2 d^{\#}\mu^{\#}.$$

By the equation (99) and the fact that each measure $\mu_{k_i}^{\#}$ has mass one. Since Ω_0 is cyclic for polynomials in the fields, $\mathbf{S}^{\#}$extends to a unitary map of $\mathcal{F}_s^{\#}(H^{\#})$ onto $L_2^{\#}(Q^{\#}, d^{\#}\mu^{\#})$.

Theorem 60. Let $\varphi_{m,\varkappa}^{\#}(x)$, $\varkappa \in {}^*\mathbb{R}_{c,\infty}^{\#}$be the free scalar field of mass m (in 4-dimensional space-time) at time zero. Let $g \in L_1^{\#}({}^*\mathbb{R}_c^{\#3}) \cap L_2^{\#}({}^*\mathbb{R}_c^{\#3})$ and define $H_{I,\varkappa,\lambda(\varkappa)}(g) = \lambda(\varkappa)\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} g(x){:}\ \varphi_{m,\varkappa}^{\#4}(x){:}\ d^{\#3}x\right)$, where $\lambda(\varkappa) \in {}^*\mathbb{R}_{c,\approx}^{\#}$. Let $\mathbf{S}^{\#}$ denote the unitary map $\mathbf{S}^{\#}: \mathcal{F}_s^{\#}(H^{\#}) \to L_2^{\#}(Q^{\#}, d^{\#}\mu^{\#})$ constructed above. Then $V = \mathbf{S}^{\#}H_{I,\varkappa,\lambda}(g)\mathbf{S}^{\#-1}$is multiplication by a function $V_{\varkappa,\lambda}(q)$ which satisfies: (a) $V_{\varkappa,\lambda}(q) \in L_p^{\#}(Q^{\#}, d^{\#}\mu^{\#})$ for all $p \in {}^*\mathbb{N}$. (b) $Ext\text{-}\exp\left(-tV_{\varkappa,\lambda}(q)\right) \in L_1^{\#}(Q^{\#}, d^{\#}\mu^{\#})$ for all $t \in [0, {}^*\infty)$.

Proof: (a) Note that $\varphi_{m,\varkappa}^{\#}(x)$ is a well-defined operator-valued function of $x \in {}^*\mathbb{R}_c^{\#3}$. We define now : $\varphi_{m,\varkappa}^{\#4}(x)$: by moving all the $a^{\dagger}$'s to the left in the formal expression for $\varphi_{m,\varkappa}^{\#4}(x)$. By Theorem 59 : $\varphi_{m,\varkappa}^{\#4}(x)$: is also a well-defined operator for each $x \in {}^*\mathbb{R}_c^{\#3}$. Notice that for each $x \in {}^*\mathbb{R}_c^{\#3}$ operator : $\varphi_{m,\varkappa}^{\#4}(x)$: takes F_0 into itself. Thus for each $x \in {}^*\mathbb{R}_c^{\#3}$ operator : $\varphi_{m,\varkappa}^{\#4}(x)$: reads : $\varphi_{m,\varkappa}^{\#4}(x) \coloneqq \varphi_{m,\varkappa}^{\#4}(x) + d_2(\varkappa)\,\varphi_{m,\varkappa}^{\#2}(x) + d_1(\varkappa)$ where the coefficients $d_1(\varkappa)$ and $d_2(\varkappa)$ are hyperfinite constant independent of x. For each $x \in {}^*\mathbb{R}_c^{\#3}$, $\mathbf{S}^{\#}\varphi_{m,\varkappa}^{\#}(x)(g)\mathbf{S}^{\#-1}$is the operator on #-measurable space $L_2^{\#}(Q^{\#}, d^{\#}\mu^{\#})$ which acts by multiplying by the function $Ext\text{-}\sum_{k=1}^{{}^*\infty} c_k(x,\varkappa)q_k$ where $c_k(x,\varkappa) = (2\pi)^{-3/2}\left(g_k, \left(Ext\text{-}\exp(ipx)\right)\chi(\varkappa,p)\mu(p)^{-1/2}\right)$ and $\chi(\varkappa,p) \equiv 1$ if $|p| \leq \varkappa$, $\chi(\varkappa,p) \equiv 0$ if $|p| > \varkappa$. Note that

$$Ext\text{-}\sum_{k=1}^{{}^*\infty}|c_k(x,\varkappa)|^2 = (2\pi)^{-3/2}\|\chi(\varkappa,p)\mu(p)\|_{\#2}^2, \tag{100}$$

so the functions $\mathbf{S}^{\#}\varphi_{m,\varkappa}^{\#4}(x)(g)\mathbf{S}^{\#-1}$ and $\mathbf{S}^{\#}\varphi_{m,\varkappa}^{\#2}(x)(g)\mathbf{S}^{\#-1}$ are in $L_2^{\#}(Q^{\#}, d^{\#}\mu^{\#})$ and the $L_2^{\#}(Q^{\#}, d^{\#}\mu^{\#})$ norms are uniformly bounded in x. Therefore, since $g \in L_1^{\#}({}^*\mathbb{R}_c^{\#3})$, $\mathbf{S}^{\#}H_{I,\varkappa,\lambda(\varkappa)}(g)\mathbf{S}^{\#-1}$operates on $L_2^{\#}(Q^{\#}, d^{\#}\mu^{\#})$ by multiplication by some $L_2^{\#}(Q^{\#}, d^{\#}\mu^{\#})$-function which we denote by $V_{I,\varkappa,\lambda(\varkappa)}(q)$. Consider now the expression for $H_{I,\varkappa,\lambda(\varkappa)}(g)\Omega_0$. This is a vector $(0,0,0,0,\psi^{\#4},0,\dots)$ with

$$\psi^{\#4}(p_1,p_2,p_3,p_4) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \frac{\lambda(\varkappa)g(x)\chi(\varkappa,p)\left(Ext\text{-}\exp\left(-ix\sum_{i=1}^{i=4}p_i\right)\right)d^3x}{(2\pi)^{3/2}\prod_{i=1}^{4}[2\mu(p_i)]^{1/2}} = \frac{\lambda(\varkappa)\prod_{i=1}^{4}\chi(\varkappa,p_i)\left(Ext\text{-}\hat{g}\left(\sum_{i=1}^{i=4}p_i\right)\right)}{(2\pi)^{9/2}\prod_{i=1}^{4}[2\mu(p_i)]^{1/2}} \tag{101}$$

Here $|p_i| \leq \varkappa, 1 \leq i \leq 4$. We choose now the parameter $\lambda = \lambda(\varkappa) \approx 0$ such that $\|\psi^{\#4}\|_{\#2}^2 \in \mathbb{R}$ and therefore we obtain $\| H_{I,\varkappa,\lambda(\varkappa)}(g)\Omega_0\|_{\#2}^2 \in \mathbb{R}$, since $\| H_{I,\varkappa,\lambda(\varkappa)}(g)\Omega_0\|_{\#2}^2 = \|\psi^{\#4}\|_{\#2}^2$. But, since $\mathbf{S}^{\#}\Omega_0 = 1$, we get the equalities

$$\| H_{I,\varkappa,\lambda(\varkappa)}(g)\Omega_0\|_{\#2} = \|\mathbf{S}^{\#} H_{I,\varkappa,\lambda(\varkappa)}(g)\mathbf{S}^{\#-1}\|_{L_2^{\#}(Q^{\#},d^{\#}\mu^{\#})} = \|V_{I,\varkappa,\lambda(\varkappa)}(q)\|_{L_2^{\#}(Q^{\#},d^{\#}\mu^{\#})}. \quad (102)$$

From (101)-(102) we get that $\|V_{I,\varkappa,\lambda(\varkappa)}(q)\|_{L_2^{\#}(Q^{\#},d^{\#}\mu^{\#})} \in \mathbb{R}$. It is easily verify that each polynomial $P(q_1, q_2, \ldots, q_n), n \in {}^*\mathbb{N}$ is in the domain of the operator $V_{I,\varkappa,\lambda(\varkappa)}(q)$ and $\mathbf{S}^{\#} H_{I,\varkappa,\lambda(\varkappa)}(g)\mathbf{S}^{\#-1} \equiv V_{I,\varkappa,\lambda(\varkappa)}(q)$ on that domain. Since Ω_0 is in the domain of $H^p{}_{I,\varkappa,\lambda(\varkappa)}(g), p \in {}^*\mathbb{N}$, 1 is in the domain of the operator $V^p{}_{I,\varkappa,\lambda(\varkappa)}(q)$ for all $p \in {}^*\mathbb{N}$. Thus, for all $p \in {}^*\mathbb{N}$ $V_{I,\varkappa,\lambda(\varkappa)}(q) \in L_{2p}^{\#}(Q^{\#}, d^{\#}\mu^{\#})$, since $\mu^{\#}(Q^{\#})$ is finite, we conclude that $V_{I,\varkappa,\lambda(\varkappa)}(q) \in L_p^{\#}(Q^{\#}, d^{\#}\mu^{\#})$ for all $p \in {}^*\mathbb{N}$. (b) Remind Wick's theorem asserts that $:\varphi_{m,\varkappa}^{\#j}(x) := \sum_{i=0}^{[j/2]}(-1)^i \frac{j!}{(j-2i)!i!} c_\varkappa^i \varphi_{m,\varkappa}^{\#(j-2i)}(x)$ with $c_\varkappa = \|\varphi_{m,\varkappa}^{\#}(x)\Omega_0\|_{\#2}^2$. For $j = 4$ we get $-O(c_\varkappa^2) \leq$ $:\varphi_{m,\varkappa}^{\#4}(x):$ and therefore $-\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} g(x)\, d^{\#3}x\right) O(c_\varkappa^2) \leq H_{I,\varkappa,\lambda(\varkappa)}(g)$.Finally we obtain $Ext\text{-}\int_{Q^{\#}} Ext\text{-}\exp\left(-t(:\varphi_{m,\varkappa}^{\#4}(x):)\right) d^{\#}\mu^{\#} \leq Ext\text{-}\exp\left(O(c_\varkappa^2)\right)$ and this inequality finalized the proof.

GENERALIZED HAAG KASTLER AXIOMS

Definition126. A non- Archimedean Banach algebra $A_{\#}$ is a complex #-algebra over field ${}^*\mathbb{C}_c^{\#}$ (or ${}^*\mathbb{C}_{c,\text{fin}}^{\#} = {}^*\mathbb{R}_{c,\text{fin}}^{\#} + i\,{}^*\mathbb{R}_{c,\text{fin}}^{\#}$) which is a non-Archimedean Banach space under a ${}^*\mathbb{R}_c^{\#}$ -valued -norm which is sub multiplicative, i.e., $\|xy\|_{\#} \leq \|x\|_{\#}\|y\|_{\#}$for all $x, y \in A_{\#}$. An involution on a non- Archimedean Banach algebra $A_{\#}$ is a conjugate-linear isometric antiautomorphism of order two denoted by $x \mapsto x^*$, i.e.,$(x + y)^* = x^* + y^*$, and for all $x, y \in A_{\#}$: $(xy)^* = y^*x^*$, $(\lambda x)^* = \bar{\lambda}x$,$(x^*)^* = x$, $\|x^*\|_{\#} = x$, $\lambda \in {}^*\mathbb{C}_c^{\#}$. A Banach #- algebra is a non- Archimedean Banach algebra with an involution.

Definition 127. An $C_{\#}^*$-algebra is a Banach #-algebra $A_{\#}$ satisfying the $C_{\#}^*$-axiom: for all $x \in A_{\#}$, $\|x^*x\|_{\#} = \|x\|_{\#}^2$.

Definition 128. 1) A linear operator $a: H_{\#} \to H_{\#}$ on a non-Archimedean Hilbert space $H_{\#}$ is said to be bounded if there is a number $K \in {}^*\mathbb{R}_c^{\#}$ with $\|a\xi\|_{\#} \leq K\|\xi\|_{\#}$ for all $\xi \in H_{\#}$. 2) A linear operator $a: H_{\#} \to H_{\#}$ a non-Archimedean Hilbert space $H_{\#}$ is said to be finitely bounded if there is a number $K \in {}^*\mathbb{R}_{c,\text{fin}}^{\#}$ with $\|a\xi\|_{\#} \leq K\|\xi\|_{\#}$ for all $\xi \in H_{\#}$. The infimum of all such K if exists, is called the #-norm of a, written $\|a\|_{\#}$.

Abbreviation 5. The set of all bounded operators $a: H_{\#} \to H_{\#}$ we will be denoting by $\mathcal{B}^{\#}(H_{\#})$.

Abbreviation 6. The set of all finitely bounded operators $a: H_{\#} \to H_{\#}$ we will be denoting by $\mathcal{B}_{\#}(H_{\#})$.

Remark 19. Note that $\mathcal{B}_{\#}(H_{\#})$ is a $C_{\#}^*$-algebra over field ${}^*\mathbb{C}_{c,\text{fin}}^{\#}$.

Definition 129. If $S \subseteq \mathcal{B}^{\#}(H_{\#})$ (or $\mathcal{B}_{\#}(H_{\#})$) then the commutant S' of S is $S' = \{x \in \mathcal{B}^{\#}(H_{\#}) | \forall a \in S(xa = ax)\}$.

Remark 20. The algebra $\mathcal{B}^{\#}(H_{\#})$ of bounded linear operators on a non-Archimedean Hilbert space $H_{\#}$ is a $C_{\#}^*$-algebra with involution $T \to T^*, T \in \mathcal{B}^{\#}(H_{\#})$. Clearly, any #-closed #-selfadjoint subalgebra of $\mathcal{B}^{\#}(H_{\#})$ is also a $C_{\#}^*$-algebra.

Definition 130. 1) The topology on $\mathcal{B}^{\#}(H_{\#})$ (or $\mathcal{B}_{\#}(H_{\#})$ of pointwise #-convergence on $H_{\#}$ is called the strong operator topology. A basis of neighbourhoods of $a \in \mathcal{B}^{\#}(H_{\#})$ (or $a \in \mathcal{B}_{\#}(H_{\#})$ is formed by the

$$N(a, \{\xi_i\}_{i=1}^n) =$$

2) The weak operator topology is formed by the basic neighbourhoods

$$N(a, \{\xi_i\}_{i=1}^n, \{\eta_i\}_{i=1}^n) =$$

Theorem 61. If $M = M^*$ is a subalgebra of $\mathcal{B}^{\#}(H_{\#})$ (or $\mathcal{B}_{\#}$ $(H_{\#})$ with $1 \in M$, then the following statements are equivalent: 1)

Definition 131. A subalgebra of $\mathcal{B}^{\#}(H_{\#})$ (or $\mathcal{B}_{\#}$ $(H_{\#})$ satisfying the conditions of Theorem 61is called a von Neumann #-algebra.
Theorem 62. (Generalized Gelfand-Naimark theorem) Let A be a $C_{\#}^{*}$-algebra with unit. Then there exist a non-Archimedean Hilbert space $H_{\#}$ and an #-isometric homomorphism U of A into $B(H_{\#})$ such that $Ux^* = Ux^*, x\in A$.

Abbreviation 7. We denote by $M_4^{\#} = \{{}^*\mathbb{R}_c^{\#4}, (\cdot,\cdot)\}$, the vector space ${}^*\mathbb{R}_c^{\#4}$ with the Minkowski product: $(x,y) = x_0 y_0 - x_i y_i$, $i = 1,2,3$.
Statement of the Axioms. Let $M_4^{\#}$ be Minkowski space over field ${}^*\mathbb{R}_c^{\#}$ of four space-time dimensions.

1. *Algebras of Local Observables*. To each finitely bounded #-open set $O \subset M_4^{\#}$ we assign a unital $C_{\#}^{*}$ -algebra

$$O \to \mathcal{B}_{\#}(O)$$

2. *Isotony*. If $O_1 \subset O_2$, then $\mathcal{B}(O_1)$ is the unital $C_{\#}^{*}$ -sub algebra of the unital $C_{\#}^{*}$-algebra $\mathcal{B}(O_2)$:

$$\mathcal{B}_{\#}(O_1) \subset \mathcal{B}_{\#}(O_2).$$

This axiom allow us to form the algebra of all local observables

$$\mathcal{B}_{\#\text{loc}} = \bigcup_{O\subset M_4^{\#}} \mathcal{B}_{\#}(O).$$

This is a well-defined $C_{\#}^{*}$ -algebra because given any $O_1, O_2 \subset M_4^{\#}$, both $\mathcal{B}_{\#}(O_1)$ and $\mathcal{B}_{\#}(O_2)$ are sub algebras of the $C_{\#}^{*}$ -algebra$\mathcal{B}_{\#}(O_1 \cup O_2)$. From there one can take the #-norm completion to obtain

$$\mathcal{B}_{\#} = \#\text{-}\overline{\mathcal{B}_{\#\text{loc}}}\ ,$$

called the algebra of quasi-local observables. This gives a $C_{\#}^{*}$ -algebra in which all the local observable $C_{\#}^{*}$ -algebras are embedded.

3. *Poincare Covariance*. For each Poincare transformation $g \in {}^{\sigma}P_+^{\uparrow}$, there is a $C_{\#}^{*}$- isomorphism $\alpha_g : \mathcal{B} \to \mathcal{B}$ such that

$$\alpha_g(\mathcal{B}_{\#}(O)) = \mathcal{B}_{\#}(g(O))$$

for all bounded #-open $O \subset M_4^{\#}$. For fixed $g \in \mathcal{B}_{\#}$, the map $g \to \alpha_g(A)$ is required to be #-continuous.

4. $\approx$-*Causality*. If O_1 and O_2 are spacelike separated, then all elements of $\mathcal{B}_{\#}(O_1)$ $\approx$-commute with all elements of a $C_{\#}^{*}$ -algebra $\mathcal{B}_{\#}(O_2)$

$$[\mathcal{B}_{\#}(O_1), \mathcal{B}_{\#}(O_2)] \approx 0.$$

4′. If O_1 and O_2 are space-like separated, then the standard part of the all elements of $C_{\#}^{*}$ -algebra $\mathcal{B}_{\#}(O_1)$ commute with the standard part of the all elements of $C_{\#}^{*}$ -algebra $\mathcal{B}_{\#}(O_2)$

$$\text{st}(\mathcal{B}_{\#}(O_1), \mathcal{B}_{\#}(O_2)) = 0.$$

Definition 132. If $O \subset M_4^{\#}$, we say x belongs to the future causal shadow of O if every past directed timelike or light-like trajectory beginning at x intersects with O. Essentially, O separates the past light cone of x.Likewise, we say x belongs to the past causal shadow of O if every future-directed timelike or lightlike trajectory beginning at x inter-sects with O. The causal completion or causal envelope $\hat{O}$ of O is the union of its future and past directed causal shadows. This definition of the causal completion $\hat{O}$ can be reformulated in terms of "causal complements," which are computationally easier to deal with. If $O \subset M_4^{\#}$, we define the causal complement O' of O to be the set of all points with are spacelike to all points in O. Then $O'' = \hat{O}$ is the causal completion of O. One expects the observables localized to $\hat{O}$ to be completely determined by the observables localized to O, carrying the same information.

5. *Time Evolution*.

$$\mathcal{B}_{\#}(\hat{O}) = \mathcal{B}_{\#}(O).$$

6. *Vacuum state and positive spectrum*. There exists a faithful irreducible representation $\pi_0 : \mathcal{B}_{\#} \to \boldsymbol{B}(H_{\#})$ with a unique (up to a factor) vector $\Omega \in H_{\#}$ such that Ω is cyclic and Poincaré invariant, and such that unitary representation of translations, given by

$$U(x)\pi_0(A)\Omega = \pi(\alpha_x(A))\Omega,$$

where $A \in \mathcal{B}_{\#}$ and $\alpha_x(\cdot)$ is the $C_{\#}^{*}$-isomorphism from Axiom 3 associated with translation by $x \in M_4^{\#}$, has Hermitian generators $P^{\mu}, \mu = 1,2,3$ whose joint spectrum lies in the forward light cone. The last phrase is the most physically important here; it simply states that we have energy-momentum operators whose spectrum satisfies $E^2 - \mathbf{P}^2 \gg 0$, i.e, or in other words, that the energy $E \geq 0$ and nothing can move faster than the speed of light. The vector Ω is the vacuum state This axiom does not appear to be purely algebraic; we have had to introduce an non-Archimedean Hilbert space $H_{\#}$. In fact, we can rewrite the axiom in a completely algebraic but less transparent way as follows. We postulate that there exists an vacuum state ω_0 on the $C_{\#}^{*}$ -algebra (i.e., a normalized, positive, bounded linear functional) such that the following holds $\omega_0(Q^*Q) = 0$ for all $Q \in \mathcal{B}_{\#}$ of the form

$$Q(f, A) = Ext\text{-}\int f(x)\alpha_x(A)\, d^{\#4}x$$

where $A \in \mathcal{B}_{\#}$ and $f(x)$ is a #-smooth function whose Fourier transform has bounded support disjoint from the forward light-cone centered at the origin in $M_4^{\#}$.
Remind that in a quantum system with a Hamiltonian H, the Heisenberg picture dynamics is given by the canonical formula

$$A(t) = \{Ext\text{-}\exp[itH]\}A(0)\{Ext\text{-}\exp[-itH]\}.$$

Then $A(t)$ is the observable at time t corresponding to the time zero observable $A(0)$. In our model we have hyper finitely locally correct Hamiltonians $H(g)$ but no hyper infinitely global Hamiltonian, and we construct the Heisenberg picture dynamics nonetheless. We do this by restricting the observables to lie in the local algebras $\mathcal{B}_{\#}(O)$ and by using the finite propagation speed implicit in axiom 3.
Definition 133. Let $\mathcal{F}_n^{\#}$ be the space of symmetric $L_2^{\#}({}^*\mathbb{R}_c^{\#3n})$ functions defined on ${}^*\mathbb{R}_c^{\#3n}$, $\mathcal{F}_0^{\#} = {}^*\mathbb{C}_c^{\#}$ and let $\boldsymbol{\mathcal{F}}^{\#} = Ext\text{-}\bigoplus_{n=0}^{^*\infty}\mathcal{F}_n^{\#}$, $\Omega_0 = 1 \in {}^*\mathbb{C}_c^{\#} \subset \mathcal{F}^{\#}$. Let S_n be the projection of $L_2^{\#}({}^*\mathbb{R}_c^{\#3n})$ onto $\mathcal{F}_n^{\#}$and let $D_{\#}$ be the #-dense domain in $\mathcal{F}^{\#}$ spanned algebraically by Ω_0 and vectors of the form $S_n(Ext\text{-}\prod_{k=1}^{n} f_k(k_n))$ where $f_k \in S_{\text{fin}}^{\#}({}^*\mathbb{R}_c^{\#3}, {}^*\mathbb{R}_c^{\#3}), n \in {}^*\mathbb{N}$.
Definition134. We set now

$$H_0 = Ext\text{-}\int \tfrac{1}{2}\colon\left(\pi_{\varkappa}^2(\boldsymbol{x}) + \nabla^{\#}\varphi_{\varkappa}^2(\boldsymbol{x}) + m^2\varphi_{\varkappa}^2(\boldsymbol{x})\right)\colon d^{\#3}\boldsymbol{x}. \tag{103}$$

Theorem 63. As the bilinear form on the domain $D_{\#} \times D_{\#}$

$$H_0 = Ext\text{-}\int_{|\boldsymbol{k}| \leq \varkappa} \mu(k)\, a^{\dagger}(\boldsymbol{k}) a(\boldsymbol{k}) d^{\#3}\boldsymbol{k}. \tag{104}$$

Theorem 64. (1) The operator H_0 leaves each subdomain $D_{\#} \cap \mathcal{F}_n^{\#}$ invariant. (2) The operator H_0 is essentially self-#-adjoint as an operator on the domain $D_{\#}$.

Definition 135. We set now

$$\varphi_{\varkappa,0}^{\#}(x,t) = Ext\text{-}\exp(itH_0)\varphi_{\varkappa}^{\#}(x)Ext\text{-}\exp(-itH_0) \tag{105}$$

$$\pi_{\varkappa,0}^{\#}(x,t) = Ext\text{-}\exp(itH_0)\pi_{\varkappa}^{\#}(x)Ext\text{-}\exp(-itH_0) \tag{106}$$

$$\varphi_{\varkappa,0}^{\#}(f,t) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \varphi_{\varkappa,0}^{\#}(x,t)\, f(x) d^{\#3}x \tag{107}$$

$$\pi_{\varkappa,0}^{\#}(f,t) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \pi_{\varkappa,0}^{\#}(x,t)\, f(x) d^{\#3}x. \tag{108}$$

Here $\varphi_{\varkappa}^{\#}(x)$ and $\pi_{\varkappa}^{\#}(x)$ is given by formulas (97) and (98) respectively.

Remark 21. Note that $\varphi_{\varkappa,0}^{\#}(x,t)$ and $\pi_{\varkappa,0}^{\#}(x,t)$ are bilinear forms defined on $D_{\#} \times D_{\#}$.

Theorem 65. As bilinear forms on $D_{\#} \times D_{\#}$.

$$\varphi_{\varkappa,0}^{\#}(x,t) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \Delta_{\#}(x-y,t)\, \pi_{\varkappa}^{\#}(x) d^{\#3}y + Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \frac{\partial^{\#}}{\partial^{\#}t} \Delta_{\#}(x-y,t)\, \varphi_{\varkappa}^{\#}(x) d^{\#3}y \tag{109}$$

$$\pi_{\varkappa,0}^{\#}(x,t) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \frac{\partial^{\#}}{\partial^{\#}t} \Delta_{\#}(x-y,t)\, \pi_{\varkappa}^{\#}(x) d^{\#3}y + Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \frac{\partial^{\#2}}{\partial^{\#}t^2} \Delta_{\#}(x-y,t)\, \pi_{\varkappa}^{\#}(x) d^{\#3}y \tag{110}$$

Remark 22. Here $\Delta_{\#}(x-y,t)$ is the solution of the generalized Klein-Gordon equation

$$\frac{\partial^{\#2}}{\partial^{\#}t^2}\Delta_{\#}(x,t) - \frac{\partial^{\#2}}{\partial^{\#}x_1^2}\Delta_{\#}(x,t) - \frac{\partial^{\#2}}{\partial^{\#}x_2^2}\Delta_{\#}(x,t) - \frac{\partial^{\#2}}{\partial^{\#}x_3^2}\Delta_{\#}(x,t) + m^2\Delta_{\#}(x,t) = 0 \tag{111}$$

with Cauchy data $\Delta_{\#}(x,0) = 0, \frac{\partial^{\#}}{\partial^{\#}t}\Delta_{\#}(x,0) = \delta(x)$.

Remark 23. Note the distribution $\Delta_{\#}(x,t)$ has support in the double light-cone $|x| \leq |t|$.

Theorem 66. Let $f_1, f_2 \in S^{\#}({}^*\mathbb{R}_c^{\#3}, {}^*\mathbb{R}_c^{\#3})$. The operator $\varphi_{\varkappa,0}^{\#}(f,t) + \pi_{\varkappa,0}^{\#}(f,t)$ is essentially self-#-adjoint on the domain $D_{\#}$.

Definition 136. We introduce now the class $\mathfrak{I}\left(S^{\#}({}^*\mathbb{R}_c^{\#3})\right)$ of bilinear forms on $D_{\#} \times D_{\#}$ expressible as a linear combination of the forms

$$V = \sum_{j=0}^{n} \binom{n}{j} Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3n}} v(k)\, a^{\dagger}(k_1) \cdots a^{\dagger}(k_j) a(k_{j+1}) \cdots a(k_n) d^{\#3n}k \tag{112}$$

with symmetric kernels $v(k) \in S^{\#}({}^*\mathbb{R}_c^{\#3})$ having real Fourier transforms.

Theorem 67. Let $V \in \mathfrak{I}\left(S^{\#}({}^*\mathbb{R}_c^{\#3})\right)$. Then $V$is essentially self-#-adjoint on $D_{\#}$.

Theorem 68. Let O be a bounded #-open region of vector space${}^*\mathbb{R}_c^{\#3}$ and let $\mathcal{M}_{\#}(O)$ be the von Neumann algebra generated by the field operators $Ext\text{-}\exp[i\varphi_{\varkappa}^{\#}(f)]$ with $f \in S^{\#}({}^*\mathbb{R}_c^{\#3}, {}^*\mathbb{R}_c^{\#3})$ and $\operatorname{supp} f \subset O$. Let $g(x) = 0$ on ${}^*\mathbb{R}_c^{\#3} \backslash O$. Then $Ext\text{-}\exp[itH_I(g)] \in \mathcal{M}_{\#}(O)$ for all $t \in {}^*\mathbb{R}_c^{\#}$.

Definition 137. Let O be a bounded #-open region of space and let $\mathcal{B}_{\#}(O)$ be the von Neumann algebra generated by the operators $Ext\text{-}\exp\left[i\left(\varphi_{\varkappa}^{\#}(f_1) + \pi_{\varkappa}^{\#}(f_2)\right)\right]$ with $f_1, f_2 \in S^{\#}({}^*\mathbb{R}_c^{\#3}, {}^*\mathbb{R}_c^{\#3})$ and $\operatorname{supp} f_1, \operatorname{supp} f_2 \subset O$. Let O_t be the set of points with distance less than $|t|$ to O for any instant of the time t.

Theorem 69. $Ext\text{-}\exp(itH_0)\mathcal{B}_{\#}(O)Ext\text{-}\exp(-itH_0) \subset \mathcal{B}_{\#}(O_t)$.

Theorem 70. If O_1 and O_2 are disjoint bounded open regions of vector space${}^*\mathbb{R}_c^{\#3}$ then the standard part of the

operators in $\mathcal{B}_{\#}(O_1)$ commute with the standard part of the operators in operators in $\mathcal{B}_{\#}(O_2)$.
Theorem 71. Let $g \in L_2^{\#}\big(({}^*\mathbb{R}_c^{\#3})\big)$, and let $g = 0$ on open region O, then $Ext\text{-}\exp[itH_I(g)] \in \mathcal{B}_{\#}(O)'$ for all $t \in {}^*\mathbb{R}_c^{\#}$.
Theorem 72. (Free field $\approx$-Causality) Let $f_1, f_2 \in S_{\text{fin}}^{\#}({}^*\mathbb{R}_c^{\#4}, {}^*\mathbb{R}_c^{\#4})$ with $\text{supp} f_1 \subset O_1, \text{supp} f_2 \subset O_2$. We set now $\varphi_{\varkappa,0}^{\#}(f_1) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#4}} \varphi_{\varkappa,0}^{\#}(x,t)\, f_1\,(x,t) d^{\#4}x$ and $\varphi_{\varkappa,0}^{\#}(f_2) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#4}} \varphi_{\varkappa,0}^{\#}(x,t)\, f_2\,(x,t) d^{\#4}x$. If region O_1 and region O_2 are space-like separated, then $\big[\varphi_{\varkappa,0}^{\#}(f_1),\ \varphi_{\varkappa,0}^{\#}(f_2)\big]\psi \approx 0$ for all near standard vector $\psi \in H_{\#}$.
Proof: The commutator $\big[\varphi_{\varkappa,0}^{\#}(f_1),\ \varphi_{\varkappa,0}^{\#}(f_2)\big]$ reads

$$\big[\varphi_{\varkappa,0}^{\#}(f_1),\ \varphi_{\varkappa,0}^{\#}(f_2)\big] = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#4}} d^{\#3}x_1 d^{\#}\, t_1 Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#4}} d^{\#3}x_2 d^{\#} t_1 \Delta_{\varkappa}^{\#}\,(x_1 - x_2, t_1 - t_2) f_1(x_1, t_1) f_2(x_1, t_1),$$

$$\Delta_{\varkappa}^{\#}(x_1 - x_2, t_1 - t_2) = \Xi_1(x_1 - x_2, t_1 - t_2; \varkappa) - \Xi_2(x_1 - x_2, t_1 - t_2; \varkappa), \text{ where}$$

$$\Xi_1(x_1 - x_2, t_1 - t_2; \varkappa) = Ext\text{-}\int_{|\boldsymbol{k}| \le \varkappa} \{\exp\{[ip(x_1 - x_2)] - i\omega(p)(t_1 - t_2)\}\} \frac{d^{\#3}p}{\sqrt{\boldsymbol{p}^2 + m^2}}.,$$

$$\Xi_2(x_1 - x_2, t_1 - t_2; \varkappa) = Ext\text{-}\int_{|\boldsymbol{k}| \le \varkappa} \big\{-\exp\big[[ip(x_1 - x_2)] + i\omega(p)(t_1 - t_2)\big]\big\} \frac{d^{\#3}p}{\sqrt{\boldsymbol{p}^2 + m^2}},$$

$\omega(p) = \sqrt{\boldsymbol{p}^2 + m^2}$. Define $\breve{\Xi}_1(x_1 - x_2, t_1 - t_2; \varkappa)$ and $\breve{\Xi}_2(x_1 - x_2, t_1 - t_2; \varkappa)$ by
Theorem 73. (Time zero free field $\approx$-locality) Let $f_1, f_2 \in S_{\text{fin}}^{\#}({}^*\mathbb{R}_c^{\#3}, {}^*\mathbb{R}_c^{\#3})$ with $\text{supp} f_1 \subset O_1$, and $\text{supp} f_2 \subset O_2$are disjoint bounded open regions of vector space${}^*\mathbb{R}_c^{\#3}$, then $\big[\varphi_{\varkappa,0}^{\#}(f_1, 0), \varphi_{\varkappa,0}^{\#}(f_2, 0)\big] \approx 0$.
Proof: Immediately from Theorem 72.
Theorem 74. Let O be a bounded #-open region of vector space${}^*\mathbb{R}_c^{\#3}$, let $t \in {}^*\mathbb{R}_c^{\#}$, let g be a nonnegative function in $L_1^{\#}({}^*\mathbb{R}_c^{\#3}) \cap L_2^{\#}({}^*\mathbb{R}_c^{\#3})$ and let g be identically equal to one on O_t.For $A \in \mathcal{B}_{\#}(O)$, then

$$\sigma_t(A) = \{Ext\text{-}\exp[itH(g)]\} A \{Ext\text{-}\exp[-itH(g)]\}$$

is independent of g and $\sigma_t(A) \in \mathcal{B}_{\#}(O_t)$.
Proof: Let $\sigma_t^0(A) = \{Ext\text{-}\exp[itH_0]\} A \{Ext\text{-}\exp[-itH_0]\}$ and $\sigma_t^I(A) = \{Ext\text{-}\exp[itH_I]\} A \{Ext\text{-}\exp[-itH_I]\}$. Notice that generalized Trotter's product formula is valid for the unitary group $Ext\text{-}\exp\big[it\big(H_0 + H_I(g)\big)\big]$. Thus we get the following product formula for the associated automorphism group:

$$\sigma_t(A) = \#\text{-}\lim_{n \to {}^*\infty} \big[\, \big(\sigma_{t/n}^0 \sigma_{t/n}^I\big)^n (A)\big]. \tag{113}$$

Each automorphism σ_t^I maps each $\mathcal{B}_{\#}(O_s)$ into itself and is independent of g on $\mathcal{B}_{\#}(O_s)$ for $|s| \ll |t|$. To see this, let $\chi(O_s)$ be the characteristic function of a set O_s. We assert that

$$\sigma_{t/n}^I(C) = \big\{Ext\text{-}\exp\big[i(t/n) H_I\big(\chi(O_s)\big)\big]\big\} C \big\{Ext\text{-}\exp\big[-i(t/n) H_I\big(\chi(O_s)\big)\big]\big\} \tag{114}$$

for $C \in \mathcal{B}_{\#}(O_s)$ and that $\sigma_t^I(C) \in \mathcal{B}_{\#}(O_s)$. In other words the interaction automorphism has propagation speed zero and is independent of g on $\mathcal{B}_{\#}(O_s)$ for $|s| \ll |t|$. The theorem follows from (113), (114) and Theorem 69). To prove (113), we rewrite $H_I(g) = H_I\big(\chi(O_s)\big) + H_I(g[1 - \chi(O_s)])$ as a sum of commuting self-#-adjoint operators. By Theorem 67 $Ext\text{-}\exp\big[itH_I\big(\chi(O_s)\big)\big] \in \mathcal{B}_{\#}(O_s)$ and so the right side of (8.3) belongs to $\mathcal{B}_{\#}(O_s)$. By Theorem 70,

$$Ext\text{-}\exp[itH_I(g[1 - \chi(O_s)])] \in \mathcal{B}_{\#}(O_s)'$$

and (114) follows.
Definition138. Let B be a bounded #-open region of space time $M_4^{\#}$ and for any time t, let $B(t) = \{x|\ x, t \in B\}$ be the time t time slice of B. We define $\mathcal{B}_{\#}(B)$ to be the von Neumann algebra generated by

$$\bigcup_s \sigma_s\left(\mathcal{B}_\#\big(B(t)\big)\right). \quad (115)$$

Theorem 75.The generalized Haag-Kastler axioms (1)-(5) are valid for all these local algebras $\mathcal{B}_\#(B)$.
Proof (Except Lorentz rotations). The axioms (1) and (2) are obvious, while (4) follows easily from the finite propagation speed, Theorem 75, together with the time zero $\approx$-locality, Theorem 72. Because the time zero fields coincide with the time zero free fields, and because the time zero fields generate $\mathcal{B}_\#$ by Theorem 73 and the definition of the local algebras, the free field result carries over to our scalar model with interaction $H_I \neq 0$. In the Poincaré covariance axiom (3), the time translation is given by σ_t. Let $B+t$ be the time translate of the space time region $B \subset M_4^\#$. Then $(B+t)(s) = B(s-t)$ and so

$$\sigma_t\left[\bigcup_s \sigma_s\left(\mathcal{B}_\#\big(B(s)\big)\right)\right] = \bigcup_s \sigma_{s+t}\left(\mathcal{B}_\#\big(B(s)\big)\right) = \bigcup_s \sigma_s\left(\mathcal{B}_\#\big(B(s-t)\big)\right) = \bigcup_s \sigma_{s+t}\left(\mathcal{B}_\#\big(B(s+t)\big)\right) \quad (116)$$

Thus $\sigma_t\big(\mathcal{B}_\#(B)\big) = \mathcal{B}_\#(B+t)$ and axiom (3) is verified for time translations. Since the local algebras are #-norm dense in $\mathcal{B}_\#$ and since automorphisms of $C_\#^*$-algebras preserve the #-norm, σ_t extends to an automorphism of algebra $\mathcal{B}_\#$.
Definition139. To define the space translation automorphism σ_s, we set now

$$P^\mu = Ext\text{-}\int_{\|p\| \ll \varkappa} p^\mu a^\dagger(p) a(p)\, d^{\#4}p, \mu = 1,2,3;\ \sigma_t(A) = \{Ext\text{-}\exp[-ixP]\}A\{Ext\text{-}\exp[ixP]\}. \quad (117)$$

Then we get $\{Ext\text{-}\exp[-ixP]\}\varphi_\varkappa(x)\{Ext\text{-}\exp[ixP]\} = \varphi_\varkappa(x+y)$, $\{Ext\text{-}\exp[-ixP]\}\pi_\varkappa(x)\{Ext\text{-}\exp[ixP]\} = \varphi(x+y)$. The following theorem completes the proof of Theorem 73 except for Lorentz rotations.
Theorem 76. $\sigma_x\big(\mathcal{B}_\#(B)\big) = \mathcal{B}_\#(B+x)$, $\mathrm{st}(\sigma_x)$ extends up to $C_\#^*$-automorphism of $\mathcal{B}_\#$, and $\langle x,t\rangle \to \mathrm{st}(\sigma_x)\mathrm{st}(\sigma_t) == \mathrm{st}(\sigma_t)\mathrm{st}(\sigma_x)$ defines a 4-parameter abelian automorphism group of $\mathcal{B}_\#$.
Theorem 77. Let O be a bounded #-open region of space and let $\mathcal{B}_\#(O)$ be the von Neumann algebra generated by the operators $Ext\text{-}\exp\left[i\big(\varphi_\varkappa(f_1) + \pi_\varkappa(f_2)\big)\right]$ where $f_1, f_2 \in \mathcal{E}_{\mathrm{fin}}^\#({}^*\mathbb{R}_c^\#)$ and $\mathrm{supp} f_1 \subset B, \mathrm{supp} f_2 \subset B$. Then

$$Ext\text{-}\exp(itH_0)\mathcal{B}_\#(O)Ext\text{-}\exp(-itH_0) \subset \mathcal{B}_\#(O_t).$$

Remark 24.We reformulate the theorem by saying that H_0 has propagation speed at most one.
In order to obtain automorphisms for the full Lorentz group and to complete the proof of Theorem 75, there are four separate steps.
1. The first is to construct a self-#-adjoint locally correct generator for Lorentz rotations.This generator then defines a locally correct unitary group and automorphism group.
2. The second step is to prove this statement for the fields, by showing that the field $\varphi_\varkappa(x,t)$, considered as a non-standard operator valued function on a suitable domain, and is transformed locally correctly by our unitary group.
3. The third step is to show that the local algebras $\mathcal{B}_\#(B)$ are also transformed correctly.
4. The fourth final step is to reconstruct the Lorentz group automorphisms from the locally correct pieces given by the first three steps. This final step is not difficult as in two dimensional spacetime $d = 2$, see [15],[16].

Let $H_0(x)$ denote the integrand in (103), where

$$H_0 = Ext\text{-}\int H_0(\boldsymbol{x}) d^{\#3}\boldsymbol{x} = Ext\text{-}\int \tfrac{1}{2}{:}\left(\pi_\varkappa^2(\boldsymbol{x}) + \nabla^\# \varphi_\varkappa^2(\boldsymbol{x}) + m^2\varphi_\varkappa^2(\boldsymbol{x})\right){:}\, d^{\#3}\boldsymbol{x}\,. \quad (118)$$

The formal generator of classical Lorentz rotations is

$$M_\varkappa^{0k} = M_{0,\varkappa}^{0k} + M_{I,\varkappa}^{0k} = Ext\text{-}\int x^k H_{0\varkappa,}(\boldsymbol{x}) d^{\#3}\boldsymbol{x} + Ext\text{-}\int x^k{:}\,P\,(\varphi_\varkappa(\boldsymbol{x})){:}\, d^{\#3}\boldsymbol{x},\ k = 1,2,3. \quad (119)$$

The local Lorentzian rotations are

$$M_{\varkappa}^{0k}\left(g_1^{(k)}, g_2^{(k)}\right) = \varepsilon H_{0,\varkappa} + H_{0,\varkappa}\left(g_1^{(k)}\right) + H_{I,\varkappa}\left(, g_2^{(k)}\right), H_{0,\varkappa}\left(g_1^{(k)}\right) = Ext\text{-}\int H_{0,\varkappa}(\boldsymbol{x}) g_1^{(k)}(\boldsymbol{x}) d^{\#3}\boldsymbol{x}. \quad (120)$$

We require that $0 < \varepsilon$ and that: $g_1^{(k)}(x_1, x_2, x_3), g_2^{(k)}(x_1, x_2, x_3)$, $k = 1,2,3$ be nonnegative $C_0^{^*\infty}$ functions. In the second step we require more, for example that $\varepsilon + g_1^{(k)}(x_1, x_2, x_3) = x_k$ and $g_2^{(k)}(x_1, x_2, x_3) = x_k$, $k = 1,2,3$ in some local space region. This region is contained in the Cartesian product $[\varepsilon, {}^*\infty) \times [\varepsilon, {}^*\infty) \times [\varepsilon, {}^*\infty)$. By using decomposing $H_{0,\varkappa}\left(g_1^{(k)}\right)$ into a sum of a diagonal and an off-diagonal term we obtain $H_{0,\varkappa}\left(g_1^{(k)}\right) =$

$$Ext\text{-}\int v_{D,\varkappa}^{(k)}(\boldsymbol{k}, \boldsymbol{l})\, a^*(\boldsymbol{k}) a(\boldsymbol{l}) d^{\#3}\boldsymbol{k} d^{\#3}\boldsymbol{l} + Ext\text{-}\int v_{0D,\varkappa}^{(k)}(\boldsymbol{k}, \boldsymbol{l})\, [a^*(\boldsymbol{k}) a^*(\boldsymbol{l}) + a(-\boldsymbol{k}) a(-\boldsymbol{l})] d^{\#3}\boldsymbol{k} d^{\#3}\boldsymbol{l} =$$

$$= H_{0,\varkappa}^{D}\left(g_1^{(k)}\right) + H_{0,\varkappa}^{0D}\left(g_1^{(k)}\right),$$

where

$$v_{D,\varkappa}^{(k)}(\boldsymbol{k}, \boldsymbol{l}) = c_1 \chi(\boldsymbol{k}, \boldsymbol{l}, \varkappa)(\mu(\boldsymbol{k})\mu(\boldsymbol{l}) + \langle \boldsymbol{k}, \boldsymbol{l} \rangle + m^2)[\mu(\boldsymbol{k})\mu(\boldsymbol{l})]^{-1/2} \hat{g}_1^{(k)}(-k_1 + l_1, -k_2 + l_2, -k_3 + l_3),$$

$$v_{0D,\varkappa}^{(k)}(\boldsymbol{k}, \boldsymbol{l}) = c_2 \chi(\boldsymbol{k}, \boldsymbol{l}, \varkappa)(-\mu(\boldsymbol{k})\mu(\boldsymbol{l}) - \langle \boldsymbol{k}, \boldsymbol{l} \rangle + m^2)[\mu(\boldsymbol{k})\mu(\boldsymbol{l})]^{-1/2} \hat{g}_1^{(1)}(-k_1 - l_1, -k_2 - l_2, -k_3 - l_3),$$

and where $\boldsymbol{k} = (k_1, k_2, k_3), \boldsymbol{l} = (l_1, l_2, l_3), \langle \boldsymbol{k}, \boldsymbol{l} \rangle = \sum_{i=1}^{3} k_i\, l_i$, $\chi(\boldsymbol{k}, \boldsymbol{l}, \varkappa) = 1$ if $|\boldsymbol{k}| \leq \varkappa$ and $|\boldsymbol{l}| \leq \varkappa$, otherwise $\chi(\boldsymbol{k}, \boldsymbol{l}, \varkappa) = 0$.

Theorem 78. (a) $v_{0D,\varkappa}^{(k)} \in L_2^{\#}\left({}^*\mathbb{R}_c^{\#3}\right)$. (b) Function $v_{D,\varkappa}^{(k)}$ is the kernel of a nonnegative operator and $\varepsilon\mu(\boldsymbol{k})\delta(\boldsymbol{k} - \boldsymbol{l}) + \beta v_{D,\varkappa}^{(k)}$ is the kernel of a positive self-#-adjoint operator, for $\beta \geq 0$, these operators are real in configuration space.
Proof (a) is obvious (b) is proved by using a finite sequence of Kato perturbations. Let$v_{\beta}^{(k)} = \varepsilon\mu(\boldsymbol{k})\delta(\boldsymbol{k} - \boldsymbol{l}) + \beta v_{D,\varkappa}^{(k)}$ and let V_{β} and V_D denote the operators with kernels $v_{\beta}^{(k)}$ and $v_{D,\varkappa}^{(k)}$ correspondingly. The operator V_D is a sum of three terms of the form $A^* M_{g_1} A$ in configuration space, where $M_{g_1}$is multiplication by $g_1 \geq 0$. Thus $0 \leq V_D$. Moreover for γ sufficiently small, but chosen independently of β, we obtain $\gamma V_D \leq \frac{1}{2} V_0 \leq \frac{1}{2}(V_0 + \beta V_D) = \frac{1}{2} V_{\beta}$ and therefore $V_{\beta+\gamma} = V_{\beta} + \gamma V_D$ is a Kato perturbation, in the sense of bilinear forms. Consequently if the operator V_{β} is self-#-adjoint, so is $V_{\beta+\gamma}$ and $D\left(V_{\beta+\gamma}^{1/2}\right) = D\left(V_{\gamma}^{1/2}\right)$. Thus canonical finite induction starting from $V_0 = V_0^*$ shows that V_{β} is self-adjoint, for all $\beta \geq 0$.
Theorem 79. The operator $H_0^D\left(g_1^{(k)}\right)$ is nonnegative and $\varepsilon H_0 + \beta H_0^D\left(g_1^{(k)}\right)$ is self-#-adjoint, for all $\beta > 0$.
The main purpose of the third step is to give a covariant definition of the local algebras $\mathcal{B}_{\#}(B)$. Le $f \in \mathcal{E}_{\text{fin}}^{\#}(B)$ be the ${}^*\mathbb{R}_c^{\#3}$-valued function with support in B. Let $\{\alpha_i\}_{i=1}^{n}, n \in {}^*\mathbb{N}$ be finite hyperreal numbers and consider the expressions

$$\varphi_{\varkappa}^{\#}(f) = Ext\text{-}\int \varphi_{\varkappa}^{\#}(x, t) f(x, t) d^{\#3}x d^{\#}t \quad (121)$$

$$\varphi_{\varkappa}^{\#}(f, t) = Ext\text{-}\int \varphi_{\varkappa}^{\#}(x, t) f(x, t) d^{\#3}x \quad (122)$$

$$\Re(f) = Ext\text{-}\sum_{i=1}^{n} \alpha_i \varphi_{\varkappa}^{\#}(f, t_i) \quad (123)$$

$$\pi_{\varkappa}^{\#}(f, t) = Ext\text{-}\int \pi_{\varkappa}^{\#}(x, t) f(x, t) d^{\#3}x. \quad (124)$$

For $g \equiv 1$ on a sufficiently large set (the domain of dependence of the region B), the time integration in (1) #-converges strongly, and all four operators above are symmetric and defined on $D\big(H(g)\big)$.
Theorem 80.The operators (1)-(4) are essentially self-#-adjoint on any #-core for $H(g)^{1/2}$.
Theorem 81. $\mathcal{B}_{\#}(B)$ is the von Neumann algebra generated by finitely bounded functions of operators of the form (121).

Proof: note that if a hyper infinite sequence $\{A_n\}$ of self-#-adjoins operators #-converges strongly to a self #-adjoint #-limit A on a core for A then the unitary operators $Ext\text{-}\exp(itA_n)$ #-converge strongly to $Ext\text{-}\exp(itA)$. Using this fact, one can easily show that the operators (1) and (4) generate the same von Neumann algebra, $\mathcal{B}_{\#1}(B)$ and that $\mathcal{B}_{\#1}(B) \supset \mathcal{B}_{\#}(B)$. To show that $\mathcal{B}_{\#1}(B) \subset \mathcal{B}_{\#}(B)$, recall that a self- #-adjoint operator A commutes with a finitely bounded operator C provided $CD \subset D(A)$ and $CA = AC$ on D, for some core D of A. Equivalently is the condition that the operator C commutes with all finitely bounded functions of A. Also equivalent is the relation $CA = AC$ on $D(A)$. We choose $D = D(H(g))$. If the operator C commutes with all operators of the form (122), it also commutes on $D(H(g))$ with all operators of the form (123). Hence we get $\mathcal{B}_{\#}(B)' \subset \mathcal{B}_{\#1}(B)'$ and so $\mathcal{B}_{\#1}(B) = \mathcal{B}_{\#1}(B)'' \subset \mathcal{B}_{\#}(B)'' = \mathcal{B}_{\#}(B)''$.

Remark 25. The Poincare group ${}^{\sigma}P_{+}^{\uparrow}$ is the semidirect product of the space-time translations group $\mathbb{R}^{1,3}$ with the Lorentz group $O(1,3)$ such that $\{a_1 + \Lambda_1\}\{a_2 + \Lambda_2\} = \{a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2\}$. Here $a \in \mathbb{R}^{1,3}$ and $\Lambda(\beta): (x_i, t) \to \left(x_i \times \cosh(\beta) + t \times \sinh(\beta), x_i \times \sinh(\beta) + t \times \cosh(\beta)\right), i = 1,2,3$. We prove that there exists a representation $\sigma(a, \Lambda)$ of the Poincare group ${}^{\sigma}P_{+}^{\uparrow}$ by $*$ - automorphisms of $\mathcal{B}_{\#}$, such that $\sigma(a, \Lambda)\left(\mathcal{B}_{\#}(O)\right) = \mathcal{B}_{\#}(\{a, \Lambda\}O)$ for all bounded open sets O and all $\{a, \Lambda\} \in {}^{\sigma}P_{+}^{\uparrow}$. The Lorentz group composition law gives $\sigma(a, \Lambda) = \sigma(a, I)\sigma(0, \Lambda)$. Obviously the existence of the automorphism representation $\sigma(a, \Lambda)$ follows directly from the construction of the pure Lorentz transformation $\sigma(0, \Lambda) = \sigma(\Lambda)$. One obtains $\sigma(\Lambda)$ by constructing locally correct infinitesimal generators. Formally, the operators,

$$M_{\varkappa}^{0k} = M_{0,\varkappa}^{0k} + M_{I,\varkappa}^{0k} = Ext\text{-}\int_{{}^{*}\mathbb{R}_c^{\#3}} \frac{1}{2}\left\{: \pi_{\varkappa}(x)^2 : + : \left(\nabla \varphi_{\varkappa}(x)\right)^2 : + m^2 : \varphi_{\varkappa}(x)^2 :\right\} x^k d^{\#3}x + H_{I,\varkappa}(x^k g) \quad (125)$$

$k = 1,2,3$ s infinitesimal generators of Lorentz transformations in a region O if the cutoff function g equals one on a sufficiently large interval. We consider now the regions O_1 contained in the sets $\{x \in {}^{*}\mathbb{R}_c^{\#3} |\, x_1, x_2, x_3 > |t| + 1\}$. Thus for such regions O_1 we may replace (1) by $M^{0k} = Ext\text{-}\int_{{}^{*}\mathbb{R}_c^{\#3}} H(x)\, x^k g(x) d^{\#3}x$, with a nonnegative functions $x^k g(x), k = 1,2,3$. Here $H(x)$ is the formally positive energy density:

$$H(x) = \frac{1}{2}\left\{: \pi_{\varkappa}(x)^2 : + : \left(\nabla^{\#} \varphi_{\varkappa}(x)\right)^2 : + m^2 : \varphi_{\varkappa}(x)^2 :\right\} + H_{I,\varkappa}(x) = H_{0,\varkappa}(x) + H_{I,\varkappa}(x).$$

Therefore M^{0k} is formally positive. In fact it is technically convenient to use different spatial cutoffs in the free and the interaction part of M^{0k}, $k = 1,2,3$. Final formulas for $M_{\varkappa}^{0k}$ reads

$$M_{\varkappa}^{0k} = M_{\varkappa}^{0k}\left(g_0^k, g^k\right) = \alpha H_{0,\varkappa} + H_{0,\varkappa}(x^k g_0^k) + H_{I,\varkappa}(x^k g^k). \quad (126)$$

Here $0 < \alpha$ and $0 \le x^k g_0(x), 0 \le x^k g(x)$ and in order that (126) be formally correct, we assume that: $\alpha + x^k g_0^k = x^k = x^k g^k$ on $[1, R]$ with R sufficiently large. For technical reasons we assume that: $\alpha + x^k g_0^k(x) = x^k, k = 1,2,3$ on $\mathrm{supp}(g)$. By above restrictions on g_0^k and g^k we have $\mathrm{supp}(g_0^k), \mathrm{supp}(g^k) \subset \{x | \alpha \le x^k, k = 1,2,3\}$ and we show that the operator $M_{\varkappa}^{0k}$ is essentially self #-adjoint and it generates Lorentz rotations in an algebra $\mathcal{B}_{\#}(O_1)$

$$Ext\text{-}\exp(i\beta M_{\varkappa}^{0k}) \mathcal{B}_{\#}(O_1) Ext\text{-}\exp(-i\beta M_{\varkappa}^{0k}) \subset \mathcal{B}_{\#}(\{a, \Lambda(\beta)\}O_1) \quad (127)$$

provided that O_1 and $\{a, \Lambda(\beta)\}O_1$ are contained in the region

$$\{x \in {}^{*}\mathbb{R}_c^{\#3}, t \in {}^{*}\mathbb{R}_c^{\#} |\, |t| + 1 < x_k < R - |t|, k = 1,2,3\} \quad (128)$$

where M^{0k} is formally correct. These results permit us to define the Lorentz rotation automorphism $\sigma(\Lambda)$ on an arbitrary local algebra $\mathcal{B}_{\#}(O)$. Using a space time translation $\sigma(a), a \in {}^{*}\mathbb{R}_c^{\#4}$ we can translate O into a region $O + a = O_1 \subset \{x \in {}^{*}\mathbb{R}_c^{\#3}, t \in {}^{*}\mathbb{R}_c^{\#3} |\, x_1, x_2, x_3 > |t| + 1\}$ and for $R \in {}^{*}\mathbb{R}_c^{\#}$ large enough, O_1 and $\{a, \Lambda(\beta)\}O_1$ are contained in the region (1) we define $\sigma\left(0, \Lambda(\beta)\right) = \sigma\left(\Lambda(\beta)\right)$ by

$$\sigma(\Lambda(\beta)) \upharpoonright \mathcal{B}_{\#}(O) = \sigma(\{-\Lambda(\beta)a, I\})\sigma(\{0, \Lambda(\beta)\})\sigma(\{a, I\}) \upharpoonright \mathcal{B}_{\#}(O).$$

Theorem 82. Let $M^{0k}(g_0, g), k = 1,2,3$ be given by (126), with $\alpha, g_0(x), g(x)$ restricted as mentioned above. Then $M^{0k}(g_0, g)$ is essentially self #-adjoint on $C^{*\infty}(H \cap H_0)$.
Theorem 83. Let O_1 and $\{0, \Lambda(\beta)\}O_1$ be contained in the set (1). Then the following identity holds between self #-adjoint operators:

$$Ext\text{-}\exp(i\beta M^{0k})\varphi_{\varkappa}^{\#}(f)Ext\text{-}\exp(i\beta M^{0k}) = \varphi_{\varkappa}^{\#}(f(\{0, \Lambda(\beta)\}x)) = \int_{{}^*\mathbb{R}_c^{\#4}} \varphi_{\varkappa}^{\#}\left(f(\{0, \Lambda(\beta)\}(x,t))\right) d^{\#3}x d^{\#}t. \quad (129)$$

Here provided $\operatorname{supp}(f) \subset O_1$.

The proof of the Theorem 83 is reduced to the verification of the following equations

$$\left\{x_k \frac{\partial^{\#}}{\partial^{\#} t} + t \frac{\partial^{\#}}{\partial^{\#} x_k}\right\} \varphi_{\varkappa}^{\#}(x,t) = [iM^{0k}, \varphi_{\varkappa}^{\#}(x,t)], k = 1,2,3. \quad (130)$$

Here (130) that is equation for bilinear forms on an appropriate domain. Since M^{0k} is self #-adjoint, we can integrate (130), thus we compute formally for $H = H_0 + H_{I,\varkappa}(g)$,

$$[iM^{0k}, \varphi_{\varkappa}^{\#}(x,t)] = [iM^{0k}, Ext\text{-}\exp(itH)\varphi_{\varkappa}^{\#}(x,t)Ext\text{-}\exp(-itH)] =$$

$$Ext\text{-}\exp(itH)[iM^{0k}(-t), \varphi_{\varkappa}^{\#}(x,0)]Ext\text{-}\exp(-itH). \quad (131)$$

Here $M^{0k}(-t) = Ext\text{-}\exp(-itH)M^{0k}Ext\text{-}\exp(itH)$. Formally one obtains that

$$M^{0k}(-t) = Ext\text{-}\sum_{n=0}^{*\infty} \frac{(-t)^n}{n!} ad^n(iH)(M^{0k}), k = 1,2,3.$$

Note that if M^{0k} and H were the correct global Lorentzian generators and Hamiltonian they would satisfy

$$[iH, M^{0k}] = ad\,(iH)(M^{0k}) = P^k, \left[iH, [iH,\ M^{0k}]\right] = 0,\ M^{0k}(-t) =\ M^{0k} - P^k t. \quad (132)$$

Here $P^k, k = 1,2,3$ are the generators of space translations. Thus from (131) we get

$$[\ iM^{0k}, \varphi_{\varkappa}^{\#}(x,0)] = [iM_0^{0k}] = x\pi_{\varkappa}^{\#}(x,0), [iP^k, \varphi_{\varkappa}^{\#}(x,0)] = -\nabla^{\#}(\varphi_{\varkappa}^{\#})(x,0).$$

Formally we have (130).However the difficulty with this formal argument is that H and M^{0k} do not obey (132) exactly, since they are correct only in O_1. We have instead (132) the equations

$$[iH, M^{0k}] = P_{loc}^k, \left[iH, [iH,\ M^{0k}]\right] = R_k^{loc}, k = 1,2,3. \quad (133)$$

Here P_{loc}^k acts like the momentum operators only in the region O_1, i.e.

$$\left[P_{loc}^k, \varphi_{\varkappa}^{\#}(x,t)\right] = [P^k, \varphi_{\varkappa}^{\#}(x,t)], (x,t) \in O_1.$$

Hence $\left[iH, P_{loc}^k\right] = R_k^{loc}, k = 1,2,3$ is not identically zero, but commutes with $\mathcal{B}_{\#}(O_1)$. Formally, further commutators of $R_k^{loc}, k = 1,2,3$ with H are localized outside region O_1, and (130) follows formally even for our approximate, but locally correct H and M^{0k}. In order to convert this formal argument into a rigorous mathematical result, we apply now generalized Taylor series expansion [13] for the quantities

$$E_k(-t) = \langle \Omega, [\ iM^{0k}(-t), \varphi_{\varkappa}^{\#}(x,0)]\Omega \rangle, k = 1,2,3. \quad (134)$$

Here $\Omega \in C^{*\infty}(H)$ and thus we obtain

$$E_k(-t) = E_k(0) - t\frac{d^{\#}E_k(0)}{d^{\#}t} + \frac{t^2}{2}\frac{d^{\#2}E_k(\xi)}{d^{\#}t^2}, \text{ where } \xi \in [-t, t].$$

From (133) we obtain

$$\frac{d^{\#2}E_k(-\xi)}{d^{\#}t^2} = \langle Ext\text{-}\exp(i\xi H)\Omega, \left[iR_k^{loc}, \varphi_\varkappa^{\#}(x,\xi)\right] Ext\text{-}\exp(i\xi H)\Omega\rangle.$$

Note that $(x,t) \in O_1$, so that with $\xi \in [-t,t]$, $(x,\xi) \in O_1$ and therefore

$$\left[R_k^{loc}, \varphi_\varkappa^{\#}(x,\xi)\right] \equiv 0. \quad (135)$$

After integration over $x \in {}^*\mathbb{R}_c^{\#3}$ with a function $f \in S_{\text{fin}}^{\#}({}^*\mathbb{R}_c^{\#3})$ we obtain the operator identity:

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \left[R_k^{loc}, \varphi_\varkappa^{\#}(x,\xi)\right] f(x) d^{\#3}x \equiv 0, k = 1,2,3. \quad (136)$$

Therefore $\frac{d^{\#2}E_k(\xi)}{d^{\#}t^2} \equiv 0$ if $|\xi| \leq |t|$ and

$$E_k(-t) = E_k(0) - t\frac{d^{\#}E_k(0)}{d^{\#}t} = \langle \Omega, \left\{[\, iM^{0k}, \varphi_\varkappa^{\#}(x,0)] - t\left[P_{loc}^k, \varphi_\varkappa^{\#}(x,0)\right]\right\}\Omega\rangle =$$

$$= \langle \Omega, \{x\pi_\varkappa^{\#}(x,0) + t\nabla^{\#}(\varphi_\varkappa^{\#})(x,0)\}\ \Omega\,\rangle.$$

Thus we get

$$[\, iM^{0k}(-t), \varphi_\varkappa^{\#}(x,0)] = x\pi_\varkappa^{\#}(x,0) + t\nabla^{\#}\varphi_\varkappa^{\#}(x,0) \quad (137)$$

Inserting the relation (137) in (131) finally we obtain (130).This completes the proof of Lorentz covariance.

CONCLUSION

A new non-Archimedean approach to interacted quantum fields is presented. In proposed approach, a field operator $\phi(x,t)$ no longer a standard tempered operator-valued distribution, but a non-classical operator-valued function. We prove using this novel approach that the quantum field theory with Hamiltonian $P(\varphi)_4$ exists and that the corresponding C^*- algebra of bounded observables satisfies all the Haag-Kastler axioms. In particular we prove that the $\lambda(\varphi^4)_4$ quantum field theory model is Lorentz covariant. For each Poincare transformation a , Λ and each bounded region O of Minkowski space we obtain a unitary operator U which correctly transforms the field bilinear forms $\varphi(x,t)$ for $(x,t) \in O$. The von Neumann algebra $\mathfrak{C}(O)$ of local observables is obtained as standard part of external nonstandard algebra $\mathcal{B}_{\#}(O)$.

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