Proofs of Four Conjectures in Number Theory : Beal's Conjecture, Riemann Hypothesis, The *abc* and $c < R^{1.63}$ Conjectures
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Abstract

This monograph presents the proofs of 4 important conjectures in the field of number theory:
- The Beal's conjecture.
- The Riemann Hypothesis.
- The $c < R^{1.63}$ conjecture.
- The abc conjecture is true.

We give in detail all the proofs.

Résumé

Cette monographie présente les preuves de 4 conjectures importantes dans le domaine de la théorie des nombres à savoir:
- La conjecture de Beal.
- L'Hypothèse de Riemann.
- La conjecture $c < R^{1.63}$.
- La conjecture abc est vraie.

Nous donnons les détails des différentes démonstrations.

Figure 1: Photo of the Author

Dedication

To the memory of my Parents, to my wife Wahida, my daughter Sinda and my son Mohamed Mazen.
To my Teachers, to my Friends.

Contents

Chapter 1

A Complete Proof of Beal's Conjecture

Abstract

In 1997, Andrew Beal announced the following conjecture: *Let $A, B, C, m, n,$ and l be positive integers with $m, n, l > 2$. If $A^m + B^n = C^l$ then $A, B,$ and C have a common factor.* We begin to construct the polynomial $P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - px + q$ with p, q integers depending of A^m, B^n and C^l. We resolve $x^3 - px + q = 0$ and we obtain the three roots x_1, x_2, x_3 as functions of p, q and a parameter θ. Since $A^m, B^n, -C^l$ are the only roots of $x^3 - px + q = 0$, we discuss the conditions that x_1, x_2, x_3 are integers and have or not a common factor. Three numerical examples are given.

Résumé

En 1997, Andrew Beal avait annoncé la conjecture suivante: *Soient $A, B, C, m, n,$ et l des entiers positifs avec $m, n, l > 2$. Si $A^m + B^n = C^l$ alors $A, B,$ et C ont un facteur commun.*

Je commence par construire le polynôme $P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - px + q$ avec p, q des entiers qui dépendent de A^m, B^n et C^l. Nous résolvons $x^3 - px + q = 0$ et nous obtenons les trois racines x_1, x_2, x_3 comme fonctions de p, q et d'un paramètre θ. Comme $A^m, B^n, -C^l$ sont les seules racines de $x^3 - px + q = 0$, nous discutons les conditions pourque x_1, x_2, x_3 soient des entiers. Trois exemples numériques sont présentés.

Contents

1.1 Introduction

In 1997, Andrew Beal [4] announced the following conjecture :

Conjecture 1.1.1. *Let $A, B, C, m, n,$ and l be positive integers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \tag{1.1.1}$$

then $A, B,$ and C have a common factor.

The purpose of this paper is to give a complete proof of Beal's conjecture. Our proof of the conjecture contains many cases to study using elementary number theory. Our idea is to construct a polynomial $P(x)$ of order three having as roots A^m, B^n and $-C^l$ with the condition (1.1.1). The paper is organized as follows. In section 1, It is an introduction of the paper. The trivial case, where $A^m = B^n$, is studied in section 2. The preliminaries needed for the proof are given in section 3 where we consider the polynomial $P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - px + q$. We express the three roots of $P(x) = x^3 - px + q = 0$ in function of two parameters p, θ that depend on A^m, B^n, C^l. The section 4 is the preamble of the proof of the main theorem. We find the expression of A^{2m} equal to $\dfrac{4p}{3}cos^2\dfrac{\theta}{3}$. As A^{2m} is an integer, it follows that $cos^2\dfrac{\theta}{3}$ must be written as $\dfrac{a}{b}$ where a, b are two positive coprime integers. We discuss the conditions of divisibility of p, a, b so that the expression of A^{2m} is an integer. Depending of each individual case, we obtain that A, B, C have or do have not a common factor. Section 5 treats the cases of the first hypothesis $3 \mid a$ and $b \mid 4p$. We study the cases of the second hypothesis $3 \mid p$ and $b \mid 4p$ in section 6. Finally, we present three numerical examples and the conclusion in section 7.

In 1997, Andrew Beal [4] announced the following conjecture :

Conjecture 1.1.2. *Let $A, B, C, m, n,$ and l be positive integers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \tag{1.1.2}$$

then $A, B,$ and C have a common factor.

1.2 Trivial Case

We consider the trivial case when $A^m = B^n$. The equation (1.1.2) becomes:

$$2A^m = C^l \tag{1.2.1}$$

then $2 \mid C^l \Longrightarrow 2 \mid C \Longrightarrow C = 2^q.C_1$ with $q \geq 1$, $2 \nmid C_1$ and $2A^m = 2^{ql}C_1^l \Longrightarrow A^m = 2^{ql-1}C_1^l$. As $l > 2$, $q \geq 1$, then $2 \mid A^m \Longrightarrow 2 \mid A \Longrightarrow A = 2^r A_1$ with $r \geq 1$ and $2 \nmid A_1$. The equation (1.2.1),becomes:

$$2 \times 2^{rm} A_1^m = 2^{ql} C_1^l \tag{1.2.2}$$

As $2 \nmid A_1$ and $2 \nmid C_1$, we obtain the first condition :

$$\text{there exists two positive integers } r, q \text{ with } r.q \geq 1 \text{ so that } \boxed{ql = mr + 1} \tag{1.2.3}$$

Then from (1.2.2):

$$A_1^m = C_1^l \tag{1.2.4}$$

1.2.1 Case 1 $A_1 = 1 \Longrightarrow C_1 = 1$

Using the condition (1.2.3) above, we obtain $2.(2^r)^m = (2^q)^l$ and the Beal conjecture is verified.

1.2.2 Case 2 $A_1 > 1 \Longrightarrow C_1 > 1$

From the fundamental theorem of the arithmetic, we can write:

$$A_1 = a_1^{\alpha_1} \ldots a_I^{\alpha_I}, \quad a_1 < a_2 < \cdots < a_I \Longrightarrow A_1^m = a_1^{m\alpha_1} \ldots a_I^{m\alpha_I} \tag{1.2.5}$$

$$C_1 = c_1^{\beta_1} \ldots c_J^{\beta_J}, \quad c_1 < c_2 < \cdots < c_J \Longrightarrow C_1^l = c_1^{l\beta_1} \ldots c_J^{l\beta_J} \tag{1.2.6}$$

where a_i (respectively c_j) are distinct positive prime numbers and α_i (respectively β_j) are integers >0.

From (1.2.4) and using the uniqueness of the factorization of A_1^m and C_1^l, we obtain necessary:

$$\begin{cases} I = J \\ a_i = c_i, \quad i = 1, 2, \ldots, I \\ m\alpha_i = l\beta_i \end{cases} \tag{1.2.7}$$

As one $a_i \mid A^m \Longrightarrow a_i \mid B^m \Longrightarrow a_i \mid B$ and in this case, the Beal conjecture is verified.

We suppose in the following that $A^m > B^n$.

1.3 Preliminaries

Let $m, n, l \in \mathbb{N}^* \ > 2$ and $A, B, C \in \mathbb{N}^*$ such:

$$A^m + B^n = C^l \tag{1.3.1}$$

We call:

$$\begin{aligned} P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - x^2(A^m + B^n - C^l) \\ +x[A^m B^n - C^l(A^m + B^n)] + C^l A^m B^n \end{aligned} \tag{1.3.2}$$

Using the equation (1.3.1), $P(x)$ can be written as:

$$\boxed{P(x) = x^3 + x[A^m B^n - (A^m + B^n)^2] + A^m B^n (A^m + B^n)} \tag{1.3.3}$$

We introduce the notations:

$$\begin{aligned} p &= (A^m + B^n)^2 - A^m B^n \\ q &= A^m B^n (A^m + B^n) \end{aligned}$$

As $A^m \neq B^n$, we have $p > (A^m - B^n)^2 > 0$. Equation (1.3.3) becomes:

$$P(x) = x^3 - px + q$$

Using the equation (1.3.2), $P(x) = 0$ has three different real roots : A^m, B^n and $-C^l$.

Now, let us resolve the equation:

$$P(x) = x^3 - px + q = 0 \tag{1.3.4}$$

To resolve (1.3.4) let:

$$x = u + v$$

Then $P(x) = 0$ gives:

$$P(x) = P(u+v) = (u+v)^3 - p(u+v) + q = 0 \Longrightarrow u^3 + v^3 + (u+v)(3uv - p) + q = 0 \tag{1.3.5}$$

To determine u and v, we obtain the conditions:

$$\begin{aligned} u^3 + v^3 &= -q \\ uv &= p/3 > 0 \end{aligned}$$

Then u^3 and v^3 are solutions of the second order equation:

$$X^2 + qX + p^3/27 = 0 \tag{1.3.6}$$

Its discriminant Δ is written as :

$$\Delta = q^2 - 4p^3/27 = \frac{27q^2 - 4p^3}{27} = \frac{\bar{\Delta}}{27}$$

Let:

$$\begin{aligned} \bar{\Delta} = 27q^2 - 4p^3 &= 27(A^m B^n (A^m + B^n))^2 - 4[(A^m + B^n)^2 - A^m B^n]^3 \\ &= 27A^{2m} B^{2n} (A^m + B^n)^2 - 4[(A^m + B^n)^2 - A^m B^n]^3 \end{aligned} \tag{1.3.7}$$

Noting :

$$\begin{aligned} \alpha &= A^m B^n > 0 \\ \beta &= (A^m + B^n)^2 \end{aligned}$$

we can write (1.3.7) as:

$$\bar{\Delta} = 27\alpha^2\beta - 4(\beta-\alpha)^3 \tag{1.3.8}$$

As $\alpha \neq 0$, we can also rewrite (1.3.8) as :

$$\bar{\Delta} = \alpha^3\left(27\frac{\beta}{\alpha} - 4\left(\frac{\beta}{\alpha}-1\right)^3\right)$$

We call t the parameter :

$$t = \frac{\beta}{\alpha}$$

$\bar{\Delta}$ becomes :

$$\bar{\Delta} = \alpha^3(27t - 4(t-1)^3)$$

Let us calling :

$$y = y(t) = 27t - 4(t-1)^3$$

Since $\alpha > 0$, the sign of $\bar{\Delta}$ is also the sign of $y(t)$. Let us study the sign of y. We obtain $y'(t)$:

$$y'(t) = y' = 3(1+2t)(5-2t)$$

$y' = 0 \Longrightarrow t_1 = -1/2$ and $t_2 = 5/2$, then the table of variations of y is given below:

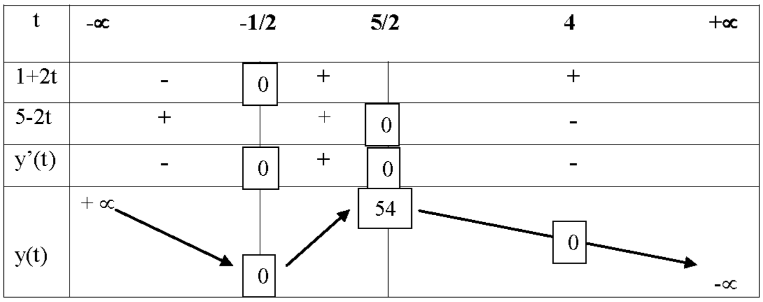

Figure 1.1: The table of variations

The table of the variations of the function y shows that $y < 0$ for $t > 4$. In our case, we are interested for $t > 0$. For $t = 4$ we obtain $y(4) = 0$ and for $t \in]0, 4[\Longrightarrow y > 0$. As we have $t = \frac{\beta}{\alpha} > 4$ because as $A^m \neq B^n$:

$$(A^m - B^n)^2 > 0 \Longrightarrow \beta = (A^m + B^n)^2 > 4\alpha = 4A^mB^n$$

Then $y < 0 \Longrightarrow \bar{\Delta} < 0 \Longrightarrow \Delta < 0$. Then, the equation (1.3.6) does not have real solutions u^3 and v^3. Let us find the solutions u and v with $x = u + v$ is a positive or a negative real and $u.v = p/3$.

1.3.1 Expressions of the roots

Proof. The solutions of (1.3.6) are:

$$X_1 = \frac{-q + i\sqrt{-\Delta}}{2}$$
$$X_2 = \overline{X_1} = \frac{-q - i\sqrt{-\Delta}}{2}$$

We may resolve:

$$u^3 = \frac{-q + i\sqrt{-\Delta}}{2}$$
$$v^3 = \frac{-q - i\sqrt{-\Delta}}{2}$$

Writing X_1 in the form:

$$X_1 = \rho e^{i\theta}$$

with:

$$\rho = \frac{\sqrt{q^2-\Delta}}{2} = \frac{p\sqrt{p}}{3\sqrt{3}}$$
$$\text{and } \ sin\theta = \frac{\sqrt{-\Delta}}{2\rho} > 0$$
$$cos\theta = -\frac{q}{2\rho} < 0$$

Then $\theta\,[2\pi] \in]+\frac{\pi}{2}, +\pi[$, let:

$$\boxed{\frac{\pi}{2} < \theta < +\pi \Rightarrow \frac{\pi}{6} < \frac{\theta}{3} < \frac{\pi}{3} \Rightarrow \frac{1}{2} < cos\frac{\theta}{3} < \frac{\sqrt{3}}{2}} \tag{1.3.9}$$

and:

$$\boxed{\frac{1}{4} < cos^2\frac{\theta}{3} < \frac{3}{4}} \tag{1.3.10}$$

hence the expression of X_2:

$$X_2 = \rho e^{-i\theta} \tag{1.3.11}$$

Let:

$$u = re^{i\psi} \tag{1.3.12}$$
$$\text{and } \ j = \frac{-1+i\sqrt{3}}{2} = e^{i\frac{2\pi}{3}} \tag{1.3.13}$$
$$j^2 = e^{i\frac{4\pi}{3}} = -\frac{1+i\sqrt{3}}{2} = \bar{j} \tag{1.3.14}$$

j is a complex cubic root of the unity $\Longleftrightarrow$ $j^3 = 1$. Then, the solutions u and v are:

$$u_1 = re^{i\psi_1} = \sqrt[3]{\rho}e^{i\frac{\theta}{3}} \tag{1.3.15}$$
$$u_2 = re^{i\psi_2} = \sqrt[3]{\rho}je^{i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{\theta+2\pi}{3}} \tag{1.3.16}$$
$$u_3 = re^{i\psi_3} = \sqrt[3]{\rho}j^2e^{i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi}{3}}e^{+i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{\theta+4\pi}{3}} \tag{1.3.17}$$

and similarly:

$$v_1 = re^{-i\psi_1} = \sqrt[3]{\rho}e^{-i\frac{\theta}{3}} \tag{1.3.18}$$
$$v_2 = re^{-i\psi_2} = \sqrt[3]{\rho}j^2e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi}{3}}e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi-\theta}{3}} \tag{1.3.19}$$
$$v_3 = re^{-i\psi_3} = \sqrt[3]{\rho}je^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{2\pi-\theta}{3}} \tag{1.3.20}$$

We may now choose u_k and v_h so that $u_k + v_h$ will be real. In this case, we have necessary :

$$v_1 = \overline{u_1} \tag{1.3.21}$$
$$v_2 = \overline{u_2} \tag{1.3.22}$$
$$v_3 = \overline{u_3} \tag{1.3.23}$$

We obtain as real solutions of the equation (1.3.5):

$$x_1 = u_1 + v_1 = 2\sqrt[3]{\rho}cos\frac{\theta}{3} > 0 \tag{1.3.24}$$

$$x_2 = u_2 + v_2 = 2\sqrt[3]{\rho}cos\tfrac{\theta+2\pi}{3} = -\sqrt[3]{\rho}\left(cos\tfrac{\theta}{3} + \sqrt{3}sin\tfrac{\theta}{3}\right) < 0 \tag{1.3.25}$$

$$x_3 = u_3 + v_3 = 2\sqrt[3]{\rho}cos\tfrac{\theta+4\pi}{3} = \sqrt[3]{\rho}\left(-cos\tfrac{\theta}{3} + \sqrt{3}sin\tfrac{\theta}{3}\right) > 0 \tag{1.3.26}$$

+ We compare the expressions of x_1 and x_3, we obtain:

$$2\sqrt[3]{p}cos\tfrac{\theta}{3} \overbrace{>}^{?} \sqrt[3]{p}\left(-cos\tfrac{\theta}{3} + \sqrt{3}sin\tfrac{\theta}{3}\right)$$

$$3cos\tfrac{\theta}{3} \overbrace{>}^{?} \sqrt{3}sin\tfrac{\theta}{3} \tag{1.3.27}$$

As $\dfrac{\theta}{3} \in]+\dfrac{\pi}{6}, +\dfrac{\pi}{3}[$, then $sin\dfrac{\theta}{3}$ and $cos\dfrac{\theta}{3}$ are > 0. Taking the square of the two members of the last equation, we get:

$$\frac{1}{4} < cos^2\frac{\theta}{3} \tag{1.3.28}$$

which is true since $\dfrac{\theta}{3} \in]+\dfrac{\pi}{6}, +\dfrac{\pi}{3}[$ then $x_1 > x_3$. As A^m, B^n and $-C^l$ are the only real solutions of (1.3.4), we consider, as A^m is supposed great than B^n, the expressions:

$$\boxed{\begin{cases} A^m = x_1 = u_1 + v_1 = 2\sqrt[3]{\rho}cos\dfrac{\theta}{3} \\[2ex] B^n = x_3 = u_3 + v_3 = 2\sqrt[3]{\rho}cos\dfrac{\theta+4\pi}{3} = \sqrt[3]{\rho}\left(-cos\dfrac{\theta}{3} + \sqrt{3}sin\dfrac{\theta}{3}\right) \\[2ex] -C^l = x_2 = u_2 + v_2 = 2\sqrt[3]{\rho}cos\dfrac{\theta+2\pi}{3} = -\sqrt[3]{\rho}\left(cos\dfrac{\theta}{3} + \sqrt{3}sin\dfrac{\theta}{3}\right) \end{cases}} \tag{1.3.29}$$

□

1.4 Preamble of the Proof of the Main Theorem

Theorem 1.4.1. *Let $A, B, C, m, n,$ and l be positive integers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \tag{1.4.1}$$

then $A, B,$ and C have a common factor.

Proof. $A^m = 2\sqrt[3]{\rho}cos\dfrac{\theta}{3}$ is an integer $\Rightarrow A^{2m} = 4\sqrt[3]{\rho^2}cos^2\dfrac{\theta}{3}$ is also an integer. But :

$$\sqrt[3]{\rho^2} = \frac{p}{3} \tag{1.4.2}$$

Then:

$$A^{2m} = 4\sqrt[3]{\rho^2}cos^2\frac{\theta}{3} = 4\frac{p}{3}.cos^2\frac{\theta}{3} = p.\frac{4}{3}.cos^2\frac{\theta}{3} \tag{1.4.3}$$

As A^{2m} is an integer and p is an integer, then $cos^2\dfrac{\theta}{3}$ must be written under the form:

$$\boxed{cos^2\frac{\theta}{3} = \frac{1}{b} \quad or \quad cos^2\frac{\theta}{3} = \frac{a}{b}} \tag{1.4.4}$$

with $b \in \mathbb{N}^*$; for the last condition $a \in \mathbb{N}^*$ and a, b coprime.

Notations: In the following of the paper, the scalars $a, b, ..., z$, $\alpha, \beta, ...$, $A, B, C, ...$ and $\Delta, \Phi, ...$ represent positive integers except the parameters θ, ρ, or others cited in the text, are reals.

1.4.1 Case $cos^2\frac{\theta}{3} = \frac{1}{b}$

We obtain:

$$A^{2m} = p.\frac{4}{3}.cos^2\frac{\theta}{3} = \frac{4.p}{3.b} \tag{1.4.5}$$

As $\frac{1}{4} < cos^2\frac{\theta}{3} < \frac{3}{4} \Rightarrow \frac{1}{4} < \frac{1}{b} < \frac{3}{4} \Rightarrow b < 4 < 3b \Rightarrow b = 1, 2, 3.$

$b = 1$

$b = 1 \Rightarrow 4 < 3$ which is impossible.

$b = 2$

$b = 2 \Rightarrow A^{2m} = p.\frac{4}{3}.\frac{1}{2} = \frac{2.p}{3} \Rightarrow 3 \mid p \Rightarrow p = 3p'$ with $p' \neq 1$ because $3 \ll p$, we obtain:

$$A^{2m} = (A^m)^2 = \frac{2p}{3} = 2.p' \Longrightarrow 2 \mid p' \Longrightarrow p' = 2^\alpha p_1^2$$
$$with \quad 2 \nmid p_1, \quad \alpha + 1 = 2\beta$$
$$A^m = 2^\beta p_1 \tag{1.4.6}$$
$$B^n C^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = p' = 2^\alpha p_1^2 \tag{1.4.7}$$

From the equation (1.4.6), it follows that $2 \mid A^m \Longrightarrow A = 2^i A_1$, $i \geq 1$ and $2 \nmid A_1$. Then, we have $\beta = i.m = im$. The equation (1.4.7) implies that $2 \mid (B^n C^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

Case $2 \mid B^n$: - If $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j B_1$ with $2 \nmid B_1$. The expression of $B^n C^l$ becomes:

$$B_1^n C^l = 2^{2im-1-jn} p_1^2$$

- If $2im - 1 - jn \geq 1$, $2 \mid C^l \Longrightarrow 2 \mid C$ according to $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (3.1.1) is verified.
- If $2im - 1 - jn \leq 0 \Longrightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

Case $2 \mid C^l$: If $2 \mid C^l$: with the same method used above, we obtain the identical results.

$b = 3$

$b = 3 \Rightarrow A^{2m} = p.\frac{4}{3}.\frac{1}{3} = \frac{4p}{9} \Rightarrow 9 \mid p \Rightarrow p = 9p'$ with $p' \neq 1$, as $9 \ll p$ then $A^{2m} = 4p'$. If p' is prime, it is impossible. We suppose that p' is not a prime, as $m \geq 3$, it follows that $2 \mid p'$, then $2 \mid A^m$. But $B^n C^l = 5p'$ and $2 \mid (B^n C^l)$. Using the same method for the case $b = 2$, we obtain the identical results.

1.4.2 Case $a > 1,\ cos^2\dfrac{\theta}{3} = \dfrac{a}{b}$

We have:

$$cos^2\frac{\theta}{3} = \frac{a}{b}; \quad A^{2m} = p.\frac{4}{3}.cos^2\frac{\theta}{3} = \frac{4.p.a}{3.b} \tag{1.4.8}$$

where a, b verify one of the two conditions:

$$\boxed{\{3 \mid a \quad and \quad b \mid 4p\}} \text{ or } \boxed{\{3 \mid p \quad and \quad b \mid 4p\}} \tag{1.4.9}$$

and using the equation (1.3.10), we obtain a third condition:

$$\boxed{b < 4a < 3b} \tag{1.4.10}$$

For these conditions, $A^{2m} = 4\sqrt[3]{\rho^2}cos^2\frac{\theta}{3} = 4\dfrac{p}{3}.cos^2\dfrac{\theta}{3}$ is an integer.

Let us study the conditions given by the equation (1.4.9) in the following two sections.

1.5 Hypothesis : $\{3 \mid a \quad and \quad b \mid 4p\}$

We obtain :

$$3 \mid a \Longrightarrow \exists a' \in \mathbb{N}^* \ / \ a = 3a' \tag{1.5.1}$$

1.5.1 Case $b = 2$ and $3 \mid a$

A^{2m} is written as:

$$A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{a}{b} = \frac{4p}{3}.\frac{a}{2} = \frac{2.p.a}{3} \tag{1.5.2}$$

Using the equation (1.5.1), A^{2m} becomes :

$$A^{2m} = \frac{2.p.3a'}{3} = 2.p.a' \tag{1.5.3}$$

but $cos^2\dfrac{\theta}{3} = \dfrac{a}{b} = \dfrac{3a'}{2} > 1$ which is impossible, then $b \neq 2$.

1.5.2 Case $b = 4$ and $3 \mid a$

A^{2m} is written :

$$A^{2m} = \frac{4.p}{3}cos^2\frac{\theta}{3} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.p}{3}.\frac{a}{4} = \frac{p.a}{3} = \frac{p.3a'}{3} = p.a' \tag{1.5.4}$$

$$\text{and} \quad cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3.a'}{4} < \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4} \Longrightarrow a' < 1 \tag{1.5.5}$$

which is impossible. Then the case $b = 4$ is impossible.

1.5.3 Case $b = p$ and $3 \mid a$

We have :

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{p} \tag{1.5.6}$$

and:

$$A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{3a'}{p} = 4a' = (A^m)^2 \tag{1.5.7}$$

$$\exists a'' \ / \ a' = a''^2 \tag{1.5.8}$$

$$and \quad B^nC^l = p - A^{2m} = b - 4a' = b - 4a''^2 \tag{1.5.9}$$

The calculation of A^mB^n gives :

$$A^mB^n = p.\frac{\sqrt{3}}{3}sin\frac{2\theta}{3} - 2a'$$
$$or \quad A^mB^n + 2a' = p.\frac{\sqrt{3}}{3}sin\frac{2\theta}{3} \tag{1.5.10}$$

The left member of (1.5.10) is an integer and p also, then $2\frac{\sqrt{3}}{3}sin\frac{2\theta}{3}$ is written under the form :

$$2\frac{\sqrt{3}}{3}sin\frac{2\theta}{3} = \frac{k_1}{k_2} \tag{1.5.11}$$

where k_1, k_2 are two coprime integers and $k_2 \mid p \Longrightarrow p = b = k_2.k_3,\, k_3 \in \mathbb{N}^*$.

We suppose that $k_3 \neq 1$

We obtain :

$$A^m(A^m + 2B^n) = k_1.k_3 \tag{1.5.12}$$

Let μ be a prime integer with $\mu \mid k_3$, then $\mu \mid b$ and $\mu \mid A^m(A^m+2B^n) \Longrightarrow \mu \mid A^m$ or $\mu \mid (A^m+2B^n)$.

** A-1-1- If $\mu \mid A^m \Longrightarrow \mu \mid A$ and $\mu \mid A^{2m}$, but $A^{2m} = 4a' \Longrightarrow \mu \mid 4a' \Longrightarrow (\mu = 2$, but $2 \mid a')$ or $(\mu \mid a')$. Then $\mu \mid a$ it follows the contradiction with a, b coprime.

** A-1-2- If $\mu \mid (A^m+2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$ then $\mu \neq 2$ and $\mu \nmid B^n$. We write $\mu \mid (A^m+2B^n)$ as:

$$A^m + 2B^n = \mu.t' \tag{1.5.13}$$

It follows :

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^mB^n = \mu^2t'^2 - 2t'\mu B^n + B^{2n}$$

Using the expression of p:

$$p = t'^2\mu^2 - 2t'B^n\mu + B^n(B^n - A^m) \tag{1.5.14}$$

As $p = b = k_2.k_3$ and $\mu \mid k_3$ then $\mu \mid b \Longrightarrow \exists \mu'$ and $b = \mu\mu'$, so we can write:

$$\mu'\mu = \mu(\mu t'^2 - 2t'B^n) + B^n(B^n - A^m) \tag{1.5.15}$$

From the last equation, we obtain $\mu \mid B^n(B^n - A^m) \Longrightarrow \mu \mid B^n$ or $\mu \mid (B^n - A^m)$.

** A-1-2-1- If $\mu \mid B^n$ which is in contradiction with $\mu \nmid B^n$.

** A-1-2-2- If $\mu \mid (B^n - A^m)$ and using that $\mu \mid (A^m + 2B^n)$, we arrive to :

$$\mu \mid 3B^n \begin{cases} \mu \mid B^n \\ or \\ \mu = 3 \end{cases} \tag{1.5.16}$$

** A-1-2-2-1- If $\mu \mid B^n \Longrightarrow \mu \mid B$, it is the contradiction with $\mu \nmid B$ cited above.

** A-1-2-2-2- If $\mu = 3$, then $3 \mid b$, but $3 \mid a$ then the contradiction with a, b coprime.

We assume now $k_3 = 1$

Then :

$$A^{2m} + 2A^m B^n = k_1 \tag{1.5.17}$$
$$b = k_2 \tag{1.5.18}$$
$$\frac{2\sqrt{3}}{3} sin \frac{2\theta}{3} = \frac{k_1}{b} \tag{1.5.19}$$

Taking the square of the last equation, we obtain:

$$\frac{4}{3} sin^2 \frac{2\theta}{3} = \frac{k_1^2}{b^2}$$
$$\frac{16}{3} sin^2 \frac{\theta}{3} cos^2 \frac{\theta}{3} = \frac{k_1^2}{b^2}$$
$$\frac{16}{3} sin^2 \frac{\theta}{3} . \frac{3a'}{b} = \frac{k_1^2}{b^2}$$

Finally:

$$4^2 a'(p-a) = k_1^2 \tag{1.5.20}$$

but $a' = a''^2$, then $p - a$ is a square. Let:

$$\lambda^2 = p - a = b - a = b - 3a''^2 \Longrightarrow \lambda^2 + 3a''^2 = b \tag{1.5.21}$$

The equation (1.5.20) becomes:

$$4^2 a''^2 \lambda^2 = k_1^2 \Longrightarrow k_1 = 4a''\lambda \tag{1.5.22}$$

taking the positive root, but $k_1 = A^m(A^m + 2B^n) = 2a''(A^m + 2B^n)$, then :

$$A^m + 2B^n = 2\lambda \Longrightarrow \lambda = a'' + B^n \tag{1.5.23}$$

** A-2-1- As $A^m = 2a'' \Longrightarrow 2 \mid A^m \Longrightarrow 2 \mid A \Longrightarrow A = 2^i A_1$, with $i \geq 1$ and $2 \nmid A_1$, then $A^m = 2a'' = 2^{im} A_1^m \Longrightarrow a'' = 2^{im-1} A_1^m$, but $im \geq 3 \Longrightarrow 4 \mid a''$. As $p = b = A^{2m} + A^m B^n + B^{2n} = \lambda = 2^{im-1} A_1^m + B^n$. Taking its square, then :

$$\lambda^2 = 2^{2im-2} A_1^{2m} + 2^{im} A_1^m B^n + B^{2n}$$

As $im \geq 3$, we can write $\lambda^2 = 4\lambda_1 + B^{2n} \Longrightarrow \lambda^2 \equiv B^{2n} (\bmod 4) \Longrightarrow \lambda^2 \equiv B^{2n} \equiv 0 (\bmod 4)$ or $\lambda^2 \equiv B^{2n} \equiv 1 (\bmod 4)$.

** A-2-1-1- We suppose that $\lambda^2 \equiv B^{2n} \equiv 0 (\bmod 4) \Longrightarrow 4 \mid \lambda^2 \Longrightarrow 2 \mid (b - a)$. But $2 \mid a$ because $a = 3a' = 3a''^2 = 3 \times 2^{2(im-1)} A_1^{2m}$ and $im \geq 3$. Then $2 \mid b$, it follows the contradiction with a, b coprime.

** A-2-1-2- We suppose now that $\lambda^2 \equiv B^{2n} \equiv 1 (\bmod 4)$. As $A^m = 2^{im-1} A_1^m$ and $im - 1 \geq 2$, then $A^m \equiv 0 (\bmod 4)$. As $B^{2n} \equiv 1 (\bmod 4)$, then B^n verifies $B^n \equiv 1 (\bmod 4)$ or $B^n \equiv 3 (\bmod 4)$ which gives for the two cases $B^n C^l \equiv 1 (\bmod 4)$.

We have also $p = b = A^{2m} + A^m B^n + B^{2n} = 4a' + B^n . C^l = 4a''^2 + B^n C^l \Longrightarrow B^n C^l = \lambda^2 - a''^2 = B^n . C^l$, then $\lambda, a'' \in \mathbb{N}^*$ are solutions of the Diophantine equation :

$$x^2 - y^2 = N \tag{1.5.24}$$

with $N = B^n C^l > 0$. Let $Q(N)$ be the number of the solutions of (1.5.24) and $\tau(N)$ is the number of suitable factorization of N, then we announce the following result concerning the solutions of the equation (1.5.24) (see theorem 27.3 in [7]):

- If $N \equiv 2 (\bmod 4)$, then $Q(N) = 0$.
- If $N \equiv 1$ or $N \equiv 3 (\bmod 4)$, then $Q(N) = [\tau(N)/2]$.
- If $N \equiv 0 (\bmod 4)$, then $Q(N) = [\tau(N/4)/2]$.
$[x]$ is the integral part of x for which $[x] \leq x < [x] + 1$.

As λ, a'' is a couple of solutions of the Diophantine equation (1.5.24), then $\exists\ d, d'$ positive integers with $d > d'$ and $N = d.d'$ so that :

$$d + d' = 2\lambda \tag{1.5.25}$$
$$d - d' = 2a'' \tag{1.5.26}$$

** A-2-1-2-1- We suppose as $C^l > B^n$ that $d = C^l$ and $d' = B^n$. It follows:

$$2\lambda = C^l + B^n = A^m + 2B^n \tag{1.5.27}$$
$$2a'' = C^l - B^n = A^m \tag{1.5.28}$$

From the paragraph A-2-1 above, we have $\lambda = p = A^{2m} + A^m B^n + B^{2n} > (A^m + 2B^n)$, then the case $d = C^l$ and $d' = B^n$ gives a contradiction.

** A-2-1-2-2- Now, we consider the case $d = c_1^{lr-1} C_1^l$ where c_1 is a prime integer with $c_1 \nmid C_1$ and $C = c_1^r C_1$, $r \geq 1$. It follows that $d' = c_1.B^n$. We rewrite the equations (1.5.25-1.5.26):

$$c_1^{lr-1} C_1^l + c_1.B^n = 2\lambda \tag{1.5.29}$$
$$c_1^{lr-1} C_1^l - c_1.B^n = 2a'' \tag{1.5.30}$$

As $l \geq 3$, from the last two equations above, it follows that $c_1 \mid (2\lambda)$ and $c_1 \mid (2a'')$. Then $c_1 = 2$, or $c_1 \mid \lambda$ and $c_1 \mid a''$.

** A-2-1-2-2-1- We suppose $c_1 = 2$. As $2 \mid A^m$ and $2 \mid C^l$ because $l \geq 3$, it follows $2 \mid B^n$, then $2 \mid (p = b)$. Then the contradiction with a, b coprime.

** A-2-1-2-2-2- We suppose $c_1 \neq 2$ and $c_1 \mid a''$ and $c_1 \mid \lambda$. $c_1 \mid a'' \Longrightarrow c_1 \mid a$ and $c_1 \mid (A^m = 2a'')$. $B^n = C^l - A^m \Longrightarrow c_1 \mid B^n$. It follows that $c_1 \mid (p = b)$. Then the contradiction with a, b coprime.

The others cases of the expressions of d and d' with d, d' not coprime so that $N = B^n C^l = d.d'$ give also contradictions.

Hence, the case $k_3 = 1$ is impossible.

Let us verify the condition (1.4.10) given by $b < 4a < 3b$. In our case, the condition becomes :

$$p < 3A^{2m} < 3p \quad with \quad p = A^{2m} + B^{2n} + A^m B^n \tag{1.5.31}$$

and $3A^{2m} < 3p \Longrightarrow A^{2m} < p$ that is verified. If :

$$p < 3A^{2m} \Longrightarrow 2A^{2m} - A^m B^n - B^{2n} \overbrace{>}^{?} 0$$

Studying the sign of the polynomial $Q(Y) = 2Y^2 - B^n Y - B^{2n}$ and taking $Y = A^m > B^n$, the condition $2A^{2m} - A^m B^n - B^{2n} > 0$ is verified, then the condition $b < 4a < 3b$ is true.

In the following of the paper, we verify easily that the condition $b < 4a < 3b$ implies to verify that $A^m > B^n$ which is true.

1.5.4 Case $b \mid p \Rightarrow \ p = b.p', p' > 1$, $b \neq 2$, $b \neq 4$ and $3 \mid a$

$$A^{2m} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.b.p'.3.a'}{3.b} = 4.p'a' \tag{1.5.32}$$

We calculate B^nC^l:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) \tag{1.5.33}$$

but $\sqrt[3]{\rho^2} = \frac{p}{3}$, using $cos^2\frac{\theta}{3} = \frac{3.a'}{b}$, we obtain:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3.a'}{b}\right) = p.\left(1 - \frac{4.a'}{b}\right) = p'(b - 4a') \tag{1.5.34}$$

As $p = b.p'$, and $p' > 1$, so we have :

$$B^nC^l = p'(b - 4a') \tag{1.5.35}$$

$$\text{and} \quad A^{2m} = 4.p'.a' \tag{1.5.36}$$

** B-1- We suppose that p' is prime, then $A^{2m} = 4a'p' = (A^m)^2 \Longrightarrow p' \mid a'$. But $B^nC^l = p'(b - 4a') \Longrightarrow p' \mid B^n$ or $p' \mid C^l$.

** B-1-1- If $p' \mid B^n \Longrightarrow p' \mid B \Longrightarrow B = p'B_1$ with $B_1 \in \mathbb{N}^*$. Hence : $p'^{n-1}B_1^nC^l = b - 4a'$. But $n > 2 \Rightarrow (n-1) > 1$ and $p' \mid a'$, then $p' \mid b \Longrightarrow a$ and b are not coprime, then the contradiction.

** B-1-2- If $p' \mid C^l \Longrightarrow p' \mid C$. The same method used above, we obtain the same results.

** B-2- We consider that p' is not a prime integer.

** B-2-1- p', a are supposed coprime: $A^{2m} = 4a'p' \Longrightarrow A^m = 2a''.p_1$ with $a' = a''^2$ and $p' = p_1^2$, then a'', p_1 are also coprime. As $A^m = 2a''.p_1$ then $2 \mid a''$ or $2 \mid p_1$.

** B-2-1-1- $2 \mid a''$, then $2 \nmid p_1$. But $p' = p_1^2$.

** B-2-1-1-1- If p_1 is prime, it is impossible with $A^m = 2a''.p_1$.

** B-2-1-1-2- We suppose that p_1 is not prime, we can write it as $p_1 = \omega^m \Longrightarrow p' = \omega^{2m}$, then: $B^nC^l = \omega^{2m}(b - 4a')$.

** B-2-1-1-2-1- If ω is prime, it is different of 2, then $\omega \mid (B^nC^l) \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** B-2-1-1-2-1-1- If $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^jB_1$ with $\omega \nmid B_1$, then $B_1^n.C^l = \omega^{2m-nj}(b - 4a')$.

** B-2-1-1-2-1-1-1- If $2m - n.j = 0$, we obtain $B_1^n.C^l = b - 4a'$. As $C^l = A^m + B^n \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$, and $\omega \mid (b - 4a')$. But $\omega \neq 2$ and ω is coprime with a' then coprime with a, then $\omega \nmid b$. The conjecture (3.1.1) is verified.

** B-2-1-1-2-1-1-2- If $2m - nj \geq 1$, in this case with the same method, we obtain $\omega \mid C^l \Longrightarrow \omega \mid C$ and $\omega \mid (b - 4a')$ and $\omega \nmid a$ and $\omega \nmid b$. The conjecture (3.1.1) is verified.

** B-2-1-1-2-1-1-3- If $2m - nj < 0 \Longrightarrow \omega^{n.j-2m}B_1^n.C^l = b - 4a'$. As $\omega \mid C$ using $C^l = A^m + B^n$ then $C = \omega^h.C_1 \Longrightarrow \omega^{n.j-2m+h.l}B_1^n.C_1^l = b - 4a'$. If $n.j - 2m + h.l < 0 \Longrightarrow \omega \mid B_1^nC_1^l$, it follows the contradiction that $\omega \nmid B_1$ or $\omega \nmid C_1$. Then if $n.j - 2m + h.l > 0$ and $\omega \mid (b - 4a')$ with ω, a, b coprime and the conjecture (3.1.1) is verified.

** B-2-1-1-2-1-2- We obtain the same results if $\omega \mid C^l$.

** B-2-1-1-2-2- Now, $p' = \omega^{2m}$ and ω not prime, we write $\omega = \omega_1^f.\Omega$ with ω_1 prime $\nmid \Omega$ and $f \geq 1$ an integer, and $\omega_1 \mid A$. Then $B^nC^l = \omega_1^{2f.m}\Omega^{2m}(b-4a') \Longrightarrow \omega_1 \mid (B^nC^l) \Longrightarrow \omega_1 \mid B^n$ or $\omega_1 \mid C^l$.

** B-2-1-1-2-2-1- If $\omega_1 \mid B^n \Longrightarrow \omega_1 \mid B \Longrightarrow B = \omega_1^j B_1$ with $\omega_1 \nmid B_1$, then $B_1^n.C^l = \omega_1^{2mf-nj}\Omega^{2m}(b-4a')$:

** B-2-1-1-2-2-1-1- If $2f.m - n.j = 0$, we obtain $B_1^n.C^l = \Omega^{2m}(b-4a')$. As $C^l = A^m + B^n \Longrightarrow \omega_1 \mid C^l \Longrightarrow \omega_1 \mid C \Longrightarrow \omega_1 \mid (b-4a')$. But $\omega_1 \neq 2$ and ω_1 is coprime with a', then coprime with a, we deduce $\omega_1 \nmid b$. Then the conjecture (3.1.1) is verified.

** B-2-1-1-2-2-1-2- If $2f.m - n.j \geq 1$, we have $\omega_1 \mid C^l \Longrightarrow \omega_1 \mid C \Longrightarrow \omega_1 \mid (b-4a')$ and $\omega_1 \nmid a$ and $\omega_1 \nmid b$. The conjecture (3.1.1) is verified.

** B-2-1-1-2-2-1-3- If $2f.m - n.j < 0 \Longrightarrow \omega_1^{n.j-2m.f}B_1^n.C^l = \Omega^{2m}(b-4a')$. As $\omega_1 \mid C$ using $C^l = A^m + B^n$, then $C = \omega_1^h.C_1 \Longrightarrow \omega^{n.j-2m.f+h.l}B_1^n.C_1^l = \Omega^{2m}(b-4a')$. If $n.j - 2m.f + h.l < 0 \Longrightarrow \omega_1 \mid B_1^nC_1^l$, it follows the contradiction with $\omega_1 \nmid B_1$ and $\omega_1 \nmid C_1$. Then if $n.j - 2m.f + h.l > 0$ and $\omega_1 \mid (b-4a')$ with ω_1, a, b coprime and the conjecture (3.1.1) is verified.

** B-2-1-1-2-2-2- We obtain the same results if $\omega_1 \mid C^l$.

** B-2-1-2- If $2 \mid p_1$, then $2 \mid p_1 \Longrightarrow 2 \nmid a' \Longrightarrow 2 \nmid a$. But $p' = p_1^2$.

** B-2-1-2-1- If $p_1 = 2$, we obtain $A^m = 4a'' \Longrightarrow 2 \mid a''$ as $m \geq 3$, then the contradiction with a, b coprime.

** B-2-1-2-2- We suppose that p_1 is not prime and $2 \mid p_1$, as $A^m = 2a''p_1$, p_1 is written as $p_1 = 2^{m-1}\omega^m \Longrightarrow p' = 2^{2m-2}\omega^{2m}$. It follows $B^nC^l = 2^{2m-2}\omega^{2m}(b-4a') \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** B-2-1-2-2-1- If $2 \mid B^n \Longrightarrow 2 \mid B$, as $2 \mid A$, then $2 \mid C$. From $B^nC^l = 2^{2m-2}\omega^{2m}(b-4a')$, it follows if $2 \mid (b-4a') \Longrightarrow 2 \mid b$ but as $2 \nmid a'$, there is no contradiction with a, b coprime and the conjecture (3.1.1) is verified.

** B-2-1-2-2-2- If $2 \mid C^l$, using the same method as above, we obtain the identical results.

** B-2-2- p', a' are supposed not coprime. Let ω be a prime integer so that $\omega \mid a'$ and $\omega \mid p'$.

** B-2-2-1- We suppose firstly $\omega = 3$. As $A^{2m} = 4a'p' \Longrightarrow 3 \mid A$, but $3 \mid p' \Longrightarrow 3 \mid p$, as $p = A^{2m} + B^{2n} + A^mB^n \Longrightarrow 3 \mid B^{2n} \Longrightarrow 3 \mid B$, then $3 \mid C^l \Longrightarrow 3 \mid C$. We write $A = 3^iA_1$, $B = 3^jB_1$, $C = 3^hC_1$ and 3 coprime with A_1, B_1 and C_1 and $p = 3^{2im}A_1^{2m} + 3^{2nj}B_1^{2n} + 3^{im+jn}A_1^mB_1^n = 3^k.g$ with $k = min(2im, 2jn, im+jn)$ and $3 \nmid g$. We have also $(\omega = 3) \mid a$ and $(\omega = 3) \mid p'$ that gives $a = 3^\alpha a_1 = 3a' \Longrightarrow a' = 3^{\alpha-1}a_1$, $3 \nmid a_1$ and $p' = 3^\mu p_1$, $3 \nmid p_1$ with $A^{2m} = 4a'p' = 3^{2im}A_1^{2m} = 4 \times 3^{\alpha-1+\mu}.a_1.p_1 \Longrightarrow \alpha + \mu - 1 = 2im$. As $p = bp' = b.3^\mu p_1 = 3^\mu.b.p_1$. The exponent of the term 3 of p is k, the exponent of the term 3 of the left member of the last equation is μ. If $3 \mid b$ it is a contradiction with a, b coprime. Then, we suppose that $3 \nmid b$, and the equality of the exponents: $min(2im, 2jn, im+jn) = \mu$, recall that $\alpha + \mu - 1 = 2im$. But $B^nC^l = p'(b-4a')$ that gives $3^{(nj+hl)}B_1^nC_1^l = 3^\mu p_1(b - 4 \times 3^{(\alpha-1)}a_1)$. We have also $A^m + B^n = C^l$ gives $3^{im}A_1^m + 3^{jn}B_1^n = 3^{hl}C_1^l$. Let $\epsilon = min(im, jn)$, we have $\epsilon = hl = min(im, jn)$. Then, we obtain the conditions:

$$k = min(2im, 2jn, im+jn) = \mu \tag{1.5.37}$$

$$\alpha + \mu - 1 = 2im \tag{1.5.38}$$

$$\epsilon = hl = min(im, jn) \tag{1.5.39}$$

$$3^{(nj+hl)}B_1^nC_1^l = 3^\mu p_1(b - 4 \times 3^{(\alpha-1)}a_1) \tag{1.5.40}$$

** B-2-2-1-1- $\alpha = 1 \Longrightarrow a = 3a_1 = 3a'$ and $3 \nmid a_1$, the equation (1.5.38) becomes:

$$\mu = 2im$$

and the first equation (1.5.37) is written as:

$$k = min(2im, 2jn, im + jn) = 2im$$

- If $k = 2im$, then $2im \leq 2jn \Longrightarrow im \leq jn \Longrightarrow hl = im$, and (1.5.40) gives $\mu = 2im = nj + hl = im + nj \Longrightarrow im = jn = hl$. Hence $3 \mid A, 3 \mid B$ and $3 \mid C$ and the conjecture (3.1.1) is verified.
- If $k = 2jn \Longrightarrow 2jn = 2im \Longrightarrow im = jn = hl$. Hence $3 \mid A, 3 \mid B$ and $3 \mid C$ and the conjecture (3.1.1) is verified.
- If $k = im + jn = 2im \Longrightarrow im = jn \Longrightarrow \epsilon = hl = im = jn$ case that is seen above and we deduce that $3 \mid A, 3 \mid B$ and $3 \mid C$, and the conjecture (3.1.1) is verified.

** B-2-2-1-2- $\alpha > 1 \Longrightarrow \alpha \geq 2$ and $a' = 3^{\alpha-1}a_1$.
- If $k = 2im \Longrightarrow 2im = \mu$, but $\mu = 2im + 1 - \alpha$ that is impossible.
- If $k = 2jn = \mu \Longrightarrow 2jn = 2im + 1 - \alpha$. We obtain $2jn < 2im \Longrightarrow jn < im \Longrightarrow 2jn < im + jn$, $k = 2jn$ is just the minimum of $(2im, 2jn, im + jn)$. We obtain $jn = hl < im$ and the equation (1.5.40) becomes:

$$B_1^n C_1^l = p_1(b - 4 \times 3^{(\alpha-1)} a_1)$$

The conjecture (3.1.1) is verified.

- If $k = im + jn \leq 2im \Longrightarrow jn \leq im$ and $k = im + jn \leq 2jn \Longrightarrow im \leq jn \Longrightarrow im = jn \Longrightarrow k = im + jn = 2im = \mu$ but $\mu = 2im + 1 - \alpha$ that is impossible.
- If $k = im + jn < 2im \Longrightarrow jn < im$ and $2jn < im + jn = k$ that is a contradiction with $k = min(2im, 2jn, im + jn)$.

** B-2-2-2- We suppose that $\omega \neq 3$. We write $a = \omega^\alpha a_1$ with $\omega \nmid a_1$ and $p' = \omega^\mu p_1$ with $\omega \nmid p_1$. As $A^{2m} = 4a'p' = 4\omega^{\alpha+\mu}.a_1.p_1 \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i A_1$, $\omega \nmid A_1$. But $B^n C^l = p'(b - 4a') = \omega^\mu p_1(b - 4a') \Longrightarrow \omega \mid B^n C^l \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** B-2-2-2-1- $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j B_1$ and $\omega \nmid B_1$. From $A^m + B^n = C^l \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$. As $p = bp' = \omega^\mu b p_1 = \omega^k(\omega^{2im-k}A_1^{2m} + \omega^{2jn-k}B_1^{2n} + \omega^{im+jn-k}A_1^m B_1^n)$ with $k = min(2im, 2jn, im + jn)$. Then :
- If $\mu = k$, then $\omega \nmid b$ and the conjecture (3.1.1) is verified.
- If $k > \mu$, then $\omega \mid b$, but $\omega \mid a$ we deduce the contradiction with a, b coprime.
- If $k < \mu$, it follows from :

$$\omega^\mu b p_1 = \omega^k(\omega^{2im-k}A_1^{2m} + \omega^{2jn-k}B_1^{2n} + \omega^{im+jn-k}A_1^m B_1^n)$$

that $\omega \mid A_1$ or $\omega \mid B_1$ that is a contradiction with the hypothesis.

** B-2-2-2-2- If $\omega \mid C^l \Longrightarrow \omega \mid C \Longrightarrow C = \omega^h C_1$ with $\omega \nmid C_1$. From $A^m + B^n = C^l \Longrightarrow \omega \mid (C^l - A^m) \Longrightarrow \omega \mid B$. Then, we obtain the same results as B-2-2-2-1- above.

1.5.5 Case $b = 2p$ and $3 \mid a$

We have :

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{2p} \Longrightarrow A^{2m} = \frac{4p.a}{3b} = \frac{4p}{3}.\frac{3a'}{2p} = 2a' = (A^m)^2 \Longrightarrow 2 \mid a' \Longrightarrow 2 \mid a$$

Then $2 \mid a$ and $2 \mid b$ that is a contradiction with a, b coprime.

1.5.6 Case $b = 4p$ and $3 \mid a$

We have :

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{4p} \Longrightarrow A^{2m} = \frac{4p.a}{3b} = \frac{4p}{3}.\frac{3a'}{4p} = a' = (A^m)^2 = a''^2$$
$$\text{with} \quad A^m = a''$$

Let us calculate A^mB^n, we obtain:

$$A^mB^n = \frac{p\sqrt{3}}{3}.sin\frac{2\theta}{3} - \frac{2p}{3}cos^2\frac{\theta}{3} = \frac{p\sqrt{3}}{3}.sin\frac{2\theta}{3} - \frac{a'}{2} \Longrightarrow$$
$$A^mB^n + \frac{A^{2m}}{2} = \frac{p\sqrt{3}}{3}.sin\frac{2\theta}{3}$$

Let:

$$A^{2m} + 2A^mB^n = \frac{2p\sqrt{3}}{3}sin\frac{2\theta}{3} \tag{1.5.41}$$

The left member of (1.5.41) is an integer and p is an integer, then $\frac{2\sqrt{3}}{3}sin\frac{2\theta}{3}$ will be written as :

$$\frac{2\sqrt{3}}{3}sin\frac{2\theta}{3} = \frac{k_1}{k_2}$$

where k_1, k_2 are two integers coprime and $k_2 \mid p \Longrightarrow p = k_2.k_3$.

** C-1- Firstly, we suppose that $k_3 \neq 1$. Then :

$$A^{2m} + 2A^mB^n = k_3.k_1$$

Let μ be a prime integer and $\mu \mid k_3$, then $\mu \mid A^m(A^m + 2B^n) \Longrightarrow \mu \mid A^m$ or $\mu \mid (A^m + 2B^n)$.

** C-1-1- If $\mu \mid (A^m = a'') \Longrightarrow \mu \mid (a''^2 = a') \Longrightarrow \mu \mid (3a' = a)$. As $\mu \mid k_3 \Longrightarrow \mu \mid p \Longrightarrow \mu \mid (4p = b)$, then the contradiction with a, b coprime.

** C-1-2- If $\mu \mid (A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$, then:

$$\mu \neq 2 \quad and \quad \mu \nmid B^n \tag{1.5.42}$$

$\mu \mid (A^m + 2B^n)$, we write:

$$A^m + 2B^n = \mu.t'$$

Then:

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^mB^n = \mu^2 t'^2 - 2t'\mu B^n + B^{2n}$$
$$\Longrightarrow p = t'^2\mu^2 - 2t'B^n\mu + B^n(B^n - A^m)$$

As $b = 4p = 4k_2.k_3$ and $\mu \mid k_3$ then $\mu \mid b \Longrightarrow \exists \mu'$ so that $b = \mu.\mu'$, we obtain:

$$\mu'.\mu = \mu(4\mu t'^2 - 8t'B^n) + 4B^n(B^n - A^m)$$

The last equation implies $\mu \mid 4B^n(B^n - A^m)$, but $\mu \neq 2$ then $\mu \mid B^n$ or $\mu \mid (B^n - A^m)$.

** C-1-1-1- If $\mu \mid B^n \Longrightarrow$ then the contradiction with (1.5.42).

** C-1-1-2- If $\mu \mid (B^n - A^m)$ and using $\mu \mid (A^m + 2B^n)$, we have :

$$\mu \mid 3B^n \Longrightarrow \begin{cases} \mu \mid B^n \\ or \\ \mu = 3 \end{cases}$$

** C-1-1-2-1- If $\mu \mid B^n$ then the contradiction with (1.5.42).

** C-1-1-2-2- If $\mu = 3$, then $3 \mid b$, but $3 \mid a$ then the contradiction with a, b coprime.

** C-2- We assume now that $k_3 = 1$, then:

$$A^{2m} + 2A^m B^n = k_1 \tag{1.5.43}$$
$$p = k_2$$
$$\frac{2\sqrt{3}}{3} sin \frac{2\theta}{3} = \frac{k_1}{p}$$

We take the square of the last equation, we obtain :

$$\frac{4}{3} sin^2 \frac{2\theta}{3} = \frac{k_1^2}{p^2}$$

$$\frac{16}{3} sin^2 \frac{\theta}{3} cos^2 \frac{\theta}{3} = \frac{k_1^2}{p^2}$$

$$\frac{16}{3} sin^2 \frac{\theta}{3} . \frac{3a'}{b} = \frac{k_1^2}{p^2}$$

Finally:

$$a'(4p - 3a') = k_1^2 \tag{1.5.44}$$

but $a' = a''^2$, then $4p - 3a'$ is a square. Let :

$$\lambda^2 = 4p - 3a' = 4p - a = b - a$$

The equation (1.5.44) becomes :

$$a''^2 \lambda^2 = k_1^2 \Longrightarrow k_1 = a''\lambda \tag{1.5.45}$$

taking the positive root. Using (1.5.43), we have:

$$k_1 = A^m(A^m + 2B^n) = a''(A^m + 2B^n)$$

Then :

$$A^m + 2B^n = \lambda$$

Now, we consider that $b - a = \lambda^2 \Longrightarrow \lambda^2 + 3a''^2 = b$, then the couple (λ, a'') is a solution of the Diophantine equation:

$$X^2 + 3Y^2 = b \tag{1.5.46}$$

with $X = \lambda$ and $Y = a''$. But using one theorem on the solutions of the equation given by (1.5.46), b is written under the form (see theorem 37.4 in [1]):

$$b = 2^{2s} \times 3^t . p_1^{t_1} \cdots p_g^{t_g} q_1^{2s_1} \cdots q_r^{2s_r}$$

where p_i are prime integers so that $p_i \equiv 1 (\mathrm{mod}\, 6)$, the q_j are also prime integers so that $q_j \equiv 5 (\mathrm{mod}\, 6)$. Then, as $b = 4p$:

- If $t \geq 1 \Longrightarrow 3 \mid b$, but $3 \mid a$, then the contradiction with a, b coprime.

** C-2-2-1- Hence, we suppose that p is written under the form:

$$p = p_1^{t_1} \cdots p_g^{t_g} q_1^{2s_1} \cdots q_r^{2s_r}$$

with $p_i \equiv 1 (\mathrm{mod}\, 6)$ and $q_j \equiv 5 (\mathrm{mod}\, 6)$. Finally, we obtain that $p \equiv 1 (\mathrm{mod}\, 6)$. We will verify if this condition does not give contradictions.

Table 1.1: Table of $p \pmod 6$

A^m, B^n	0	1	2	3	4	5
0	0	**1**	4	3	4	**1**
1	**1**	3	**1**	**1**	3	**1**
2	4	**1**	0	**1**	4	3
3	3	**1**	**1**	3	**1**	**1**
4	4	3	4	**1**	0	**1**
5	**1**	**1**	3	**1**	**1**	3

We will present the table of the value modulo 6 of $p = A^{2m} + A^mB^n + B^{2n}$ in function of the values of $A^m, B^n \pmod 6$. We obtain the table below:

** C-2-2-1-1- Case $A^m \equiv 0 \pmod 6 \Longrightarrow 2 \mid (A^m = a'') \Longrightarrow 2 \mid (a' = a''^2) \Longrightarrow 2 \mid a$, but $2 \mid b$, then the contradiction with a, b coprime. All the cases of the first line of the table 1.1 are to reject.

** C-2-2-1-2- Case $A^m \equiv 1 \pmod 6$ and $B^n \equiv 0 \pmod 6$, then $2 \mid B^n \Longrightarrow B^n = 2B'$ and p is written as $p = (A^m + B')^2 + 3B'^2$ with $(p, 3) = 1$, if not $3 \mid p$, then $3 \mid b$, but $3 \mid a$, then the contradiction with a, b coprime. Hence, the pair $(A^m + B', B')$ is a solution of the Diophantine equation:

$$x^2 + 3y^2 = p \tag{1.5.47}$$

The solution $x = A^m + B', y = B'$ is unique because $x - y$ verifies $x - y = A^m$. If (u, v) is another pair solution of (1.5.47), with $u, v \in \mathbb{N}^*$, then we obtain:

$$\begin{aligned} u^2 + 3v^2 &= p \\ u - v &= A^m \end{aligned}$$

Then $u = v + A^m$ and we obtain the equation of second degree $4v^2 + 2vA^m - 2B'(A^m + 2B') = 0$ that gives as positive root $v_1 = B' = y$, then $u = A^m + B' = x$. It follows that p in (1.5.47) has an unique representation under the form $X^2 + 3Y^2$ with $X, 3Y$ coprime. As p is an odd integer number, we applique one of Euler's theorems on convenient numbers "numerus idoneus" (see [2, 3]) : *Let m be an odd number relatively prime to n which is properly represented by $x^2 + ny^2$. If the equation $m = x^2 + ny^2$ has only one solution with $x, y > 0$, then m is a prime number*. Then p is prime and $4p$ has an unique representation (we put $U = 2u$, $V = 2v$, with $U^2 + 3V^2 = 4p$ and $U - V = 2A^m$). But $b = 4p \Longrightarrow \lambda^2 + 3a''^2 = (2(A^m + B'))^2 + 3(2B')^2$, the representation of $4p$ is unique gives:

$$\begin{aligned} \lambda &= 2(A^m + B') = 2a'' + B^n \\ \textit{and} \quad a'' &= 2B' = B^n = A^m \end{aligned}$$

But $A^m > B^n$, then the contradiction.

** C-2-2-1-3- Case $A^m \equiv 1 \pmod 6$ and $B^n \equiv 2 \pmod 6$, then B^n is even, see C-2-2-1-2-.

** C-2-2-1-4- Case $A^m \equiv 1 \pmod 6$ and $B^n \equiv 3 \pmod 6$, then $3 \mid B^n \Longrightarrow B^n = 3B'$. We can write $b = 4p = (2A^m + 3B')^2 + 3(3B')^2 = \lambda^2 + 3a''^2$. The unique representation of b as $x^2 + 3y^2 = \lambda^2 + 3a''^2 \Longrightarrow a'' = A^m = 3B' = B^n$, then the contradiction with $A^m > B^n$.

** C-2-2-1-5- Case $A^m \equiv 1 \pmod 6$ and $B^n \equiv 5 \pmod 6$, then $C^l \equiv 0 \pmod 6 \Longrightarrow 2 \mid C^l$, see C-2-2-1-2-.

** C-2-2-1-6- Case $A^m \equiv 2(\bmod 6) \Longrightarrow 2 \mid a'' \Longrightarrow 2 \mid a$, but $2 \mid b$, then the contradiction with a, b coprime.

** C-2-2-1-7- Case $A^m \equiv 3(\bmod 6)$ and $B^n \equiv 1(\bmod 6)$, then $C^l \equiv 4(\bmod 6) \Longrightarrow 2 \mid C^l \Longrightarrow C^l = 2C'$, we can write that $p = (C' - B^n)^2 + 3C'^2$, see C-2-2-1-2-.

** C-2-2-1-8- Case $A^m \equiv 3(\bmod 6)$ and $B^n \equiv 2(\bmod 6)$, then B^n is even, see C-2-2-1-2-.

** C-2-2-1-9- Case $A^m \equiv 3(\bmod 6)$ and $B^n \equiv 4(\bmod 6)$, then B^n is even, see C-2-2-1-2-.

** C-2-2-1-10- Case $A^m \equiv 3(\bmod 6)$ and $B^n \equiv 5(\bmod 6)$, then $C^l \equiv 2(\bmod 6) \Longrightarrow 2 \mid C^l$, see C-2-2-1-2-.

** C-2-2-1-11- Case $A^m \equiv 4(\bmod 6) \Longrightarrow 2 \mid a'' \Longrightarrow 2 \mid a$, but $2 \mid b$, then the contradiction with a, b coprime.

** C-2-2-1-12- Case $A^m \equiv 5(\bmod 6)$ and $B^n \equiv 0(\bmod 6)$, then B^n is even, see C-2-2-1-2-.

** C-2-2-1-13- Case $A^m \equiv 5(\bmod 6)$ and $B^n \equiv 1(\bmod 6)$, then $C^l \equiv 0(\bmod 6) \Longrightarrow 2 \mid C^l$, see C-2-2-1-2-.

** C-2-2-1-14- Case $A^m \equiv 5(\bmod 6)$ and $B^n \equiv 3(\bmod 6)$, then $C^l \equiv 2(\bmod 6) \Longrightarrow 2 \mid C^l \Longrightarrow C^l = 2C'$, p is written as $p = (C' - B^n)^2 + 3C'^2$, see C-2-2-1-2-.

** C-2-2-1-15- Case $A^m \equiv 5(\bmod 6)$ and $B^n \equiv 4(\bmod 6)$, then B^n is even, see C-2-2-1-2-.

We have achieved the study all the cases of the table 1.1 giving contradictions.

Then the case $k_3 = 1$ is impossible.

1.5.7 Case $3 \mid a$ and $b = 2p',\ b \neq 2$ with $p' \mid p$

$3 \mid a \Longrightarrow a = 3a'$, $b = 2p'$ with $p = k.p'$, then:

$$A^{2m} = \frac{4.p}{3} \cdot \frac{a}{b} = \frac{4.k.p'.3.a'}{6p'} = 2.k.a'$$

We calculate B^nC^l:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right)$$

but $\sqrt[3]{\rho^2} = \dfrac{p}{3}$, then using $cos^2\dfrac{\theta}{3} = \dfrac{3.a'}{b}$:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3.a'}{b}\right) = p.\left(1 - \frac{4.a'}{b}\right) = k(p' - 2a')$$

As $p = b.p'$, and $p' > 1$, then we have:

$$B^nC^l = k(p' - 2a') \tag{1.5.48}$$
$$\text{and} \quad A^{2m} = 2k.a' \tag{1.5.49}$$

** D-1- We suppose that k is prime.

** D-1-1- If $k = 2$, then we have $p = 2p' = b \Longrightarrow 2 \mid b$, but $A^{2m} = 4a' = (A^m)^2 \Longrightarrow A^m = 2a''$ with $a' = a''^2$, then $2 \mid a'' \Longrightarrow 2 \mid (a = 3a''^2)$, it follows the contradiction with a, b coprime.

** D-1-2- We suppose $k \neq 2$. From $A^{2m} = 2k.a' = (A^m)^2 \Longrightarrow k \mid a'$ and $2 \mid a' \Longrightarrow a' = 2.k.a''^2 \Longrightarrow A^m = 2.k.a''$. Then $k \mid A^m \Longrightarrow k \mid A \Longrightarrow A = k^i.A_1$ with $i \geq 1$ and $k \nmid A_1$. $k^{im}A_1^m = 2ka'' \Longrightarrow 2a'' = k^{im-1}A_1^m$. From $B^nC^l = k(p'-2a') \Longrightarrow k \mid (B^nC^l) \Longrightarrow k \mid B^n$ or $k \mid C^l$.

** D-1-2-1- We suppose that $k \mid B^n \Longrightarrow k \mid B \Longrightarrow B = k^j.B_1$ with $j \geq 1$ and $k \nmid B_1$. It follows $k^{nj-1}B_1^nC^l = p' - 2a' = p' - 4ka''^2$. As $n \geq 3 \Longrightarrow nj-1 \geq 2$, then $k \mid p'$ but $k \neq 2 \Longrightarrow k \mid (2p' = b)$, but $k \mid a' \Longrightarrow k \mid (3a' = a)$. It follows the contradiction with a, b coprime.

** D-1-2-2- If $k \mid C^l$ we obtain the identical results.

** D-2- We suppose that k is not prime. Let ω be an integer prime so that $k = \omega^s.k_1$, with $s \geq 1$, $\omega \nmid k_1$. The equations (1.5.48-1.5.49) become:

$$B^nC^l = \omega^s.k_1(p'-2a')$$
$$\text{and} \quad A^{2m} = 2\omega^s.k_1.a'$$

** D-2-1- We suppose that $\omega = 2$, then we have the equations:

$$A^{2m} = 2^{s+1}.k_1.a' \tag{1.5.50}$$
$$B^nC^l = 2^s.k_1(p'-2a') \tag{1.5.51}$$

** D-2-1-1- Case: $2 \mid a' \Longrightarrow 2 \mid a$, but $2 \mid b$, then the contradiction with a, b coprime.

** D-2-1-2- Case: $2 \nmid a'$. As $2 \nmid k_1$, the equation (1.5.50) gives $2 \mid A^{2m} \Longrightarrow A = 2^iA_1$, with $i \geq 1$ and $2 \nmid A_1$. It follows that $2im = s+1$.

** D-2-1-2-1- We suppose that $2 \nmid (p'-2a') \Longrightarrow 2 \nmid p'$. From the equation (1.5.51), we obtain that $2 \mid B^nC^l \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** D-2-1-2-1-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^jB_1$ with $2 \nmid B_1$ and $j \geq 1$, then $B_1^nC^l = 2^{s-jn}k_1(p'-2a')$:

- If $s - jn \geq 1$, then $2 \mid C^l \Longrightarrow 2 \mid C$, and no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$, and the conjecture (3.1.1) is verified.
- If $s - jn \leq 0$, from $B_1^nC^l = 2^{s-jn}k_1(p'-2a') \Longrightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n \Longrightarrow 2 \mid C^l$.

** D-2-1-2-1-2- Using the same method of the proof above, we obtain the identical results if $2 \mid C^l$.

** D-2-1-2-2- We suppose now that $2 \mid (p'-2a') \Longrightarrow p' - 2a' = 2^\mu.\Omega$, with $\mu \geq 1$ and $2 \nmid \Omega$. We recall that $2 \nmid a'$. The equation (1.5.51) is written as:

$$B^nC^l = 2^{s+\mu}.k_1.\Omega$$

This last equation implies that $2 \mid (B^nC^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** D-2-1-2-2-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^jB_1$ with $j \geq 1$ and $2 \nmid B_1$. Then $B_1^nC^l = 2^{s+\mu-jn}.k_1.\Omega$:

- If $s + \mu - jn \geq 1$, then $2 \mid C^l \Longrightarrow 2 \mid C$, no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$, and the conjecture (3.1.1) is verified.
- If $s + \mu - jn \leq 0$, from $B_1^nC^l = 2^{s+\mu-jn}k_1.\Omega \Longrightarrow 2 \nmid C^l$, then contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n \Longrightarrow 2 \mid C^l$.

** D-2-1-2-2-2- We obtain the identical results if $2 \mid C^l$.

** D-2-2- We suppose that $\omega \neq 2$. We have then the equations:

$$A^{2m} = 2\omega^s.k_1.a' \tag{1.5.52}$$
$$B^nC^l = \omega^s.k_1.(p' - 2a') \tag{1.5.53}$$

As $\omega \neq 2$, from the equation (1.5.52), we have $2 \mid (k_1.a')$. If $2 \mid a' \Longrightarrow 2 \mid a$, but $2 \mid b$, then the contradiction with a, b coprime.

** D-2-2-1- Case: $2 \nmid a'$ and $2 \mid k_1 \Longrightarrow k_1 = 2^\mu.\Omega$ with $\mu \geq 1$ and $2 \nmid \Omega$. From the equation (1.5.52), we have $2 \mid A^{2m} \Longrightarrow 2 \mid A \Longrightarrow A = 2^iA_1$ with $i \geq 1$ and $2 \nmid A_1$, then $2im = 1 + \mu$. The equation (1.5.53) becomes:

$$B^nC^l = \omega^s.2^\mu.\Omega.(p' - 2a') \tag{1.5.54}$$

From the equation (1.5.54), we obtain $2 \mid (B^nC^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** D-2-2-1-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^jB_1$, with $j \in \mathbb{N}^*$ and $2 \nmid B_1$.

** D-2-2-1-1-1- We suppose that $2 \nmid (p' - 2a')$, then we have $B_1^nC^l = \omega^s2^{\mu-jn}\Omega(p' - 2a')$:
- If $\mu - jn \geq 1 \Longrightarrow 2 \mid C^l \Longrightarrow 2 \mid C$, no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1.1) is verified.
- If $\mu - jn \leq 0 \Longrightarrow 2 \nmid C^l$ then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

** D-2-2-1-1-2- We suppose that $2 \mid (p' - 2a') \Longrightarrow p' - 2a' = 2^\alpha.P$, with $\alpha \in \mathbb{N}^*$ and $2 \nmid P$. It follows that $B_1^nC^l = \omega^s2^{\mu+\alpha-jn}\Omega.P$:
- If $\mu + \alpha - jn \geq 1 \Longrightarrow 2 \mid C^l \Longrightarrow 2 \mid C$, no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1.1) is verified.
- If $\mu + \alpha - jn \leq 0 \Longrightarrow 2 \nmid C^l$ then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

** D-2-2-1-2- We suppose now that $2 \mid C^n \Longrightarrow 2 \mid C$. Using the same method described above, we obtain the identical results.

1.5.8 Case $3 \mid a$ and $b = 4p',\ b \neq 4$ with $p' \mid p$

$3 \mid a \Longrightarrow a = 3a', b = 4p'$ with $p = k.p', k \neq 1$ if not $b = 4p$ this case has been studied (see paragraph 1.5.6), then we have :

$$A^{2m} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.k.p'.3.a'}{12p'} = k.a'$$

We calculate B^nC^l:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right)$$

but $\sqrt[3]{\rho^2} = \dfrac{p}{3}$, then using $cos^2\dfrac{\theta}{3} = \dfrac{3.a'}{b}$:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3.a'}{b}\right) = p.\left(1 - \frac{4.a'}{b}\right) = k(p' - a')$$

As $p = b.p'$, and $p' > 1$, we have :

$$B^nC^l = k(p' - a') \tag{1.5.55}$$
$$\text{and} \quad A^{2m} = k.a' \tag{1.5.56}$$

** E-1- We suppose that k is prime. From $A^{2m} = k.a' = (A^m)^2 \Longrightarrow k \mid a'$ and $a' = k.a''^2 \Longrightarrow A^m = k.a''$. Then $k \mid A^m \Longrightarrow k \mid A \Longrightarrow A = k^i.A_1$ with $i \geq 1$ and $k \nmid A_1$. $k^{mi}A_1^m = ka'' \Longrightarrow a'' = k^{mi-1}A_1^m$. From $B^nC^l = k(p' - a') \Longrightarrow k \mid (B^nC^l) \Longrightarrow k \mid B^n$ or $k \mid C^l$.

** E-1-1- We suppose that $k \mid B^n \Longrightarrow k \mid B \Longrightarrow B = k^j.B_1$ with $j \geq 1$ and $k \nmid B_1$. Then $k^{n.j-1}B_1^nC^l = p' - a'$. As $n.j - 1 \geq 2 \Longrightarrow k \mid (p' - a')$. But $k \mid a' \Longrightarrow k \mid a$, then $k \mid p' \Longrightarrow k \mid (4p' = b)$ and we arrive to the contradiction that a, b are coprime.

** E-1-2- We suppose that $k \mid C^l$, using the same method with the above hypothesis $k \mid B^n$, we obtain the identical results.

** E-2- We suppose that k is not prime.

** E-2-1- We take $k = 4 \Longrightarrow p = 4p' = b$, it is the case 1.5.3 studied above.

** E-2-2- We suppose that $k \geq 6$ not prime. Let ω be a prime so that $k = \omega^s.k_1$, with $s \geq 1$, $\omega \nmid k_1$. The equations (1.5.55-1.5.56) become:

$$B^nC^l = \omega^s.k_1(p' - a') \tag{1.5.57}$$
$$\text{and} \quad A^{2m} = \omega^s.k_1.a' \tag{1.5.58}$$

** E-2-2-1- We suppose that $\omega = 2$.

** E-2-2-1-1- If $2 \mid a' \Longrightarrow 2 \mid (3a' = a)$, but $2 \mid (4p' = b)$, then the contradiction with a, b coprime.

** E-2-2-1-2- We consider that $2 \nmid a'$. From the equation (1.5.58), it follows that $2 \mid A^{2m} \Longrightarrow 2 \mid A \Longrightarrow A = 2^iA_1$ with $2 \nmid A_1$ and:

$$B^nC^l = 2^sk_1(p' - a')$$

** E-2-2-1-2-1- We suppose that $2 \nmid (p' - a')$, from the above expression, we have $2 \mid (B^nC^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** E-2-2-1-2-1-1- If $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^jB_1$ with $2 \nmid B_1$. Then $B_1^nC^l = 2^{2im-jn}k_1(p' - a')$:
- If $2im - jn \geq 1 \Longrightarrow 2 \mid C^l \Longrightarrow 2 \mid C$, no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1.1) is verified.
- If $2im - jn \leq 0 \Longrightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n \Longrightarrow 2 \mid C^l$.

** E-2-2-1-2-1-2- If $2 \mid C^l \Longrightarrow 2 \mid C$, using the same method described above, we obtain the identical results.

** E-2-2-1-2-2- We suppose that $2 \mid (p' - a')$. As $2 \nmid a' \Longrightarrow 2 \nmid p'$, $2 \mid (p' - a') \Longrightarrow p' - a' = 2^\alpha.P$ with $\alpha \geq 1$ and $2 \nmid P$. The equation (1.5.57) is written as :

$$B^nC^l = 2^{s+\alpha}k_1.P = 2^{2im+\alpha}k_1.P \tag{1.5.59}$$

then $2 \mid (B^nC^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** E-2-2-1-2-2-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^jB_1$, with $2 \nmid B_1$. The equation (1.5.59) becomes $B_1^nC^l = 2^{2im+\alpha-jn}k_1P$:
- If $2im + \alpha - jn \geq 1 \Longrightarrow 2 \mid C^l \Longrightarrow 2 \mid C$, no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1.1) is verified.
- If $2im + \alpha - jn \leq 0 \Longrightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n \Longrightarrow 2 \mid C^l$.

** E-2-2-1-2-2-2- We suppose that $2 \mid C^l \Longrightarrow 2 \mid C$. Using the same method described above, we obtain the identical results.

** E-2-2-2- We suppose that $\omega \neq 2$. We recall the equations:

$$A^{2m} = \omega^s.k_1.a' \tag{1.5.60}$$
$$B^nC^l = \omega^s.k_1(p' - a') \tag{1.5.61}$$

** E-2-2-2-1- We suppose that ω, a' are coprime, then $\omega \nmid a'$. From the equation (1.5.60), we have $\omega \mid A^{2m} \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i A_1$ with $\omega \nmid A_1$ and $s = 2im$.

** E-2-2-2-1-1- We suppose that $\omega \nmid (p' - a')$. From the equation (1.5.61) above, we have $\omega \mid (B^n C^l) \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** E-2-2-2-1-1-1- If $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j B_1$ with $\omega \nmid B_1$. Then $B_1^n C^l = 2^{2im-jn} k_1 (p' - a')$:

- If $2im - jn \geq 1 \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$, no contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$ and the conjecture (3.1.1) is verified.
- If $2im - jn \leq 0 \Longrightarrow \omega \nmid C^l$, then the contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n \Longrightarrow \omega \mid C^l$.

** E-2-2-2-1-1-2- If $\omega \mid C^l \Longrightarrow \omega \mid C$, using the same method described above, we obtain the identical results.

** E-2-2-2-1-2- We suppose that $\omega \mid (p' - a') \Longrightarrow \omega \nmid p'$ as ω and a' are coprime. $\omega \mid (p' - a') \Longrightarrow p' - a' = \omega^\alpha . P$ with $\alpha \geq 1$ and $\omega \nmid P$. The equation (1.5.61) becomes :

$$B^n C^l = \omega^{s+\alpha} k_1 . P = \omega^{2im+\alpha} k_1 . P \tag{1.5.62}$$

then $\omega \mid (B^n C^l) \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** E-2-2-2-1-2-1- We suppose that $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j B_1$, with $\omega \nmid B_1$. The equation (1.5.62) is written as $B_1^n C^l = 2^{2im+\alpha-jn} k_1 P$:

- If $2im + \alpha - jn \geq 1 \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$, no contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$ and the conjecture (3.1.1) is verified.
- If $2im + \alpha - jn \leq 0 \Longrightarrow \omega \nmid C^l$, then the contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n \Longrightarrow \omega \mid C^l$.

** E-2-2-2-1-2-2- We suppose that $\omega \mid C^l \Longrightarrow \omega \mid C$, using the same method described above, we obtain the identical results.

** E-2-2-2-2- We suppose that ω, a' are not coprime, then $a' = \omega^\beta . a''$ with $\omega \nmid a''$. The equation (1.5.60) becomes:

$$A^{2m} = \omega^s k_1 a' = \omega^{s+\beta} k_1 . a''$$

We have $\omega \mid A^{2m} \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i A_1$ with $\omega \nmid A_1$ and $s + \beta = 2im$.

** E-2-2-2-2-1- We suppose that $\omega \nmid (p' - a') \Longrightarrow \omega \nmid p' \Longrightarrow \omega \nmid (b = 4p')$. From the equation (1.5.61), we obtain $\omega \mid (B^n C^l) \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** E-2-2-2-2-1-1- If $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j B_1$ with $\omega \nmid B_1$. Then $B_1^n C^l = 2^{s-jn} k_1 (p' - a')$:

- If $s - jn \geq 1 \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$, no contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$ and the conjecture (3.1.1) is verified.
- If $s - jn \leq 0 \Longrightarrow \omega \nmid C^l$, then the contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n \Longrightarrow \omega \mid C^l$.

** E-2-2-2-2-1-2- If $\omega \mid C^l \Longrightarrow \omega \mid C$, using the same method described above, we obtain the identical results.

** E-2-2-2-2-2- We suppose that $\omega \mid (p' - a' = p' - \omega^\beta . a'') \Longrightarrow \omega \mid p' \Longrightarrow \omega \mid (4p' = b)$, but $\omega \mid a' \Longrightarrow \omega \mid a$. Then the contradiction with a, b coprime.

The study of the cases of 1.5.8 is achieved.

1.5.9 Case $3 \mid a$ and $b \mid 4p$

$a = 3a'$ and $4p = k_1 b$. As $A^{2m} = \dfrac{4p}{3} cos^2\dfrac{\theta}{3} = \dfrac{4p}{3}\dfrac{3a'}{b} = k_1 a'$ and $B^n C^l$:

$$B^n C^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3a'}{b}\right) = \frac{k_1}{4}(b - 4a')$$

As $B^n C^l$ is an integer, we must obtain $4 \mid k_1$, or $4 \mid (b - 4a')$ or ($2 \mid k_1$ and $2 \mid (b - 4a')$).
** F-1- If $k_1 = 1 \Rightarrow b = 4p$: it is the case 1.5.6.

** F-2- If $k_1 = 4 \Rightarrow p = b$: it is the case 1.5.3.

** F-3- If $k_1 = 2$ and $2 \mid (b - 4a')$: in this case, we have $A^{2m} = 2a' \Longrightarrow 2 \mid a' \Longrightarrow 2 \mid a$. $2 \mid (b - 4a') \Longrightarrow 2 \mid b$ then the contradiction with a, b coprime.

** F-4- If $2 \mid k_1$ and $2 \mid (b - 4a')$: $2 \mid (b - 4a') \Longrightarrow b - 4a' = 2^\alpha \lambda$, α and $\lambda \in \mathbb{N}^* \geq 1$ with $2 \nmid \lambda$; $2 \mid k_1 \Longrightarrow k_1 = 2^t k_1'$ with $t \geq 1 \in \mathbb{N}^*$ with $2 \nmid k_1'$ and we have:

$$A^{2m} = 2^t k_1' a' \tag{1.5.63}$$
$$B^n C^l = 2^{t+\alpha-2} k_1' \lambda \tag{1.5.64}$$

From the equation (1.5.63), we have $2 \mid A^{2m} \Longrightarrow 2 \mid A \Longrightarrow A = 2^i A_1$, $i \geq 1$ and $2 \nmid A_1$.
** F-4-1- We suppose that $t = \alpha = 1$, then the equations (1.5.63-1.5.64) become :

$$A^{2m} = 2k_1' a' \tag{1.5.65}$$
$$B^n C^l = k_1' \lambda \tag{1.5.66}$$

From the equation (1.5.65) it follows that $2 \mid a' \Longrightarrow 2 \mid (a = 3a')$. But $b = 4a' + 2\lambda \Longrightarrow 2 \mid b$, then the contradiction with a, b coprime.

** F-4-2- We suppose that $t + \alpha - 2 \geq 1$ and we have the expressions:

$$A^{2m} = 2^t k_1' a' \tag{1.5.67}$$
$$B^n C^l = 2^{t+\alpha-2} k_1' \lambda \tag{1.5.68}$$

** F-4-2-1- We suppose that $2 \mid a' \Longrightarrow 2 \mid a$, but $b = 2^\alpha \lambda + 4a' \Longrightarrow 2 \mid b$, then the contradiction with a, b coprime.

** F-4-2-2- We suppose that $2 \nmid a'$. From (1.5.67), we have $2 \mid A^{2m} \Longrightarrow 2 \mid A \Longrightarrow A = 2^i A_1$ and $B^n C^l = 2^{t+\alpha-2} k_1' \lambda \Longrightarrow 2 \mid B^n C^l \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** F-4-2-2-1- We suppose that $2 \mid B^n$. We have $2 \mid B \Longrightarrow B = 2^j B_1$, $j \geq 1$ and $2 \nmid B_1$. The equation (1.5.68) becomes $B_1^n C^l = 2^{t+\alpha-2-jn} k_1' \lambda$:

- If $t + \alpha - 2 - jn > 0 \Longrightarrow 2 \mid C^l \Longrightarrow 2 \mid C$, no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (3.1.1) is verified.
- If $t + \alpha - 2 - jn < 0 \Longrightarrow 2 \mid k_1' \lambda$, but $2 \nmid k_1'$ and $2 \nmid \lambda$. Then this case is impossible.
- If $t + \alpha - 2 - jn = 0 \Longrightarrow B_1^n C^l = k_1' \lambda \Longrightarrow 2 \nmid C^l$ then it is a contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

** F-4-2-2-2- We suppose that $2 \mid C^l$. We use the same method described above, we obtain the identical results.

** F-5- We suppose that $4 \mid k_1$ with $k_1 > 4 \Rightarrow k_1 = 4k_2'$, we have :

$$A^{2m} = 4k_2' a' \tag{1.5.69}$$
$$B^n C^l = k_2'(b - 4a') \tag{1.5.70}$$

** F-5-1- We suppose that k'_2 is prime, from (1.5.69), we have $k'_2 \mid a'$. From (1.5.70), $k'_2 \mid (B^nC^l) \Longrightarrow k'_2 \mid B^n$ or $k'_2 \mid C^l$.

** F-5-1-1- We suppose that $k'_2 \mid B^n \Longrightarrow k'_2 \mid B \Longrightarrow B = k'^{\beta}_2.B_1$ with $\beta \geq 1$ and $k'_2 \nmid B_1$. It follows that we have $k'^{n\beta-1}_2 B^n_1C^l = b - 4a' \Longrightarrow k'_2 \mid b$ then the contradiction with a, b coprime.

** F-5-1-2- We obtain identical results if we suppose that $k'_2 \mid C^l$.

** F-5-2- We suppose that k'_2 is not prime.

** F-5-2-1- We suppose that k'_2 and a' are coprime. From (1.5.69), k'_2 can be written under the form $k'_2 = q^{2j}_1.q^2_2$ and $q_1 \nmid q_2$ and q_1 prime. We have $A^{2m} = 4q^{2j}_1.q^2_2a' \Longrightarrow q_1 \mid A$ and $B^nC^l = q^{2j}_1.q^2_2(b - 4a') \Longrightarrow q_1 \mid B^n$ or $q_1 \mid C^l$.

** F-5-2-1-1- We suppose that $q_1 \mid B^n \Longrightarrow q_1 \mid B \Longrightarrow B = q^f_1.B_1$ with $q_1 \nmid B_1$. We obtain $B^n_1C^l = q^{2j-fn}_1 q^2_2(b - 4a')$:
- If $2j - f.n \geq 1 \Longrightarrow q_1 \mid C^l \Longrightarrow q_1 \mid C$ but $C^l = A^m + B^n$ gives also $q_1 \mid C$ and the conjecture (3.1.1) is verified.
- If $2j - f.n = 0$, we have $B^n_1C^l = q^2_2(b - 4a')$, but $C^l = A^m + B^n$ gives $q_1 \mid C$, then $q_1 \mid (b - 4a')$. As q_1 and a' are coprime, then $q_1 \nmid b$, and the conjecture (3.1.1) is verified.
- If $2j - f.n < 0 \Longrightarrow q_1 \mid (b - 4a') \Longrightarrow q_1 \nmid b$ because a' is coprime with q_1, and $C^l = A^m + B^n$ gives $q_1 \mid C$, and the conjecture (3.1.1) is verified.

** F-5-2-1-2- We obtain identical results if we suppose that $q_1 \mid C^l$.

** F-5-2-2- We suppose that k'_2, a' are not coprime. Let q_1 be a prime so that $q_1 \mid k'_2$ and $q_1 \mid a'$. We write k'_2 under the form $q^j_1.q_2$ with $j \geq 1$, $q_1 \nmid q_2$. From $A^{2m} = 4k'_2a' \Longrightarrow q_1 \mid A^{2m} \Longrightarrow q_1 \mid A$. Then from $B^nC^l = q^j_1q_2(b - 4a')$, it follows that $q_1 \mid (B^nC^l) \Longrightarrow q_1 \mid B^n$ or $q_1 \mid C^l$.

** F-5-2-2-1- We suppose that $q_1 \mid B^n \Longrightarrow q_1 \mid B \Longrightarrow B = q^{\beta}_1.B_1$ with $\beta \geq 1$ and $q_1 \nmid B_1$. Then, we have $q^{n\beta}_1 B^n_1C^l = q^j_1q_2(b - 4a') \Longrightarrow B^n_1C^l = q^{j-n\beta}_1 q_2(b - 4a')$.
- If $j - n\beta \geq 1$, then $q_1 \mid C^l \Longrightarrow q_1 \mid C$, but $C^l = A^m + B^n$ gives $q_1 \mid C$, then the conjecture (3.1.1) is verified.
- If $j - n\beta = 0$, we obtain $B^n_1C^l = q_2(b - 4a')$, but $C^l = A^m + B^n$ gives $q_1 \mid C$, then $q_1 \mid (b - 4a') \Longrightarrow q_1 \mid b$ because $q_1 \mid a' \Longrightarrow q_1 \mid a$, then the contradiction with a, b coprime.
- If $j - n\beta < 0 \Longrightarrow q_1 \mid (b - 4a') \Longrightarrow q_1 \mid b$, because $q_1 \mid a' \Longrightarrow q_1 \mid a$, then the contradiction with a, b coprime.

** F-5-2-2-2- We obtain identical results if we suppose that $q_1 \mid C^l$.

** F-6- If $4 \nmid (b - 4a')$ and $4 \nmid k_1$ it is impossible. We suppose that $4 \mid (b - 4a') \Rightarrow 4 \mid b$, and $b - 4a' = 4^t.g$, $t \geq 1$ with $4 \nmid g$, then we have :

$$A^{2m} = k_1a'$$
$$B^nC^l = k_1.4^{t-1}.g$$

** F-6-1- We suppose that k_1 is prime. From $A^{2m} = k_1a'$ we deduce easily that $k_1 \mid a'$. From $B^nC^l = k_1.4^{t-1}.g$ we obtain that $k_1 \mid (B^nC^l) \Longrightarrow k_1 \mid B^n$ or $k_1 \mid C^l$.

** F-6-1-1- We suppose that $k_1 \mid B^n \Longrightarrow k_1 \mid B \Longrightarrow B = k^j_1.B_1$ with $j > 0$ and $k_1 \nmid B_1$, then $k^{n.j}_1 B^n_1C^l = k_1.4^{t-1}.g \Longrightarrow k^{n.j-1}_1 B^n_1C^l = 4^{t-1}.g$. But $n \geq 3$ and $j \geq 1$, then $n.j - 1 \geq 2$. We deduce as $k_1 \neq 2$ that $k_1 \mid g \Longrightarrow k_1 \mid (b - 4a')$, but $k_1 \mid a' \Longrightarrow k_1 \mid b$, then the contradiction with a, b coprime.

** F-6-1-2- We obtain identical results if we suppose that $k_1 \mid C^l$.

** F-6-2- We suppose that k_1 is not prime $\neq 4$, ($k_1 = 4$ see case F-2, above) with $4 \nmid k_1$.

** F-6-2-1- If $k_1 = 2k'$ with k' odd > 1. Then $A^{2m} = 2k'a' \Longrightarrow 2 \mid a' \Longrightarrow 2 \mid a$, as $4 \mid b$ it follows the contradiction with a, b coprime.

** F-6-2-2- We suppose that k_1 is odd with k_1 and a' coprime. We write k_1 under the form $k_1 = q_1^j.q_2$ with $q_1 \nmid q_2$, q_1 prime and $j \geq 1$. $B^nC^l = q_1^j.q_2 4^{t-1}g \Longrightarrow q_1 \mid B^n$ or $q_1 \mid C^l$.

** F-6-2-2-1- We suppose that $q_1 \mid B^n \Longrightarrow q_1 \mid B \Longrightarrow B = q_1^f.B_1$ with $q_1 \nmid B_1$. We obtain $B_1^nC^l = q_1^{j-f.n}q_2 4^{t-1}g$.
- If $j - f.n \geq 1 \Longrightarrow q_1 \mid C^l \Longrightarrow q_1 \mid C$, but $C^l = A^m + B^n$ gives also $q_1 \mid C$ and the conjecture (3.1.1) is verified.
- If $j - f.n = 0$, we have $B_1^nC^l = q_2 4^{t-1}g$, but $C^l = A^m + B^n$ gives $q_1 \mid C$, then $q_1 \mid (b - 4a')$. As q_1 and a' are coprime then $q_1 \nmid b$ and the conjecture (3.1.1) is verified.
- If $j - f.n < 0 \Longrightarrow q_1 \mid (b - 4a') \Longrightarrow q_1 \nmid b$ because q_1, a' are primes. $C^l = A^m + B^n$ gives $q_1 \mid C$ and the conjecture (3.1.1) is verified.

** F-6-2-2-2- We obtain identical results if we suppose that $q_1 \mid C^l$.

** F-6-2-3- We suppose that k_1 and a' are not coprime. Let q_1 be a prime so that $q_1 \mid k_1$ and $q_1 \mid a'$. We write k_1 under the form $q_1^j.q_2$ with $q_1 \nmid q_2$. From $A^{2m} = k_1 a' \Longrightarrow q_1 \mid A^{2m} \Longrightarrow q_1 \mid A$. From $B^nC^l = q_1^j q_2(b - 4a')$, it follows that $q_1 \mid (B^nC^l) \Longrightarrow q_1 \mid B^n$ or $q_1 \mid C^l$.

** F-6-2-3-1- We suppose that $q_1 \mid B^n \Longrightarrow q_1 \mid B \Longrightarrow B = q_1^\beta.B_1$ with $\beta \geq 1$ and $q_1 \nmid B_1$. Then we have $q_1^{n\beta}B_1^nC^l = q_1^j q_2(b - 4a') \Longrightarrow B_1^nC^l = q_1^{j-n\beta}q_2(b - 4a')$:
- If $j - n\beta \geq 1$, then $q_1 \mid C^l \Longrightarrow q_1 \mid C$, but $C^l = A^m + B^n$ gives $q_1 \mid C$, and the conjecture (3.1.1) is verified.
- If $j - n\beta = 0$, we obtain $B_1^nC^l = q_2(b - 4a')$, but $q_1 \mid A$ and $q_1 \mid B$ then $q_1 \mid C$ and we obtain $q_1 \mid (b - 4a') \Longrightarrow q_1 \mid b$ because $q_1 \mid a' \Longrightarrow q_1 \mid a$, then the contradiction with a, b coprime.
- If $j - n\beta < 0 \Longrightarrow q_1 \mid (b - 4a') \Longrightarrow q_1 \mid b$, then the contradiction with a, b coprime.

** F-6-2-3-2- We obtain identical results as above if we suppose that $q_1 \mid C^l$.

1.6 Hypothèse: $\{3 \mid p \quad and \quad b \mid 4p\}$

1.6.1 Case $b = 2$ and $3 \mid p$

$3 \mid p \Rightarrow p = 3p'$ with $p' \neq 1$ because $3 \ll p$, and $b = 2$, we obtain:

$$A^{2m} = \frac{4p.a}{3b} = \frac{4.3p'.a}{3b} = \frac{4.p'.a}{2} = 2.p'.a$$

As:

$$\frac{1}{4} < cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{2} < \frac{3}{4} \Rightarrow 1 < 2a < 3 \Rightarrow a = 1 \Longrightarrow cos^2\frac{\theta}{3} = \frac{1}{2}$$

but this case was studied (see case 1.4.1).

1.6.2 Case $b = 4$ and $3 \mid p$

we have $3 \mid p \Longrightarrow p = 3p'$ with $p' \in \mathbb{N}^*$, it follows :

$$A^{2m} = \frac{4p.a}{3b} = \frac{4.3p'.a}{3 \times 4} = p'.a$$

and:
$$\frac{1}{4} < cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{4} < \frac{3}{4} \Rightarrow 1 < a < 3 \Rightarrow a = 2$$
as a, b are coprime, then the case $b = 4$ and $3 \mid p$ is impossible.

1.6.3 Case: $b \neq 2, b \neq 4, b \neq 3$, $b \mid p$ and $3 \mid p$

As $3 \mid p$, then $p = 3p'$ and :
$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{4\times 3p'}{3}\frac{a}{b} = \frac{4p'a}{b}$$
We consider the case: $b \mid p' \Longrightarrow p' = bp''$ and $p'' \neq 1$ (If $p'' = 1$, then $p = 3b$, see paragraph 1.6.8 Case $k' = 1$). Finally, we obtain:
$$A^{2m} = \frac{4bp''a}{b} = 4ap''; \quad B^nC^l = p''.(3b-4a)$$

** G-1- We suppose that p'' is prime, then $A^{2m} = 4ap'' = (A^m)^2 \Longrightarrow p'' \mid a$. But $B^nC^l = p''(3b-4a) \Longrightarrow p'' \mid B^n$ or $p'' \mid C^l$.

** G-1-1- If $p'' \mid B^n \Longrightarrow p'' \mid B \Longrightarrow B = p''B_1$ with $B_1 \in \mathbb{N}^*$. Then $p''^{n-1}B_1^nC^l = 3b-4a$. As $n > 2$, then $(n-1) > 1$ and $p'' \mid a$, then $p'' \mid 3b \Longrightarrow p'' = 3$ or $p'' \mid b$.

** G-1-1-1- If $p'' = 3 \Longrightarrow 3 \mid a$, with a that we write as $a = 3a'^2$, but $A^m = 6a' \Longrightarrow 3 \mid A^m \Longrightarrow 3 \mid A \Longrightarrow A = 3A_1$, then $3^{m-1}A_1^m = 2a' \Longrightarrow 3 \mid a' \Longrightarrow a' = 3a''$. As $p''^{n-1}B_1^nC^l = 3^{n-1}B_1^nC^l = 3b-4a \Longrightarrow 3^{n-2}B_1^nC^l = b - 36a''^2$. As $n > 2 \Longrightarrow n-2 \geq 1$, then $3 \mid b$ and the contradiction with a, b coprime.

** G-1-1-2- We suppose that $p'' \mid b$, as $p'' \mid a$, then the contradiction with a, b coprime.

** G-1-2- If we suppose $p'' \mid C^l$, we obtain identical results (contradictions).

** G-2- We consider now that p'' is not prime.

** G-2-1- p'', a coprime: $A^{2m} = 4ap'' \Longrightarrow A^m = 2a'.p_1$ with $a = a'^2$ and $p'' = p_1^2$, then a', p_1 are also coprime. As $A^m = 2a'.p_1$, then $2 \mid a'$ or $2 \mid p_1$.

** G-2-1-1- We suppose that $2 \mid a'$, then $2 \mid a' \Longrightarrow 2 \nmid p_1$, but $p'' = p_1^2$.

** G-2-1-1-1- If p_1 is prime, it is impossible with $A^m = 2a'.p_1$.

** G-2-1-1-2- We suppose that p_1 is not prime so we can write $p_1 = \omega^m \Longrightarrow p'' = \omega^{2m}$. Then $B^nC^l = \omega^{2m}(3b-4a)$.

** G-2-1-1-2-1- If ω is prime, $\omega \neq 2$, then $\omega \mid (B^nC^l) \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** G-2-1-1-2-1-1- If $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^jB_1$ with $\omega \nmid B_1$, then $B_1^n.C^l = \omega^{2m-nj}(3b-4a)$.

** G-2-1-1-2-1-1-1- If $2m - n.j = 0$, we obtain $B_1^n.C^l = 3b-4a$. As $C^l = A^m + B^n \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$, and $\omega \mid (3b-4a)$. But $\omega \neq 2$ and ω, a' are coprime, then ω, a are coprime, it follows $\omega \nmid (3b)$, then $\omega \neq 3$ and $\omega \nmid b$, the conjecture (3.1.1) is verified.

** G-2-1-1-2-1-1-2- If $2m - nj \geq 1$, using the method as above, we obtain $\omega \mid C^l \Longrightarrow \omega \mid C$ and $\omega \mid (3b-4a)$ and $\omega \nmid a$ and $\omega \neq 3$ and $\omega \nmid b$, then the conjecture (3.1.1) is verified.

** G-2-1-1-2-1-1-3- If $2m - nj < 0 \Longrightarrow \omega^{n.j-2m}B_1^n.C^l = 3b - 4a$. From $A^m + B^n = C^l \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$, then $C = \omega^h.C_1$, with $\omega \nmid C_1$, we obtain $\omega^{n.j-2m+h.l}B_1^n.C_1^l = 3b - 4a$. If $n.j - 2m + h.l < 0 \Longrightarrow \omega \mid B_1^n C_1^l$ then the contradiction with $\omega \nmid B_1$ or $\omega \nmid C_1$. It follows $n.j - 2m + h.l > 0$ and $\omega \mid (3b - 4a)$ with ω, a, b coprime and the conjecture is verified.

** G-2-1-1-2-1-2- Using the same method above, we obtain identical results if $\omega \mid C^l$.

** G-2-1-1-2-2- We suppose that $p'' = \omega^{2m}$ and ω is not prime. We write $\omega = \omega_1^f.\Omega$ with ω_1 prime $\nmid \Omega$, $f \geq 1$, and $\omega_1 \mid A$. Then $B^nC^l = \omega_1^{2f.m}\Omega^{2m}(3b - 4a) \Longrightarrow \omega_1 \mid (B^nC^l) \Longrightarrow \omega_1 \mid B^n$ or $\omega_1 \mid C^l$.

** G-2-1-1-2-2-1- If $\omega_1 \mid B^n \Longrightarrow \omega_1 \mid B \Longrightarrow B = \omega_1^j B_1$ with $\omega_1 \nmid B_1$, then $B_1^n.C^l = \omega_1^{2.m-nj}\Omega^{2m}(3b - 4a)$:

** G-2-1-1-2-2-1-1- If $2f.m - n.j = 0$, we obtain $B_1^n.C^l = \Omega^{2m}(3b - 4a)$. As $C^l = A^m + B^n \Longrightarrow \omega_1 \mid C^l \Longrightarrow \omega_1 \mid C$, and $\omega_1 \mid (3b - 4a)$. But $\omega_1 \neq 2$ and ω_1, a' are coprime, then ω, a are coprime, it follows $\omega_1 \nmid (3b)$, then $\omega_1 \neq 3$ and $\omega_1 \nmid b$, and the conjecture (3.1.1) is verified.

** G-2-1-1-2-2-1-2- If $2f.m - n.j \geq 1$, we have $\omega_1 \mid C^l \Longrightarrow \omega_1 \mid C$ and $\omega_1 \mid (3b - 4a)$ and $\omega_1 \nmid a$ and $\omega_1 \neq 3$ and $\omega_1 \nmid b$, it follows that the conjecture (3.1.1) is verified.

** G-2-1-1-2-2-1-3- If $2f.m - n.j < 0 \Longrightarrow \omega_1^{n.j-2m.f}B_1^n.C^l = \Omega^{2m}(3b - 4a)$. As $\omega_1 \mid C$ using $C^l = A^m + B^n$, then $C = \omega_1^h.C_1 \Longrightarrow \omega^{n.j-2m.f+h.l}B_1^n.C_1^l = \Omega^{2m}(3b - 4a)$. If $n.j - 2m.f + h.l < 0 \Longrightarrow \omega_1 \mid B_1^nC_1^l$, then the contradiction with $\omega_1 \nmid B_1$ and $\omega_1 \nmid C_1$. Then if $n.j - 2m.f + h.l > 0$ and $\omega_1 \mid (3b - 4a)$ with ω_1, a, b coprime and the conjecture (3.1.1) is verified.

** G-2-1-1-2-2-2- Using the same method above, we obtain identical results if $\omega_1 \mid C^l$.

** G-2-1-2- We suppose that $2 \mid p_1$: then $2 \mid p_1 \Longrightarrow 2 \nmid a' \Longrightarrow 2 \nmid a$, but $p'' = p_1^2$.

** G-2-1-2-1- We suppose that $p_1 = 2$, we obtain $A^m = 4a' \Longrightarrow 2 \mid a'$, then the contradiction with a, b coprime.

** G-2-1-2-2- We suppose that p_1 is not prime and $2 \mid p_1$. As $A^m = 2a'p_1$, p_1 can written as $p_1 = 2^{m-1}\omega^m \Longrightarrow p'' = 2^{2m-2}\omega^{2m}$. Then $B^nC^l = 2^{2m-2}\omega^{2m}(3b - 4a) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** G-2-1-2-2-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B$. As $2 \mid A$, then $2 \mid C$. From $B^nC^l = 2^{2m-2}\omega^{2m}(3b - 4a)$ it follows that if $2 \mid (3b - 4a) \Longrightarrow 2 \mid b$ but as $2 \nmid a$ there is no contradiction with a, b coprime and the conjecture (3.1.1) is verified.

** G-2-1-2-2-2- We suppose that $2 \mid C^l$, using the same method above, we obtain identical results.

** G-2-2- We suppose that p'', a are not coprime: let ω be a prime integer so that $\omega \mid a$ and $\omega \mid p''$.

** G-2-2-1- We suppose that $\omega = 3$. As $A^{2m} = 4ap'' \Longrightarrow 3 \mid A$, but $3 \mid p$. As $p = A^{2m} + B^{2n} + A^mB^n \Longrightarrow 3 \mid B^{2n} \Longrightarrow 3 \mid B$, then $3 \mid C^l \Longrightarrow 3 \mid C$. We write $A = 3^iA_1$, $B = 3^jB_1$, $C = 3^hC_1$ with 3 coprime with A_1, B_1 and C_1 and $p = 3^{2im}A_1^{2m} + 3^{2nj}B_1^{2n} + 3^{im+jn}A_1^mB_1^n = 3^k.g$ with $k = min(2im, 2jn, im + jn)$ and $3 \nmid g$. We have also $(\omega = 3) \mid a$ and $(\omega = 3) \mid p''$ that gives $a = 3^\alpha a_1$, $3 \nmid a_1$ and $p'' = 3^\mu p_1$, $3 \nmid p_1$ with $A^{2m} = 4ap'' = 3^{2im}A_1^{2m} = 4 \times 3^{\alpha+\mu}.a_1.p_1 \Longrightarrow \alpha + \mu = 2im$. As $p = 3p' = 3b.p'' = 3b.3^\mu p_1 = 3^{\mu+1}.b.p_1$, the exponent of the factor 3 of p is k, the exponent of the factor 3 of the left member of the last equation is $\mu + 1$ added of the exponent β of 3 of the term b, with $\beta \geq 0$, let $min(2im, 2jn, im + jn) = \mu + 1 + \beta$ and we recall that $\alpha + \mu = 2im$. But $B^nC^l = p''(3b - 4a)$, we obtain $3^{(nj+hl)}B_1^nC_1^l = 3^{\mu+1}p_1(b - 4 \times 3^{(\alpha-1)}a_1) = 3^{\mu+1}p_1(3^\beta b_1 - 4 \times 3^{(\alpha-1)}a_1)$, $3 \nmid b_1$. We have also $A^m + B^n = C^l \Longrightarrow 3^{im}A_1^m + 3^{jn}B_1^n = 3^{hl}C_1^l$. We call $\epsilon = min(im, jn)$, we have

$\epsilon = hl = min(im, jn)$. We obtain the conditions:

$$k = min(2im, 2jn, im + jn) = \mu + 1 + \beta \tag{1.6.1}$$
$$\alpha + \mu = 2im \tag{1.6.2}$$
$$\epsilon = hl = min(im, jn)$$
$$3^{(nj+hl)} B_1^n C_1^l = 3^{\mu+1} p_1 (3^\beta b_1 - 4 \times 3^{(\alpha-1)} a_1)$$

** G-2-2-1-1- $\alpha = 1 \Longrightarrow a = 3a_1$ and $3 \nmid a_1$, the equation (1.6.2) becomes:

$$1 + \mu = 2im$$

and the first equation (1.6.1) is written as:

$$k = min(2im, 2jn, im + jn) = 2im + \beta$$

- If $k = 2im \Longrightarrow \beta = 0$ then $3 \nmid b$. We obtain $2im \leq 2jn \Longrightarrow im \leq jn$, and $2im \leq im + jn \Longrightarrow im \leq jn$. The third equation gives $hl = im$ and the last equation gives $nj + hl = \mu + 1 = 2im \Longrightarrow im = nj$, then $im = nj = hl$ and $B_1^n C_1^l = p_1(b - 4a_1)$. As a, b are coprime, the conjecture (3.1.1) is verified.

- If $k = 2jn$ or $k = im + jn$, we obtain $\beta = 0$, $im = jn = hl$ and $B_1^n C_1^l = p_1(b - 4a_1)$. As a, b are coprime, the conjecture (3.1.1) is verified.

** G-2-2-1-2- $\alpha > 1 \Longrightarrow \alpha \geq 2$.

- If $k = 2im \Longrightarrow 2im = \mu + 1 + \beta$, but $\mu = 2im - \alpha$ that gives $\alpha = 1 + \beta \geq 2 \Longrightarrow \beta \neq 0 \Longrightarrow 3 \mid b$, but $3 \mid a$ then the contradiction with a, b coprime.

- If $k = 2jn = \mu + 1 + \beta \leq 2im \Longrightarrow \mu + 1 + \beta \leq \mu + \alpha \Longrightarrow 1 + \beta \leq \alpha \Longrightarrow \beta \geq 1$. If $\beta \geq 1 \Longrightarrow 3 \mid b$ but $3 \mid a$, then the contradiction with a, b coprime.

- If $k = im + jn \Longrightarrow im + jn \leq 2im \Longrightarrow jn \leq im$, and $im + jn \leq 2jn \Longrightarrow im \leq jn$, then $im = jn$. As $k = im + jn = 2im = 1 + \mu + \beta$ and $\alpha + \mu = 2im$, we obtain $\alpha = 1 + \beta \geq 2 \Longrightarrow \beta \geq 1 \Longrightarrow 3 \mid b$, then the contradiction with a, b coprime.

** G-2-2-2- We suppose that $\omega \neq 3$. We write $a = \omega^\alpha a_1$ with $\omega \nmid a_1$ and $p'' = \omega^\mu p_1$ with $\omega \nmid p_1$. As $A^{2m} = 4ap'' = 4\omega^{\alpha+\mu}.a_1.p_1 \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i A_1$, $\omega \nmid A_1$. But $B^n C^l = p''(3b - 4a) = \omega^\mu p_1(3b - 4a) \Longrightarrow \omega \mid B^n C^l \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** G-2-2-2-1- We suppose that $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j B_1$ and $\omega \nmid B_1$. From $A^m + B^n = C^l \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$. As $p = bp' = 3bp'' = 3\omega^\mu b p_1 = \omega^k(\omega^{2im-k}A_1^{2m} + \omega^{2jn-k}B_1^{2n} + \omega^{im+jn-k}A_1^m B_1^n)$ with $k = min(2im, 2jn, im + jn)$. Then:

- If $k = \mu$, then $\omega \nmid b$ and the conjecture (3.1.1) is verified.
- If $k > \mu$, then $\omega \mid b$, but $\omega \mid a$ then the contradiction with a, b coprime.
- If $k < \mu$, it follows from:

$$3\omega^\mu b p_1 = \omega^k(\omega^{2im-k}A_1^{2m} + \omega^{2jn-k}B_1^{2n} + \omega^{im+jn-k}A_1^m B_1^n)$$

that $\omega \mid A_1$ or $\omega \mid B_1$ then the contradiction with $\omega \nmid A_1$ or $\omega \nmid B_1$.

** G-2-2-2-2- If $\omega \mid C^l \Longrightarrow \omega \mid C \Longrightarrow C = \omega^h C_1$ with $\omega \nmid C_1$. From $A^m + B^n = C^l \Longrightarrow \omega \mid (C^l - A^m) \Longrightarrow \omega \mid B$. Then, using the same method as for the case G-2-2-2-1-, we obtain identical results.

1.6.4 Case $b = 3$ and $3 \mid p$

As $3 \mid p \Longrightarrow p = 3p'$, We write :

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{4 \times 3p'}{3}\frac{a}{3} = \frac{4p'a}{3}$$

As A^{2m} is an integer and a, b are coprime and $cos^2\frac{\theta}{3} < 1$ (see equation (1.3.9)), then we have necessary $3 \mid p' \Longrightarrow p' = 3p''$ with $p'' \neq 1$, if not $p = 3p' = 3 \times 3p'' = 9$, but $9 \ll (p = A^{2m} + B^{2n} + A^m B^n)$, the hypothesis $p'' = 1$ is impossible, then $p'' > 1$, and we obtain:

$$A^{2m} = \frac{4p'a}{3} = \frac{4 \times 3p''a}{3} = 4p''a\,; \quad B^n C^l = p''.(9 - 4a)$$

As $\frac{1}{4} < cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{3} < \frac{3}{4} \Longrightarrow 3 < 4a < 9 \Longrightarrow$ as $a > 1, a = 2$ and we obtain:

$$A^{2m} = 4p''a = 8p''\,; \quad B^n C^l = \frac{3p''(9 - 4a)}{3} = p'' \tag{1.6.3}$$

The two last equations above imply that p'' is not a prime. We can write p'' as : $p'' = \prod_{i \in I} p_i^{\alpha_i}$ where p_i are distinct primes, α_i elements of $\mathbb{N}^*$ and $i \in I$ a finite set of indexes. We can write also $p'' = p_1^{\alpha_1}.q_1$ with $p_1 \nmid q_1$. From (1.6.3), we have $p_1 \mid A$ and $p_1 \mid B^n C^l \Longrightarrow p_1 \mid B^n$ or $p_1 \mid C^l$.

** H-1- We suppose that $p_1 \mid B^n \Longrightarrow B = p_1^{\beta_1}.B_1$ with $p_1 \nmid B_1$ and $\beta_1 \geq 1$. Then, we obtain $B_1^n C^l = p_1^{\alpha_1 - n\beta_1}.q_1$ with the following cases:

- If $\alpha_1 - n\beta_1 \geq 1 \Longrightarrow p_1 \mid C^l \Longrightarrow p_1 \mid C$, in accord with $p_1 \mid (C^l = A^m + B^n)$, it follows that the conjecture (3.1.1) is verified.
- If $\alpha_1 - n\beta_1 = 0 \Longrightarrow B_1^n C^l = q_1 \Longrightarrow p_1 \nmid C^l$, it is a contradiction with $p_1 \mid (A^m - B^n) \Longrightarrow p_1 \mid C^l$. Then this case is impossible.
- If $\alpha_1 - n\beta_1 < 0$, we obtain $p_1^{n\beta_1 - \alpha_1} B_1^n C^l = q_1 \Longrightarrow p_1 \mid q_1$, it is a contradiction with $p_1 \nmid q_1$. Then this case is impossible.

** H-2- We suppose that $p_1 \mid C^l$, using the same method as for the case $p_1 \mid B^n$, we obtain identical results.

1.6.5 Case $3 \mid p$ and $b = p$

We have $cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{p}$ and:

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{a}{p} = \frac{4a}{3}$$

As A^{2m} is an integer, it implies that $3 \mid a$, but $3 \mid p \Longrightarrow 3 \mid b$. As a and b are coprime, then the contradiction and the case $3 \mid p$ and $b = p$ is impossible.

1.6.6 Case $3 \mid p$ and $b = 4p$

$3 \mid p \Longrightarrow p = 3p'$, $p' \neq 1$ because $3 \ll p$, then $b = 4p = 12p'$.

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{a}{3} \Longrightarrow 3 \mid a$$

as A^{2m} is an integer. But $3 \mid p \Longrightarrow 3 \mid [(4p) = b]$, then the contradiction with a, b coprime and the case $b = 4p$ is impossible.

1.6.7 Case $3 \mid p$ and $b = 2p$

$3 \mid p \Longrightarrow p = 3p'$, $p' \neq 1$ because $3 \ll p$, then $b = 2p = 6p'$.

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{2a}{3} \Longrightarrow 3 \mid a$$

as A^{2m} is an integer. But $3 \mid p \Longrightarrow 3 \mid (2p) \Longrightarrow 3 \mid b$, then the contradiction with a, b coprime and the case $b = 2p$ is impossible.

1.6.8 Case $3 \mid p$ and $b \neq 3$ a divisor of p

We have $b = p' \neq 3$, and p is written as $p = kp'$ with $3 \mid k \Longrightarrow k = 3k'$ and :

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{a}{b} = 4ak'$$
$$B^nC^l = \frac{p}{3}.\left(3 - 4cos^2\frac{\theta}{3}\right) = k'(3p' - 4a) = k'(3b - 4a)$$

** I-1- $k' \neq 1$:

** I-1-1- We suppose that k' is prime, then $A^{2m} = 4ak' = (A^m)^2 \Longrightarrow k' \mid a$. But $B^nC^l = k'(3b - 4a) \Longrightarrow k' \mid B^n$ or $k' \mid C^l$.

** I-1-1-1- If $k' \mid B^n \Longrightarrow k' \mid B \Longrightarrow B = k'B_1$ with $B_1 \in \mathbb{N}^*$. Then $k'^{n-1}B_1^nC^l = 3b - 4a$. As $n > 2$, then $(n-1) > 1$ and $k' \mid a$, then $k' \mid 3b \Longrightarrow k' = 3$ or $k' \mid b$.

** I-1-1-1-1- If $k' = 3 \Longrightarrow 3 \mid a$, with a that we can write it under the form $a = 3a'^2$. But $A^m = 6a' \Longrightarrow 3 \mid A^m \Longrightarrow 3 \mid A \Longrightarrow A = 3A_1$ with $A_1 \in \mathbb{N}^*$. Then $3^{m-1}A_1^m = 2a' \Longrightarrow 3 \mid a' \Longrightarrow a' = 3a''$. But $k'^{n-1}B_1^nC^l = 3^{n-1}B_1^nC^l = 3b - 4a \Longrightarrow 3^{n-2}B_1^nC^l = b - 36a''^2$. As $n \geq 3 \Longrightarrow n - 2 \geq 1$, then $3 \mid b$. Hence the contradiction with a, b coprime.

** I-1-1-1-2- We suppose that $k' \mid b$, but $k' \mid a$, then the contradiction with a, b coprime.

** I-1-1-2- We suppose that $k' \mid C^l$, using the same method as for the case $k' \mid B^n$, we obtain identical results.

** I-1-2- We consider that k' is not a prime.

** I-1-2-1- We suppose that k', a coprime: $A^{2m} = 4ak' \Longrightarrow A^m = 2a'.p_1$ with $a = a'^2$ and $k' = p_1^2$, then a', p_1 are also coprime. As $A^m = 2a'.p_1$ then $2 \mid a'$ or $2 \mid p_1$.

** I-1-2-1-1- We suppose that $2 \mid a'$, then $2 \mid a' \Longrightarrow 2 \nmid p_1$, but $k' = p_1^2$.

** I-1-2-1-1-1- If p_1 is prime, it is impossible with $A^m = 2a'.p_1$.

** I-1-2-1-1-2- We suppose that p_1 is not prime and it can be written as $p_1 = \omega^m \Longrightarrow k' = \omega^{2m}$. Then $B^nC^l = \omega^{2m}(3b - 4a)$.

** I-1-2-1-1-2-1- If ω is prime $\neq 2$, then $\omega \mid (B^nC^l) \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** I-1-2-1-1-2-1-1- If $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^jB_1$ with $\omega \nmid B_1$, then $B_1^n.C^l = \omega^{2m-nj}(3b - 4a)$.

- If $2m - n.j = 0$, we obtain $B_1^n.C^l = 3b - 4a$, as $C^l = A^m + B^n \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$, and $\omega \mid (3b - 4a)$. But $\omega \neq 2$ and ω, a' are coprime, then $\omega \nmid (3b) \Longrightarrow \omega \neq 3$ and $\omega \nmid b$. Hence, the conjecture (3.1.1) is verified.
- If $2m - nj \geq 1$, using the same method, we have $\omega \mid C^l \Longrightarrow \omega \mid C$ and $\omega \mid (3b - 4a)$ and $\omega \nmid a$ and $\omega \neq 3$ and $\omega \nmid b$. Then the conjecture (3.1.1) is verified.
- If $2m - nj < 0 \Longrightarrow \omega^{n.j-2m}B_1^n.C^l = 3b - 4a$. As $C^l = A^m + B^n \Longrightarrow \omega \mid C$ then $C = \omega^h.C_1 \Longrightarrow \omega^{n.j-2m+h.l}B_1^n.C_1^l = 3b - 4a$. If $n.j - 2m + h.l < 0 \Longrightarrow \omega \mid B_1^nC_1^l$, then the contradiction with $\omega \nmid B_1$ or $\omega \nmid C_1$. If $n.j - 2m + h.l > 0 \Longrightarrow \omega \mid (3b - 4a)$ with ω, a, b coprime, it implies that the conjecture (3.1.1) is verified.

** I-1-2-1-1-2-1-2- We suppose that $\omega \mid C^l$, using the same method as for the case $\omega \mid B^n$, we obtain identical results.

** I-1-2-1-1-2-2- Now $k' = \omega^{2m}$ and ω not a prime, we write $\omega = \omega_1^f.\Omega$ with ω_1 a prime $\nmid \Omega$ and $f \geq 1$ an integer, and $\omega_1 \mid A$, then $B^nC^l = \omega_1^{2f.m}\Omega^{2m}(3b-4a) \Longrightarrow \omega_1 \mid (B^nC^l) \Longrightarrow \omega_1 \mid B^n$ or $\omega_1 \mid C^l$.

** I-1-2-1-1-2-2-1- If $\omega_1 \mid B^n \Longrightarrow \omega_1 \mid B \Longrightarrow B = \omega_1^j B_1$ with $\omega_1 \nmid B_1$, then $B_1^n.C^l = \omega_1^{2.fm-nj}\Omega^{2m}(3b-4a)$.

- If $2f.m - n.j = 0$, we obtain $B_1^n.C^l = \Omega^{2m}(3b-4a)$. As $C^l = A^m + B^n \Longrightarrow \omega_1 \mid C^l \Longrightarrow \omega_1 \mid C$, and $\omega_1 \mid (3b-4a)$. But $\omega_1 \neq 2$ and ω_1, a' are coprime, then ω, a are coprime, then $\omega_1 \nmid (3b) \Longrightarrow \omega_1 \neq 3$ and $\omega_1 \nmid b$. Hence, the conjecture (3.1.1) is verified.

- If $2f.m - n.j \geq 1$, we have $\omega_1 \mid C^l \Longrightarrow \omega_1 \mid C$ and $\omega_1 \mid (3b-4a)$ and $\omega_1 \nmid a$ and $\omega_1 \neq 3$ and $\omega_1 \nmid b$, then the conjecture (3.1.1) is verified.

- If $2f.m - n.j < 0 \Longrightarrow \omega_1^{n.j-2m.f}B_1^n.C^l = \Omega^{2m}(3b-4a)$. As $C^l = A^m + B^n \Longrightarrow \omega_1 \mid C$, then $C = \omega_1^h.C_1 \Longrightarrow \omega^{n.j-2m.f+h.l}B_1^n.C_1^l = \Omega^{2m}(3b-4a)$. If $n.j - 2m.f + h.l < 0 \Longrightarrow \omega_1 \mid B_1^nC_1^l$, then the contradiction with $\omega_1 \nmid B_1$ and $\omega_1 \nmid C_1$. Then if $n.j - 2m.f + h.l > 0$ and $\omega_1 \mid (3b-4a)$ with ω_1, a, b coprime, then the conjecture (3.1.1) is verified.

** I-1-2-1-1-2-2-2- As in the case $\omega_1 \mid B^n$, we obtain identical results if $\omega_1 \mid C^l$.

** I-1-2-1-2- If $2 \mid p_1$: then $2 \mid p_1 \Longrightarrow 2 \nmid a' \Longrightarrow 2 \nmid a$, but $k' = p_1^2$.

** I-1-2-1-2-1- If $p_1 = 2$, we obtain $A^m = 4a' \Longrightarrow 2 \mid a'$, then the contradiction with $2 \nmid a'$. Case to reject.

** I-1-2-1-2-2- We suppose that p_1 is not prime and $2 \mid p_1$. As $A^m = 2a'p_1$, p_1 is written under the form $p_1 = 2^{m-1}\omega^m \Longrightarrow p_1^2 = 2^{2m-2}\omega^{2m}$. Then $B^nC^l = k'(3b-4a) = 2^{2m-2}\omega^{2m}(3b-4a) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** I-1-2-1-2-2-1- If $2 \mid B^n \Longrightarrow 2 \mid B$, as $2 \mid A \Longrightarrow 2 \mid C$. From $B^nC^l = 2^{2m-2}\omega^{2m}(3b-4a)$ it follows that if $2 \mid (3b-4a) \Longrightarrow 2 \mid b$ but as $2 \nmid a$, there is no contradiction with a, b coprime and the conjecture (3.1.1) is verified.

** I-1-2-1-2-2-2- We obtain identical results as above if $2 \mid C^l$.

** I-1-2-2- We suppose that k', a are not coprime: let ω be a prime integer so that $\omega \mid a$ and $\omega \mid p_1^2$.

** I-1-2-2-1- We suppose that $\omega = 3$. As $A^{2m} = 4ak' \Longrightarrow 3 \mid A$, but $3 \mid p$. As $p = A^{2m} + B^{2n} + A^mB^n \Longrightarrow 3 \mid B^{2n} \Longrightarrow 3 \mid B$, then $3 \mid C^l \Longrightarrow 3 \mid C$. We write $A = 3^iA_1$, $B = 3^jB_1$, $C = 3^hC_1$ with 3 coprime with A_1, B_1 and C_1 and $p = 3^{2im}A_1^{2m} + 3^{2nj}B_1^{2n} + 3^{im+jn}A_1^mB_1^n = 3^s.g$ with $s = min(2im, 2jn, im+jn)$ and $3 \nmid g$. We have also $(\omega = 3) \mid a$ and $(\omega = 3) \mid k'$ that give $a = 3^\alpha a_1, 3 \nmid a_1$ and $k' = 3^\mu p_2, 3 \nmid p_2$ with $A^{2m} = 4ak' = 3^{2im}A_1^{2m} = 4 \times 3^{\alpha+\mu}.a_1.p_2 \Longrightarrow \alpha + \mu = 2im$. As $p = 3p' = 3b.k' = 3b.3^\mu p_2 = 3^{\mu+1}.b.p_2$. The exponent of the factor 3 of p is s, the exponent of the factor 3 of the left member of the last equation is $\mu + 1$ added of the exponent β of 3 of the factor b, with $\beta \geq 0$, let $min(2im, 2jn, im+jn) = \mu + 1 + \beta$, we recall that $\alpha + \mu = 2im$. But $B^nC^l = k'(4b-3a)$ that gives $3^{(nj+hl)}B_1^nC_1^l = 3^{\mu+1}p_2(b - 4 \times 3^{(\alpha-1)}a_1) = 3^{\mu+1}p_2(3^\beta b_1 - 4 \times 3^{(\alpha-1)}a_1)$, $3 \nmid b_1$. We have also $A^m + B^n = C^l$ that gives $3^{im}A_1^m + 3^{jn}B_1^n = 3^{hl}C_1^l$. We call $\epsilon = min(im, jn)$, we obtain $\epsilon = hl = min(im, jn)$. We have then the conditions:

$$s = min(2im, 2jn, im+jn) = \mu + 1 + \beta \tag{1.6.4}$$

$$\alpha + \mu = 2im \tag{1.6.5}$$

$$\epsilon = hl = min(im, jn) \tag{1.6.6}$$

$$3^{(nj+hl)}B_1^nC_1^l = 3^{\mu+1}p_2(3^\beta b_1 - 4 \times 3^{(\alpha-1)}a_1) \tag{1.6.7}$$

** I-1-2-2-1-1- $\alpha = 1 \Longrightarrow a = 3a_1$ and $3 \nmid a_1$, the equation (1.6.5) becomes:

$$1 + \mu = 2im$$

and the first equation (1.6.4) is written as :

$$s = min(2im, 2jn, im + jn) = 2im + \beta$$

- If $s = 2im \Longrightarrow \beta = 0 \Longrightarrow 3 \nmid b$. We obtain $2im \leq 2jn \Longrightarrow im \leq jn$, and $2im \leq im + jn \Longrightarrow im \leq jn$. The third equation (1.6.6) gives $hl = im$. The last equation (1.6.7) gives $nj + hl = \mu + 1 = 2im \Longrightarrow im = jn$, then $im = jn = hl$ and $B_1^n C_1^l = p_2(b - 4a_1)$. As a, b are coprime, the conjecture (3.1.1) is verified.

- If $s = 2jn$ or $s = im + jn$, we obtain $\beta = 0$, $im = jn = hl$ and $B_1^n C_1^l = p_2(b - 4a_1)$. Then as a, b are coprime, the conjecture (3.1.1) is verified.

** I-1-2-2-1-2- $\alpha > 1 \Longrightarrow \alpha \geq 2$.

- If $s = 2im \Longrightarrow 2im = \mu + 1 + \beta$, but $\mu = 2im - \alpha$ it gives $\alpha = 1 + \beta \geq 2 \Longrightarrow \beta \neq 0 \Longrightarrow 3 \mid b$, but $3 \mid a$ then the contradiction with a, b coprime and the conjecture (3.1.1) is not verified.

- If $s = 2jn = \mu + 1 + \beta \leq 2im \Longrightarrow \mu + 1 + \beta \leq \mu + \alpha \Longrightarrow 1 + \beta \leq \alpha \Longrightarrow \beta = 1$. If $\beta = 1 \Longrightarrow 3 \mid b$ but $3 \mid a$, then the contradiction with a, b coprime and the conjecture (3.1.1) is not verified.

- If $s = im + jn \Longrightarrow im + jn \leq 2im \Longrightarrow jn \leq im$, and $im + jn \leq 2jn \Longrightarrow im \leq jn$, then $im = jn$. As $s = im + jn = 2im = 1 + \mu + \beta$ and $\alpha + \mu = 2im$ it gives $\alpha = 1 + \beta \geq 2 \Longrightarrow \beta \geq 1 \Longrightarrow 3 \mid b$, then the contradiction with a, b coprime and the conjecture (3.1.1) is not verified.

** I-1-2-2-2- We suppose that $\omega \neq 3$. We write $a = \omega^\alpha a_1$ with $\omega \nmid a_1$ and $k' = \omega^\mu p_2$ with $\omega \nmid p_2$. As $A^{2m} = 4ak' = 4\omega^{\alpha+\mu}.a_1.p_2 \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i A_1$, $\omega \nmid A_1$. But $B^n C^l = k'(3b - 4a) = \omega^\mu p_2(3b - 4a) \Longrightarrow \omega \mid B^n C^l \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** I-1-2-2-2-1- $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B^n = \omega^j B_1$ and $\omega \nmid B_1$. From $A^m + B^n = C^l \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$. As $p = bp' = 3bk' = 3\omega^\mu bp_2 = \omega^s(\omega^{2im-s}A_1^{2m} + \omega^{2jn-s}B_1^{2n} + \omega^{im+jn-s}A_1^m B_1^n)$ with $s = min(2im, 2jn, im + jn)$. Then:

- If $s = \mu$, then $\omega \nmid b$ and the conjecture (3.1.1) is verified.

- If $s > \mu$, then $\omega \mid b$, but $\omega \mid a$ then the contradiction with a, b coprime and the conjecture (3.1.1) is not verified.

- If $s < \mu$, it follows from:

$$3\omega^\mu bp_1 = \omega^s(\omega^{2im-s}A_1^{2m} + \omega^{2jn-s}B_1^{2n} + \omega^{im+jn-s}A_1^m B_1^n)$$

that $\omega \mid A_1$ or $\omega \mid B_1$ that is the contradiction with the hypothesis and the conjecture (3.1.1) is not verified.

** I-1-2-2-2-2- If $\omega \mid C^l \Longrightarrow \omega \mid C \Longrightarrow C = \omega^h C_1$ with $\omega \nmid C_1$. From $A^m + B^n = C^l \Longrightarrow \omega \mid (C^l - A^m) \Longrightarrow \omega \mid B$. Then we obtain identical results as the case above I-1-2-2-2-1-.

** I-2- We suppose $k' = 1$: then $k' = 1 \Longrightarrow p = 3b$, then we have $A^{2m} = 4a = (2a')^2 \Longrightarrow A^m = 2a'$, then $a = a'^2$ is even and :

$$A^m B^n = 2\sqrt[3]{\rho}cos\frac{\theta}{3}.\sqrt[3]{\rho}\left(\sqrt{3}sin\frac{\theta}{3} - cos\frac{\theta}{3}\right) = \frac{p\sqrt{3}}{3}sin\frac{2\theta}{3} - 2a$$

and we have also:

$$A^{2m} + 2A^m B^n = \frac{2p\sqrt{3}}{3}sin\frac{2\theta}{3} = 2b\sqrt{3}sin\frac{2\theta}{3} \tag{1.6.8}$$

The left member of the equation (1.6.8) is a naturel number and also b, then $2\sqrt{3}sin\dfrac{2\theta}{3}$ can be written under the form :

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_1}{k_2}$$

where k_1, k_2 are two natural numbers coprime and $k_2 \mid b \Longrightarrow b = k_2.k_3$.

** I-2-1- $k' = 1$ and $k_3 \neq 1$: then $A^{2m} + 2A^mB^n = k_3.k_1$. Let μ be a prime integer so that $\mu \mid k_3$. If $\mu = 2 \Rightarrow 2 \mid b$, but $2 \mid a$, it is a contradiction with a, b coprime. We suppose that $\mu \neq 2$ and $\mu \mid k_3$, then $\mu \mid A^m(A^m + 2B^n) \Longrightarrow \mu \mid A^m$ or $\mu \mid (A^m + 2B^n)$.

** I-2-1-1- $\mu \mid A^m$: If $\mu \mid A^m \Longrightarrow \mu \mid A^{2m} \Longrightarrow \mu \mid 4a \Longrightarrow \mu \mid a$. As $\mu \mid k_3 \Longrightarrow \mu \mid b$, the contradiction with a, b coprime.

** I-2-1-2- $\mu \mid (A^m + 2B^n)$: If $\mu \mid (A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$, then $\mu \neq 2$ and $\mu \nmid B^n$. $\mu \mid (A^m + 2B^n)$, we can write $A^m + 2B^n = \mu.t'$. It follows:

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^mB^n = \mu^2 t'^2 - 2t'\mu B^n + B^{2n}$$

Using the expression of p, we obtain:

$$p = t'^2\mu^2 - 2t'B^n\mu + B^n(B^n - A^m)$$

As $p = 3b = 3k_2.k_3$ and $\mu \mid k_3$ then $\mu \mid p \Longrightarrow \ p = \mu.\mu'$, then we obtain:

$$\mu'.\mu = \mu(\mu t'^2 - 2t'B^n) + B^n(B^n - A^m)$$

and $\mu \mid B^n(B^n - A^m) \Longrightarrow \mu \mid B^n$ or $\mu \mid (B^n - A^m)$.

** I-2-1-2-1- $\mu \mid B^n$: If $\mu \mid B^n \Longrightarrow \mu \mid B$, that is the contradiction with I-2-1-2- above.

** I-2-1-2-2- $\mu \mid (B^n - A^m)$: If $\mu \mid (B^n - A^m)$ and using that $\mu \mid (A^m + 2B^n)$, we obtain :

$$\mu \mid 3B^n \Longrightarrow \begin{cases} \mu \mid B^n \Longrightarrow \mu \mid B \\ or \\ \mu = 3 \end{cases}$$

** I-2-1-2-2-1- $\mu \mid B^n$: If $\mu \mid B^n \Longrightarrow \mu \mid B$, that is the contradiction with I-2-1-2- above.

** I-2-1-2-2-2- $\mu = 3$: If $\mu = 3 \Longrightarrow 3 \mid k_3 \Longrightarrow k_3 = 3k'_3$, and we have $b = k_2k_3 = 3k_2k'_3$, it follows $p = 3b = 9k_2k'_3$, then $9 \mid p$, but $p = (A^m - B^n)^2 + 3A^mB^n$ then:

$$9k_2k'_3 - 3A^mB^n = (A^m - B^n)^2$$

that we write as:

$$3(3k_2k'_3 - A^mB^n) = (A^m - B^n)^2 \tag{1.6.9}$$

then:

$$3 \mid (3k_2k'_3 - A^mB^n) \Longrightarrow \ 3 \mid A^mB^n \Longrightarrow 3 \mid A^m \ or \ 3 \mid B^n$$

** I-2-1-2-2-2-1- $3 \mid A^m$: If $3 \mid A^m \Longrightarrow 3 \mid A$ and we have also $3 \mid A^{2m}$, but $A^{2m} = 4a \Longrightarrow 3 \mid 4a \Longrightarrow 3 \mid a$. As $b = 3k_2k'_3$ then $3 \mid b$, but a, b are coprime, then the contradiction and $3 \nmid A$.

** I-2-1-2-2-2-2- $3 \mid B^m$: If $3 \mid B^n \Longrightarrow 3 \mid B$, but the equation (1.6.9) implies $3 \mid (A^m - B^n)^2 \Longrightarrow 3 \mid (A^m - B^n) \Longrightarrow 3 \mid A^m \Longrightarrow 3 \mid A$. The last case above has given that $3 \nmid A$. Then the case $3 \mid B^m$ is to reject.

Finally the hypothesis $k_3 \neq 1$ is impossible.

** I-2-2- Now, we suppose that $k_3 = 1 \Longrightarrow b = k_2$ and $p = 3b = 3k_2$, then we have:

$$2\sqrt{3} sin\frac{2\theta}{3} = \frac{k_1}{b} \tag{1.6.10}$$

with k_1, b coprime. We write (1.6.10) as :

$$4\sqrt{3}sin\frac{\theta}{3}cos\frac{\theta}{3} = \frac{k_1}{b}$$

Taking the square of the two members and replacing $cos^2\frac{\theta}{3}$ by $\frac{a}{b}$, we obtain:

$$3 \times 4^2.a(b-a) = k_1^2 \Longrightarrow k_1^2 = 3 \times 4^2.a'^2(b-a)$$

it implies that :

$$b - a = 3\alpha^2 \Longrightarrow b = a'^2 + 3\alpha^2 \Longrightarrow k_1 = 12a'\alpha$$

As:

$$k_1 = 12a'\alpha = A^m(A^m + 2B^n) \Longrightarrow 3\alpha = a' + B^n$$

We consider now that $3 \mid (b-a)$ with $b = a'^2 + 3\alpha^2$. The case $\alpha = 1$ gives $a' + B^n = 3$ that is impossible. We suppose $\alpha > 1$, the pair (a', α) is a solution of the Diophantine equation:

$$X^2 + 3Y^2 = b \tag{1.6.11}$$

with $X = a'$ and $Y = \alpha$. But using a theorem on the solutions of the equation given by (1.6.11), b is written as (see theorem in [7]):

$$b = 2^{2s} \times 3^t.p_1^{t_1} \cdots p_g^{t_g} q_1^{2s_1} \cdots q_r^{2s_r}$$

where p_i are prime numbers verifying $p_i \equiv 1(\text{mod } 6)$, the q_j are also prime numbers so that $q_j \equiv 5(\text{mod } 6)$, then :

- If $s \geq 1 \Longrightarrow 2 \mid b$, as $2 \mid a$, then the contradiction with a, b coprime.
- If $t \geq 1 \Longrightarrow 3 \mid b$, but $3 \mid (b-a) \Longrightarrow 3 \mid a$, then the contradiction with a, b coprime.

** I-2-2-1- We suppose that b is written as :

$$b = p_1^{t_1} \cdots p_g^{t_g} q_1^{2s_1} \cdots q_r^{2s_r}$$

with $p_i \equiv 1(\text{mod } 6)$ and $q_j \equiv 5(\text{mod } 6)$. Finally, we obtain that $b \equiv 1(\text{mod } 6)$. We will verify then this condition.

** I-2-2-1-1- We present the table below giving the value of $A^m + B^n = C^l$ modulo 6 in function of the value of $A^m, B^n(\text{mod } 6)$. We obtain the table below after retiring the lines (respectively the colones) of $A^m \equiv 0(\text{mod } 6)$ and $A^m \equiv 3(\text{mod } 6)$ (respectively of $B^n \equiv 0(\text{mod } 6)$ and $B^n \equiv 3(\text{mod } 6)$), they present cases with contradictions:

Table 1.2: Table of $C^l(\text{mod } 6)$

A^m, B^n	1	2	4	5
1	2	3	5	0
2	3	4	0	1
4	5	0	2	3
5	0	1	3	4

** I-2-2-1-1-1- For the case $C^l \equiv 0(\text{mod } 6)$ and $C^l \equiv 3(\text{mod } 6)$, we deduce that $3 \mid C^l \Longrightarrow 3 \mid C \Longrightarrow C = 3^h C_1$, with $h \geq 1$ and $3 \nmid C_1$. It follows that $p - B^n C^l = 3b - 3^{lh} C_1^l B^n = A^{2m} \Longrightarrow 3 \mid (A^{2m} =$

$4a) \Longrightarrow 3 \mid a \Longrightarrow 3 \mid b$, then the contradiction with a, b coprime.

** I-2-2-1-1-2- For the case $C^l \equiv 0(\bmod 6)$, $C^l \equiv 2(\bmod 6)$ and $C^l \equiv 4(\bmod 6)$, we deduce that $2 \mid C^l \Longrightarrow 2 \mid C \Longrightarrow C = 2^h C_1$, with $h \geq 1$ and $2 \nmid C_1$. It follows that $p = 3b = A^{2m} + B^n C^l = 4a + 2^{lh} C_1^l B^n \Longrightarrow 2 \mid 3b \Longrightarrow 2 \mid b$, then the contradiction with a, b coprime.

** I-2-2-1-1-3- We consider the cases $A^m \equiv 1(\bmod 6)$ and $B^n \equiv 4(\bmod 6)$ (respectively $B^n \equiv 2(\bmod 6)$): then $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j B_1$ with $j \geq 1$ and $2 \nmid B_1$. It follows from $3b = A^{2m} + B^n C^l = 4a + 2^{jn} B_1^n C^l$ that $2 \mid b$, then the contradiction with a, b coprime.

** I-2-2-1-1-4- We consider the case $A^m \equiv 5(\bmod 6)$ and $B^n \equiv 2(\bmod 6)$: then $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j B_1$ with $j \geq 1$ and $2 \nmid B_1$. It follows that $3b = A^{2m} + B^n C^l = 4a + 2^{jn} B_1^n C^l$, then $2 \mid b$ and we obtain the contradiction with a, b coprime.

** I-2-2-1-1-5- We consider the case $A^m \equiv 2(\bmod 6)$ and $B^n \equiv 5(\bmod 6)$: as $A^m \equiv 2(\bmod 6) \Longrightarrow A^m \equiv 2(\bmod 3)$, then A^m is not a square and also for B^n. Hence, we can write A^m and B^n as:

$$A^m = a_0.\mu A^2$$
$$B^n = b_0 \mu B^2$$

where a_0 (respectively b_0) regroups the product of the prime numbers of A^m with exponent 1 (respectively of B^n) with not necessary $(a_0, \mu A) = 1$ and $(b_0, \mu B) = 1$. We have also $p = 3b = A^{2m} + A^m B^n + B^{2n} = (A^m - B^n)^2 + 3A^m B^n \Longrightarrow 3 \mid (b - A^m B^n) \Longrightarrow A^m B^n \equiv b(\bmod 3)$ but $b = a + 3\alpha^2 \Longrightarrow b \equiv a \equiv a'^2(\bmod 3)$, then $A^m B^n \equiv a'^2(\bmod 3)$. But $A^m \equiv 2(\bmod 6) \Longrightarrow 2a' \equiv 2(\bmod 6) \Longrightarrow 4a'^2 \equiv 4(\bmod 6) \Longrightarrow a'^2 \equiv 1(\bmod 3)$. It follows that $A^m B^n$ is a square, let $A^m B^n = \mu N^2 = \mu A^2.\mu B^2.a_0.b_0$. We call $\mu N_1^2 = a_0.b_0$. Let p_1 be a prime number so that $p_1 \mid a_0 \Longrightarrow a_0 = p_1.a_1$ with $p_1 \nmid a_1$. $p_1 \mid \mu N_1^2 \Longrightarrow p_1 \mid \mu N_1 \Longrightarrow \mu N_1 = p_1^t \mu N_1'$ with $t \geq 1$ and $p_1 \nmid \mu N_1'$, then $p_1^{2t-1} \mu N_1'^2 = a_1.b_0$. As $2t \geq 2 \Longrightarrow 2t - 1 \geq 1 \Longrightarrow p_1 \mid a_1.b_0$ but $(p_1, a_1) = 1$, then $p_1 \mid b_0 \Longrightarrow p_1 \mid B^n \Longrightarrow p_1 \mid B$. But $p_1 \mid (A^m = 2a')$, and $p_1 \neq 2$ because $p_1 \mid B^n$ and B^n is odd, then the contradiction. Hence, $p_1 \mid a' \Longrightarrow p_1 \mid a$. If $p_1 = 3$, from $3 \mid (b - a) \Longrightarrow 3 \mid b$ then the contradiction with a, b coprime. Then $p_1 > 3$ a prime that divides A^m and B^n, then $p_1 \mid (p = 3b) \Longrightarrow p_1 \mid b$, it follows the contradiction with a, b coprime, knowing that $p = 3b \equiv 3(\bmod 6)$ and we choose the case $b \equiv 1(\bmod 6)$ of our interest.

** I-2-2-1-1-6- We consider the last case of the table above $A^m \equiv 4(\bmod 6)$ and $B^n \equiv 1(\bmod 6)$. We return to the equation (1.6.11) that b verifies :

$$b = X^2 + 3Y^2 \tag{1.6.12}$$
$$with \quad X = a'; \quad Y = \alpha$$
$$and \quad 3\alpha = a' + B^n$$

Suppose that it exists another solution of (1.6.12):

$$b = X^2 + 3Y^3 = u^2 + 3v^2 \Longrightarrow 2u \neq A^m, 3v \neq a' + B^n$$

But $B^n = \dfrac{6\alpha - A^m}{2} = 3\alpha - a'$ and b verifies also :$3b = p = A^{2m} + A^m B^n + B^{2n}$, it is impossible that u, v verify:

$$6v = 2u + 2B^n$$
$$3b = 4u^2 + 2uB^n + B^{2n}$$

If we consider that : $6v - 2u = 6\alpha - 2a' \Longrightarrow u = 3v - 3\alpha + a'$, then $b = u^2 + 3v^2 = (3v - 3\alpha + a')^2 + 3v^2$, it gives:

$$2v^2 - B^n v + \alpha^2 - a'\alpha = 0$$
$$2v^2 - B^n v - \frac{(a' + B^n)(A^m - B^n)}{9} = 0$$

The resolution of the last equation gives with taking the positive root (because $A^m > B^n$), $v_1 = \alpha$, then $u = a'$. It follows that b in (1.6.12) has an unique representation under the form $X^2 + 3Y^2$ with $X, 3Y$ coprime. As b is odd, we applique one of Euler's theorems on the convenient numbers "numerus idoneus" as cited above (Case C-2-2-1-2). It follows that b is prime.

We have also $p = 3b = A^{2m} + A^m B^n + B^{2n} = 4a'^2 + B^n.C^l \Longrightarrow 9\alpha^2 - a'^2 = B^n.C^l$, then $3\alpha, a' \in \mathbb{N}^*$ are solutions of the Diophantine equation:

$$x^2 - y^2 = N \tag{1.6.13}$$

with $N = B^n C^l > 0$. Let $Q(N)$ be the number of the solutions of (1.6.13) and $\tau(N)$ the number of ways to write the factors of N, then we announce the following result concerning the number of the solutions of (1.6.13) (see theorem 27.3 in [7]):
- If $N \equiv 2(\mathrm{mod}\, 4)$, then $Q(N) = 0$.
- If $N \equiv 1$ or $N \equiv 3(\mathrm{mod}\, 4)$, then $Q(N) = [\tau(N)/2]$.
- If $N \equiv 0(\mathrm{mod}\, 4)$, then $Q(N) = [\tau(N/4)/2]$.

We recall that $A^m \equiv 0(\mathrm{mod}\, 4)$. Concerning B^n, for $B^n \equiv 0(\mathrm{mod}\, 4)$ or $B^n \equiv 2(\mathrm{mod}\, 4)$, we find that $2 \mid B^n \Longrightarrow 2 \mid \alpha \Longrightarrow 2 \mid b$, then the contradiction with a, b coprime.

For the last case $B^n \equiv 3(\mathrm{mod}\, 4) \Longrightarrow C^l \equiv 3(\mathrm{mod}\, 4) \Longrightarrow N = B^n C^l \equiv 1(\mathrm{mod}\, 4) \Longrightarrow Q(N) = [\tau(N)/2] > 1$.

As $(3\alpha, a')$ is a couple of solutions of the Diophantine equation (1.6.13) and $3\alpha > a'$, then $\exists\, d, d'$ positive integers with $d > d'$ and $N = d.d'$ so that :

$$d + d' = 6\alpha \tag{1.6.14}$$
$$d - d' = 2a' \tag{1.6.15}$$

** I-2-2-1-1-6-1 Now, we consider the case $d = c_1^{lr-1} C_1^l$ where c_1 is a prime integer with $c_1 \nmid C_1$ and $C = c_1^r C_1$, $r \geq 1$. It follows that $d' = c_1.B^n$. We rewrite the equations (3.2.36-3.2.37):

$$c_1^{lr-1} C_1^l + c_1.B^n = 6\alpha \tag{1.6.16}$$
$$c_1^{lr-1} C_1^l - c_1.B^n = 2a' \tag{1.6.17}$$

As $l \geq 3$, from the last two equations above, it follows that $c_1 \mid (6\alpha)$ and $c_1 \mid (2a')$. Then $c_1 = 2$, or $c_1 = 3$ and $3 \mid a'$ or $c_1 \neq 3 \mid \alpha$ and $c_1 \mid a'$.

** I-2-2-1-1-6-1-1 We suppose $c_1 = 2$. As $2 \mid (A^m = 2a') \Rightarrow 2 \mid a$ and $2 \mid C^l$ because $l \geq 3$, it follows $2 \mid B^n$, then $2 \mid (p = 3b)$. Then the contradiction with a, b coprime.

** I-2-2-1-1-6-1-2 We suppose $c_1 = 3 \Rightarrow c_1 \mid (a = 3a')$ and $c_1 = 3 \mid a'$. It follows that $(c_1 = 3) \mid (b = a'^2 + 3\alpha^2)$, then the contradiction with a, b coprime.
** I-2-2-1-1-6-1-3 We suppose $c_1 \neq 3$ and $c_1 \mid 3\alpha$ and $c_1 \mid a'$. It follows that $c_1 \mid a$ and $c_1 \mid b$, then the contradiction with a, b coprime.

The others cases of the expressions of d and d' not coprime so that $N = B^n C^l = d.d'$ give also contradictions.

** I-2-2-1-1-6-2 The last case is to consider $d = C^l$ and $d' = B^n$, so we obtain the only solution $(3\alpha, a')$ of the Diophantine equation (1.6.13). It follows that $Q(N) = 1$, then the contradiction with $Q(N) = [\tau(N)/2] > 1$ the number of the solution of (1.6.13).

It follows that the condition $3 \mid (b - a)$ is a contradiction.

The study of the case 1.6.8 is achieved.

1.6.9 Case $3 \mid p$ and $b \mid 4p$

The following cases have been soon studied:
* $3 \mid p$, $b = 2 \Longrightarrow b \mid 4p$: case 1.6.1,
* $3 \mid p$, $b = 4 \Longrightarrow b \mid 4p$: case 1.6.2,
* $3 \mid p \Longrightarrow p = 3p'$, $b \mid p' \Longrightarrow p' = bp''$, $p'' \neq 1$: case 1.6.3,
* $3 \mid p$, $b = 3 \Longrightarrow b \mid 4p$: case 1.6.4,
* $3 \mid p \Longrightarrow p = 3p'$, $b = p' \Longrightarrow b \mid 4p$: case 1.6.8.

** J-1- Particular case: $b = 12$. In fact $3 \mid p \Longrightarrow p = 3p'$ and $4p = 12p'$. Taking $b = 12$, we have $b \mid 4p$. But $b < 4a < 3b$, that gives $12 < 4a < 36 \Longrightarrow 3 < a < 9$. As $2 \mid b$ and $3 \mid b$, the possible values of a are 5 and 7.

** J-1-1- $a = 5$ and $b = 12 \Longrightarrow 4p = 12p' = bp'$. But $A^{2m} = \dfrac{4p}{3}.\dfrac{a}{b} = \dfrac{5bp'}{3b} = \dfrac{5p'}{3} \Longrightarrow 3 \mid p' \Longrightarrow p' = 3p''$ with $p'' \in \mathbb{N}^*$, then $p = 9p''$, we obtain the expressions:

$$A^{2m} = 5p'' \tag{1.6.18}$$

$$B^nC^l = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = 4p'' \tag{1.6.19}$$

As $n, l \geq 3$, we deduce from the equation (1.6.19) that $2 \mid p'' \Longrightarrow p'' = 2^{\alpha}p_1$ with $\alpha \geq 1$ and $2 \nmid p_1$. Then (1.6.18) becomes: $A^{2m} = 5p'' = 5 \times 2^{\alpha}p_1 \Longrightarrow 2 \mid A \Longrightarrow A = 2^iA_1$, $i \geq 1$ and $2 \nmid A_1$. We have also $B^nC^l = 2^{\alpha+2}p_1 \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** J-1-1-1- We suppose that $2 \mid B^n \Longrightarrow B = 2^jB_1$, $j \geq 1$ and $2 \nmid B_1$. We obtain $B_1^nC^l = 2^{\alpha+2-jn}p_1$:

- If $\alpha + 2 - jn > 0 \Longrightarrow 2 \mid C^l$, there is no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n \Longrightarrow 2 \mid C^l$ and the conjecture (3.1.1) is verified.
- If $\alpha + 2 - jn = 0 \Longrightarrow B_1^nC^l = p_1$. From $C^{=}2^{im}A_1^m + 2^{jn}B_1^n \Longrightarrow 2 \mid C^l$ that implies that $2 \mid p_1$, then the contradiction with $2 \nmid p_1$.
- If $\alpha + 2 - jn < 0 \Longrightarrow 2^{jn-\alpha-2}B_1^nC^l = p_1$, it implies that $2 \mid p_1$, then the contradiction as above.

** J-1-1-2- We suppose that $2 \mid C^l$, using the same method above, we obtain the identical results.

** J-1-2- We suppose that $a = 7$ and $b = 12 \Longrightarrow 4p = 12p' = bp'$. But $A^{2m} = \dfrac{4p}{3}.\dfrac{a}{b} = \dfrac{12p'}{3}.\dfrac{7}{12} = \dfrac{7p'}{3} \Longrightarrow 3 \mid p' \Longrightarrow p = 9p''$, we obtain:

$$A^{2m} = 7p''$$

$$B^nC^l = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = 2p''$$

The last equation implies that $2 \mid B^nC^l$. Using the same method as for the case J-1-1- above, we obtain the identical results.

We study now the general case. As $3 \mid p \Rightarrow p = 3p'$ and $b \mid 4p \Rightarrow \exists k_1 \in \mathbb{N}^*$ and $4p = 12p' = k_1b$.

** J-2- $k_1 = 1$: If $k_1 = 1$ then $b = 12p'$, ($p' \neq 1$, if not $p = 3 \ll A^{2m} + B^{2n} + A^mB^n$). But $A^{2m} = \dfrac{4p}{3}.cos^2\dfrac{\theta}{3} = \dfrac{12p'}{3}\dfrac{a}{b} = \dfrac{4p'.a}{12p'} = \dfrac{a}{3} \Rightarrow 3 \mid a$ because A^{2m} is a natural number, then the contradiction with a, b coprime.

** J-3- $k_1 = 3$: If $k_1 = 3$, then $b = 4p'$ and $A^{2m} = \dfrac{4p}{3}.cos^2\dfrac{\theta}{3} = \dfrac{k_1.a}{3} = a = (A^m)^2 = a'^2 \Longrightarrow A^m = a'$. The term A^mB^n gives $A^mB^n = \dfrac{p\sqrt{3}}{3}sin\dfrac{2\theta}{3} - \dfrac{a}{2}$, then:

$$A^{2m} + 2A^mB^n = \frac{2p\sqrt{3}}{3}sin\frac{2\theta}{3} = 2p'\sqrt{3}sin\frac{2\theta}{3} \tag{1.6.20}$$

The left member of (1.6.20) is an integer number and also p', then $2\sqrt{3}sin\dfrac{2\theta}{3}$ can be written under the form:

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_2}{k_3}$$

where k_2, k_3 are two integer numbers and are coprime and $k_3 \mid p' \Longrightarrow p' = k_3.k_4$.

** J-3-1- $k_4 \neq 1$: We suppose that $k_4 \neq 1$, then:

$$A^{2m} + 2A^m B^n = k_2.k_4 \tag{1.6.21}$$

Let μ be a prime number so that $\mu \mid k_4$, then $\mu \mid A^m(A^m + 2B^n) \Longrightarrow \mu \mid A^m$ or $\mu \mid (A^m + 2B^n)$.

** J-3-1-1- $\mu \mid A^m$: If $\mu \mid A^m \Longrightarrow \mu \mid A^{2m} \Longrightarrow \mu \mid a$. As $\mu \mid k_4 \Longrightarrow \mu \mid p' \Rightarrow \mu \mid (4p' = b)$. But a, b are coprime, then the contradiction.

** J-3-1-2- $\mu \mid (A^m + 2B^n)$: If $\mu \mid (A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$, then $\mu \neq 2$ and $\mu \nmid B^n$. $\mu \mid (A^m + 2B^n)$, we can write $A^m + 2B^n = \mu.t'$. It follows:

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t'\mu B^n + B^{2n}$$

Using the expression of p, we obtain $p = t'^2\mu^2 - 2t'B^n\mu + B^n(B^n - A^m)$. As $p = 3p'$ and $\mu \mid p' \Rightarrow \mu \mid (3p') \Rightarrow \mu \mid p$, we can write : $\exists \mu'$ and $p = \mu\mu'$, then we arrive to:

$$\mu'.\mu = \mu(\mu t'^2 - 2t'B^n) + B^n(B^n - A^m)$$

and $\mu \mid B^n(B^n - A^m) \Longrightarrow \mu \mid B^n$ or $\mu \mid (B^n - A^m)$.

** J-3-1-2-1- $\mu \mid B^n$: If $\mu \mid B^n \Longrightarrow \mu \mid B$, it is in contradiction with J-3-1-2-.

** J-3-1-2-2- $\mu \mid (B^n - A^m)$: If $\mu \mid (B^n - A^m)$ and using $\mu \mid (A^m + 2B^n)$, we obtain :

$$\mu \mid 3B^n \Longrightarrow \begin{cases} \mu \mid B^n \\ or \\ \mu = 3 \end{cases}$$

** J-3-1-2-2-1- $\mu \mid B^n$: If $\mu \mid B^n \Longrightarrow \mu \mid B$, it is in contradiction with J-3-1-2-.

** J-3-1-2-2-2- $\mu = 3$: If $\mu = 3 \Longrightarrow 3 \mid k_4 \Longrightarrow k_4 = 3k_4'$, and we have $p' = k_3 k_4 = 3k_3 k_4'$, it follows that $p = 3p' = 9k_3 k_4'$, then $9 \mid p$, but $p = (A^m - B^n)^2 + 3A^m B^n$, then we obtain:

$$9k_3k_4' - 3A^m B^n = (A^m - B^n)^2$$

that we write : $3(3k_3k_4' - A^m B^n) = (A^m - B^n)^2$, then : $3 \mid (3k_3k_4' - A^m B^n) \Longrightarrow 3 \mid A^m B^n \Longrightarrow 3 \mid A^m$ or $3 \mid B^n$.

** J-3-1-2-2-2-1- $3 \mid A^m$: If $3 \mid A^m \Longrightarrow 3 \mid A^{2m} \Rightarrow 3 \mid a$, but $3 \mid p' \Rightarrow 3 \mid (4p') \Rightarrow 3 \mid b$, then the contradiction with a, b coprime and $3 \nmid A$.

** J-3-1-2-2-2-2- $3 \mid B^n$: If $3 \mid B^n$ but $A^m = \mu t' - 2B^n = 3t' - 2B^n \Longrightarrow 3 \mid A^m$, it is in contradiction with $3 \nmid A$.

Then the hypothesis $k_4 \neq 1$ is impossible.

** J-3-2- $k_4 = 1$: We suppose now that $k_4 = 1 \Longrightarrow p' = k_3k_4 = k_3$. Then we have:

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_2}{p'} \tag{1.6.22}$$

with k_2, p' coprime, we write (1.6.22) as :

$$4\sqrt{3}sin\frac{\theta}{3}cos\frac{\theta}{3} = \frac{k_2}{p'}$$

Taking the square of the two members and replacing $cos^2\frac{\theta}{3}$ by $\frac{a}{b}$ and $b = 4p'$, we obtain:

$$3.a(b-a) = k_2^2$$

As $A^{2m} = a = a'^2$, it implies that :

$$3 \mid (b-a), \quad and \quad b - a = b - a'^2 = 3\alpha^2$$

As $k_2 = A^m(A^m + 2B^n)$ following the equation (1.6.21) and that $3 \mid k_2 \Longrightarrow 3 \mid A^m(A^m + 2B^n) \Longrightarrow 3 \mid A^m$ or $3 \mid (A^m + 2B^n)$.

** J-3-2-1- $3 \mid A^m$: If $3 \mid A^m \Longrightarrow 3 \mid A^{2m} \Longrightarrow 3 \mid a$, but $3 \mid (b-a) \Longrightarrow 3 \mid b$, then the contradiction with a, b coprime.

** J-3-2-2- $3 \mid (A^m + 2B^n) \Longrightarrow 3 \nmid A^m$ and $3 \nmid B^n$. As $k_2^2 = 9a\alpha^2 = 9a'^2\alpha^2 \Longrightarrow k_2 = 3a'\alpha = A^m(A^m + 2B^n)$, then :

$$3\alpha = A^m + 2B^n \tag{1.6.23}$$

As b can be written under the form $b = a'^2 + 3\alpha^2$, then the pair (a', α) is a solution of the Diophantine equation:

$$x^2 + 3y^2 = b \tag{1.6.24}$$

As $b = 4p'$, then :

** J-3-2-2-1- If x, y are even, then $2 \mid a' \Longrightarrow 2 \mid a$, it is a contradiction with a, b coprime.

** J-3-2-2-2- If x, y are odd, then a', α are odd, it implies $A^m = a' \equiv 1(\text{mod } 4)$ or $A^m \equiv 3(\text{mod } 4)$. If u, v verify (1.6.24), then $b = u^2 + 3v^2$, with $u \neq a'$ and $v \neq \alpha$, then u, v do not verify (1.6.23): $3v \neq u + 2B^n$, if not, $u = 3v - 2B^n \Longrightarrow b = (3v - 2B^n)^2 + 3v^2 = a'^2 + 3\alpha$, the resolution of the obtained equation of second degree in v gives the positive root $v_1 = \alpha$, then $u = 3\alpha - 2B^n = a'$, then the uniqueness of the representation of b by the equation (1.6.24).

** J-3-2-2-2-1- We suppose that $A^m \equiv 1(\text{mod } 4)$ and $B^n \equiv 0(\text{mod } 4)$, then B^n is even and $B^n = 2B'$. The expression of p becomes:

$$p = a'^2 + 2a'B' + 4B'^2 = (a' + B')^2 + 3B'^2 = 3p' \Longrightarrow 3 \mid (a' + B') \Longrightarrow a' + B' = 3B''$$
$$p' = B'^2 + 3B''^2 \Longrightarrow b = 4p' = (2B')^2 + 3(2B'')^2 = a'^2 + 3\alpha^2$$

that gives $2B' = B^n = a' = A^m$, then the contradiction with $A^m > B^n$.

** J-3-2-2-2-2- We suppose that $A^m \equiv 1(\text{mod } 4)$ and $B^n \equiv 1(\text{mod } 4)$, then C^l is even and $C^l = 2C'$. The expression of p becomes:

$$p = C^{2l} - C^lB^n + B^{2n} = 4C'^2 - 2C'B^n + B^{2n} = (C' - B^n)^2 + 3C'^2 = 3p'$$
$$\Longrightarrow 3 \mid (C' - B^n) \Longrightarrow C' - B^n = 3C''$$
$$p' = C'^2 + 3C''^2 \Longrightarrow b = 4p' = (2C')^2 + 3(2C'')^2 = a'^2 + 3\alpha^2$$

We obtain $2C' = C^l = a' = A^m$, then the contradiction.

** J-3-2-2-2-3- We suppose that $A^m \equiv 1(\bmod 4)$ and $B^n \equiv 2(\bmod 4)$, then B^n is even, see J-3-2-2-2-1-.

** J-3-2-2-2-4- We suppose that $A^m \equiv 1(\bmod 4)$ and $B^n \equiv 3(\bmod 4)$, then C^l is even, see J-3-2-2-2-2-.

** J-3-2-2-2-5- We suppose that $A^m \equiv 3(\bmod 4)$ and $B^n \equiv 0(\bmod 4)$, then B^n is even, see J-3-2-2-2-1-.

** J-3-2-2-2-6- We suppose that $A^m \equiv 3(\bmod 4)$ and $B^n \equiv 1(\bmod 4)$, then C^l is even, see J-3-2-2-2-2-.

** J-3-2-2-2-7- We suppose that $A^m \equiv 3(\bmod 4)$ and $B^n \equiv 2(\bmod 4)$, then B^n is even, see J-3-2-2-2-1-.

** J-3-2-2-2-8- We suppose that $A^m \equiv 3(\bmod 4)$ and $B^n \equiv 3(\bmod 4)$, then C^l is even, see J-3-2-2-2-2-.

We have achieved the study of the case J-3-2-2- . It gives contradictions.

** J-4- We suppose that $k_1 \neq 3$ and $3 \mid k_1 \Longrightarrow k_1 = 3k'_1$ with $k'_1 \neq 1$, then $4p = 12p' = k_1 b = 3k'_1 b \Rightarrow 4p' = k'_1 b$. A^{2m} can be written as $A^{2m} = \dfrac{4p}{3} cos^2 \dfrac{\theta}{3} = \dfrac{3k'_1 b}{3} \dfrac{a}{b} = k'_1 a$ and $B^n C^l = \dfrac{p}{3}\left(3 - 4cos^2 \dfrac{\theta}{3}\right) = \dfrac{k'_1}{4}(3b-4a)$. As $B^n C^l$ is an integer number, we must have $4 \mid (3b-4a)$ or $4 \mid k'_1$ or $[2 \mid k'_1$ and $2 \mid (3b-4a)]$.

** J-4-1- We suppose that $4 \mid (3b-4a)$.

** J-4-1-1- We suppose that $3b - 4a = 4 \Longrightarrow 4 \mid b \Longrightarrow 2 \mid b$. Then, we have:

$$A^{2m} = k'_1 a$$
$$B^n C^l = k'_1$$

** J-4-1-1-1- If k'_1 is prime, from $B^n C^l = k'_1$, it is impossible.

** J-4-1-1-2- We suppose that $k'_1 > 1$ is not prime. Let ω be a prime number so that $\omega \mid k'_1$.

** J-4-1-1-2-1- We suppose that $k'_1 = \omega^s$, with $s \geq 6$. Then we have :

$$A^{2m} = \omega^s.a \tag{1.6.25}$$
$$B^n C^l = \omega^s \tag{1.6.26}$$

** J-4-1-1-2-1-1- We suppose that $\omega = 2$. If a, k'_1 are not coprime , then $2 \mid a$, as $2 \mid b$, it is the contradiction with a, b coprime.

** J-4-1-1-2-1-2- We suppose $\omega = 2$ and a, k'_1 are coprime, then $2 \nmid a$. From (1.6.26), we deduce that $B = C = 2$ and $n + l = s$, and $A^{2m} = 2^s.a$, but $A^m = 2^l - 2^n \Longrightarrow A^{2m} = (2^l - 2^n)^2 = 2^{2l} + 2^{2n} - 2(2^{l+n}) = 2^{2l} + 2^{2n} - 2 \times 2^s = 2^s.a \Longrightarrow 2^{2l} + 2^{2n} = 2^s(a+2)$. If $l = n$, we obtain $a = 0$ then the contradiction. If $l \neq n$, as $A^m = 2^l - 2^n > 0 \Longrightarrow n < l \Longrightarrow 2n < s$, then $2^{2n}(1 + 2^{2l-2n} - 2^{s+1-2n}) = 2^n 2^l.a$. We call $l = n + n_1 \Longrightarrow 1 + 2^{2l-2n} - 2^{s+1-2n} = 2^{n_1}.a$, but the left member is odd and the right member is even, then the contradiction. Then the case $\omega = 2$ is impossible.

** J-4-1-1-2-1-3- We suppose that $k'_1 = \omega^s$ with $\omega \neq 2$:

** J-4-1-1-2-1-3-1- Suppose that a, k'_1 are not coprime, then $\omega \mid a \Longrightarrow a = \omega^t.a_1$ and $t \nmid a_1$. Then, we have:

$$A^{2m} = \omega^{s+t}.a_1 \tag{1.6.27}$$
$$B^n C^l = \omega^s \tag{1.6.28}$$

From (1.6.28), we deduce that $B^n = \omega^n$, $C^n = \omega^l$, $s = n + l$ and $A^m = \omega^l - \omega^n > 0 \Longrightarrow l > n$. We have also $A^{2m} = \omega^{s+t}.a_1 = (\omega^l - \omega^n)^2 = \omega^{2l} + \omega^{2n} - 2 \times \omega^s$. As $\omega \neq 2 \Longrightarrow \omega$ is odd, then $A^{2m} = \omega^{s+t}.a_1 = (\omega^l - \omega^n)^2$ is even, then $2 \mid a_1 \Longrightarrow 2 \mid a$, it is in contradiction with a, b coprime, then this case is impossible.

** J-4-1-1-2-1-3-2- Suppose that a, k'_1 are coprime, with :

$$A^{2m} = \omega^s.a \tag{1.6.29}$$
$$B^n C^l = \omega^s \tag{1.6.30}$$

From (1.6.30), we deduce that $B^n = \omega^n$, $C^l = \omega^l$ and $s = n + l$. As $\omega \neq 2 \Longrightarrow \omega$ is odd and $A^{2m} = \omega^s.a = (\omega^l - \omega^n)^2$ is even, then $2 \mid a$. It follows the contradiction with a, b coprime and this case is impossible.

** J-4-1-1-2-2- We suppose that $k'_1 = \omega^s.k_2$, with $s \geq 6$, $\omega \nmid k_2$. We have :

$$A^{2m} = \omega^s.k_2.a$$
$$B^n C^l = \omega^s.k_2$$

** J-4-1-1-2-2-1- If k_2 is prime, from the last equation above, $\omega = k_2$, it is in contradiction with $\omega \nmid k_2$. Then this case is impossible.

** J-4-1-1-2-2-2- We suppose that $k'_1 = \omega^s.k_2$, with $s \geq 6$, $\omega \nmid k_2$ and k_2 not a prime. Then, we have:

$$A^{2m} = \omega^s.k_2.a$$
$$B^n C^l = \omega^s.k_2 \tag{1.6.31}$$

** J-4-1-1-2-2-2-1- We suppose that ω, a are coprime, then $\omega \nmid a$. As $A^{2m} = \omega^s.k_2.a \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i.A_1$ with $i \geq 1$ and $\omega \nmid A_1$, then $s = 2i.m$. From (1.6.31), we have $\omega \mid (B^n C^l) \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** J-4-1-1-2-2-2-1-1- We suppose that $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j.B_1$ with $j \geq 1$ and $\omega \nmid B_1$. then :

$$B_1^n C^l = \omega^{2im - jn} k_2$$

- If $2im - jn > 0$, $\omega \mid C^l \Longrightarrow \omega \mid C$, no contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$ and the conjecture (3.1.1) is verified.
- If $2im - jn = 0 \Longrightarrow B_1^n C^l = k_2$, as $\omega \nmid k_2 \Longrightarrow \omega \nmid C^l$, then the contradiction with $\omega \mid (C^l = A^m + B^n)$.
- If $2im - jn < 0 \Longrightarrow \omega^{jn-2im} B_1^n C^l = k_2 \Longrightarrow \omega \mid k_2$, then the contradiction with $\omega \nmid k_2$.

** J-4-1-1-2-2-2-1-2- We suppose that $\omega \mid C^l$. Using the same method used above, we obtain identical results.

** J-4-1-1-2-2-2-2- We suppose that a, ω are not coprime, then $\omega \mid a \Longrightarrow a = \omega^t.a_1$ and $\omega \nmid a_1$. So we have :

$$A^{2m} = \omega^{s+t}.k_2.a_1 \tag{1.6.32}$$
$$B^n C^l = \omega^s.k_2 \tag{1.6.33}$$

As $A^{2m} = \omega^{s+t}.k_2.a_1 \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i A_1$ with $i \geq 1$ and $\omega \nmid A_1$, then $s + t = 2im$. From (1.6.33), we have $\omega \mid (B^n C^l) \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** J-4-1-1-2-2-2-2-1- We suppose that $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j B_1$ with $j \geq 1$ and $\omega \nmid B_1$. then:

$$B_1^n C^l = \omega^{2im-t-jn} k_2$$

- If $2im - t - jn > 0, \omega \mid C^l \Longrightarrow \omega \mid C$, no contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$ and the conjecture (3.1.1) is verified.

- If $2im - t - jn = 0 \Longrightarrow B_1^n C^l = k_2$, As $\omega \nmid k_2 \Longrightarrow \omega \nmid C^l$, then the contradiction with $\omega \mid (C^l = A^m + B^n)$.

- If $2im - t - jn < 0 \Longrightarrow \omega^{jn+t-2im} B_1^n C^l = k_2 \Longrightarrow \omega \mid k_2$, then the contradiction with $\omega \nmid k_2$.

** J-4-1-1-2-2-2-2-2- We suppose that $\omega \mid C^l$. Using the same method used above, we obtain identical results.

** J-4-1-2- $3b - 4a \neq 4$ and $4 \mid (3b - 4a) \Longrightarrow 3b - 4a = 4^s\Omega$ with $s \geq 1$ and $4 \nmid \Omega$. We obtain:

$$A^{2m} = k_1' a \tag{1.6.34}$$
$$B^n C^l = 4^{s-1} k_1' \Omega \tag{1.6.35}$$

** J-4-1-2-1- We suppose that $k_1' = 2$. From (1.6.34), we deduce that $2 \mid a$. As $4 \mid (3b - 4a) \Longrightarrow 2 \mid b$, then the contradiction with a, b coprime and this case is impossible.

** J-4-1-2-2- We suppose that $k_1' = 3$. From (1.6.34) we deduce that $3^3 \mid A^{2m}$. From (1.6.35), it follows that $3^3 \mid B^n$ or $3^3 \mid C^l$. In the last two cases, we obtain $3^3 \mid p$. But $4p = 3k_1' b = 9b \Longrightarrow 3 \mid b$, then the contradiction with a, b coprime. Then this case is impossible.

** J-4-1-2-3- We suppose that k_1' is prime ≥ 5:

** J-4-1-2-3-1- Suppose that k_1' and a are coprime. The equation (1.6.34) gives $(A^m)^2 = k_1'.a$, that is impossible with $k_1' \nmid a$. Then this case is impossible.

** J-4-1-2-3-2- Suppose that k_1' and a are not coprime. Let $k_1' \mid a \Longrightarrow a = k_1'^{\alpha} a_1$ with $\alpha \geq 1$ and $k_1' \nmid a_1$. The equation (1.6.34) is written as :

$$A^{2m} = k_1' a = k_1'^{\alpha+1} a_1$$

The last equation gives $k_1' \mid A^{2m} \Longrightarrow k_1' \mid A \Longrightarrow A = k_1'^{i}.A_1$, with $k_1' \nmid A_1$. If $2i.m \neq (\alpha + 1)$, it is impossible. We suppose that $2i.m = \alpha + 1$, then $k_1' \mid A^m$. We return to the equation (1.6.35). If k_1' and Ω are coprime, it is impossible. We suppose that k_1' and Ω are not coprime, then $k_1' \mid \Omega$ and the exponent of k_1' in Ω is so the equation (1.6.35) is satisfying. We deduce easily that $k_1' \mid B^n$. Then $k_1'^2 \mid (p = A^{2m} + B^{2n} + A^m B^n)$, but $4p = 3k_1' b \Longrightarrow k_1' \mid b$, then the contradiction with a, b coprime.

** J-4-1-2-4- We suppose that $k_1' \geq 4$ is not a prime.

** J-4-1-2-4-1- We suppose that $k_1' = 4$, we obtain then $A^{2m} = 4a$ and $B^n C^l = 3b - 4a = 3p' - 4a$. This case was studied in the paragraph 1.6.8, case ** I-2-.

** J-4-1-2-4-2- We suppose that $k_1' > 4$ is not a prime.

** J-4-1-2-4-2-1- We suppose that a, k_1' are coprime. From the expression $A^{2m} = k_1'.a$, we deduce that $a = a_1^2$ and $k_1' = k''^2_1$. It gives :

$$A^m = a_1.k''_1$$
$$B^n C^l = 4^{s-1} k''^2_1.\Omega$$

Let ω be a prime so that $\omega \mid k''_1$ and $k''_1 = \omega^t.k''_2$ with $\omega \nmid k''_2$. The last two equations become :

$$A^m = a_1.\omega^t.k''_2 \tag{1.6.36}$$
$$B^nC^l = 4^{s-1}\omega^{2t}.k''^2_2.\Omega \tag{1.6.37}$$

From (1.6.36), $\omega \mid A^m \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i.A_1$ with $\omega \nmid A_1$ and $im = t$. From (1.6.37), we obtain $\omega \mid B^nC^l \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** J-4-1-2-4-2-1-1- If $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j.B_1$ with $\omega \nmid B_1$. From (1.6.36), we have $B_1^nC^l = \omega^{2t-j.n}4^{s-1}.k''^2_2.\Omega$.

** J-4-1-2-4-2-1-1-1- If $\omega = 2$ and $2 \nmid \Omega$, we have $B_1^nC^l = 2^{2t+2s-j.n-2}k''^2_2.\Omega$:
- If $2t + 2s - jn - 2 \leq 0$ then $2 \nmid C^l$, then the contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n$.
- If $2t + 2s - jn - 2 \geq 1 \Longrightarrow 2 \mid C^l \Longrightarrow 2 \mid C$ and the conjecture (3.1.1) is verified.

** J-4-1-2-4-2-1-1-2- If $\omega = 2$ and if $2 \mid \Omega \Longrightarrow \Omega = 2.\Omega_1$ because $4 \nmid \Omega$, we have $B_1^nC^l = 2^{2t+2s+1-j.n-2}k''^2_2\Omega_1$:
- If $2t + 2s - jn - 3 \leq 0$ then $2 \nmid C^l$, then the contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n$.
- If $2t + 2s - jn - 3 \geq 1 \Longrightarrow 2 \mid C^l \Longrightarrow 2 \mid C$ and the conjecture (3.1.1) is verified.

** J-4-1-2-4-2-1-1-3- If $\omega \neq 2$, we have $B_1^nC^l = \omega^{2t-j.n}4^{s-1}.k''^2_2.\Omega$:
-If $2t - jn \leq 0 \Longrightarrow \omega \nmid C^l$ it is in contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n$.
-If $2t - jn \geq 1 \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$ and the conjecture (3.1.1) is verified.

** J-4-1-2-4-2-1-2- If $\omega \mid C^l \Longrightarrow \omega \mid C \Longrightarrow C = \omega^h.C_1$, with $\omega \nmid C_1$. Using the same method as in the case J-4-1-2-4-2-1-1 above, we obtain identical results.

** J-4-1-2-4-2-2- We suppose that a, k'_1 are not coprime. Let ω be a prime so that $\omega \mid a$ and $\omega \mid k'_1$. We write:

$$a = \omega^\alpha.a_1$$
$$k'_1 = \omega^\mu.k''_1$$

with a_1, k''_1 coprime. The expression of A^{2m} becomes $A^{2m} = \omega^{\alpha+\mu}.a_1.k''_1$. The term B^nC^l becomes:

$$B^nC^l = 4^{s-1}.\omega^\mu.k''_1.\Omega \tag{1.6.38}$$

** J-4-1-2-4-2-2-1- If $\omega = 2 \Longrightarrow 2 \mid a$, but $2 \mid b$, then the contradiction with a, b coprime, this case is impossible.

** J-4-1-2-4-2-2-2- If $\omega \geq 3$, we have $\omega \mid a$. If $\omega \mid b$ then the contradiction with a, b coprime. We suppose that $\omega \nmid b$. From the expression of A^{2m}, we obtain $\omega \mid A^{2m} \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i.A_1$ with $\omega \nmid A_1$, $i \geq 1$ and $2i.m = \alpha + \mu$. From (1.6.38), we deduce that $\omega \mid B^n$ or $\omega \mid C^l$.

** J-4-1-2-4-2-2-2-1- We suppose that $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^jB_1$ with $\omega \nmid B_1$ and $j \geq 1$. Then, $B_1^nC^l = 4^{s-1}\omega^{\mu-jn}.k''_1.\Omega$:

* $\omega \nmid \Omega$:
- If $\mu - jn \geq 1$, we have $\omega \mid C^l \Longrightarrow \omega \mid C$, there is no contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n$ and the conjecture (3.1.1) is verified.
- If $\mu - jn \leq 0$, then $\omega \nmid C^l$ and it is a contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n$. Then this case is impossible.

* $\omega \mid \Omega$: we write $\Omega = \omega^\beta.\Omega_1$ with $\beta \geq 1$ and $\omega \nmid \Omega_1$. As $3b - 4a = 4^s.\Omega = 4^s.\omega^\beta.\Omega_1 \Longrightarrow 3b = 4a + 4^s.\omega^\beta.\Omega_1 = 4\omega^\alpha.a_1 + 4^s.\omega^\beta.\Omega_1 \Longrightarrow 3b = 4\omega(\omega^{\alpha-1}.a_1 + 4^{s-1}.\omega^{\beta-1}.\Omega_1)$. If $\omega = 3$ and $\beta = 1$, we obtain $b = 4(3^{\alpha-1}a_1 + 4^{s-1}\Omega_1)$ and $B_1^nC^l = 4^{s-1}3^{\mu+1-jn}.k''_1\Omega_1$.

- If $\mu - jn + 1 \geq 1$, then $3 \mid C^l$ and the conjecture (3.1.1) is verified.
- If $\mu - jn + 1 \leq 0$, then $3 \nmid C^l$ and it is the contradiction with $C^l = 3^{im}A_1^m + 3^{jn}B_1^n$.

Now, if $\beta \geq 2$ and $\alpha = im \geq 3$, we obtain $3b = 4\omega^2(\omega^{\alpha-2}a_1 + 4^{s-1}\omega^{\beta-2}\Omega_1)$. If $\omega = 3$ or not, then $\omega \mid b$, but $\omega \mid a$, then the contradiction with a, b coprime.

** J-4-1-2-4-2-2-2-2- We suppose that $\omega \mid C^l \Longrightarrow \omega \mid C \Longrightarrow C = \omega^h C_1$ with $\omega \nmid C_1$ and $h \geq 1$. Then, $B^n C_1^l = 4^{s-1}\omega^{\mu-hl}.k''_1.\Omega$. Using the same method as above, we obtain identical results.

** J-4-2- We suppose that $4 \mid k'_1$.

** J-4-2-1- $k'_1 = 4 \Longrightarrow 4p = 3k'_1 b = 12b \Longrightarrow p = 3b = 3p'$, this case has been studied (see case I-2-paragraph 1.6.8).

** J-4-2-2- $k'_1 > 4$ with $4 \mid k'_1 \Longrightarrow k'_1 = 4^s k''_1$ and $s \geq 1$, $4 \nmid k''_1$. Then, we obtain:

$$A^{2m} = 4^s k''_1 a = 2^{2s} k''_1 a$$
$$B^n C^l = 4^{s-1} k''_1 (3b - 4a) = 2^{2s-2} k''_1 (3b - 4a)$$

** J-4-2-2-1- We suppose that $s = 1$ and $k'_1 = 4k''_1$ with $k''_1 > 1$, so $p = 3p'$ and $p' = k''_1 b$, this is the case 1.6.3 already studied.

** J-4-2-2-2- We suppose that $s > 1$, then $k'_1 = 4^s k''_1 \Longrightarrow 4p = 3 \times 4^s k''_1 b$ and we obtain:

$$A^{2m} = 4^s k''_1 a \tag{1.6.39}$$
$$B^n C^l = 4^{s-1} k''_1 (3b - 4a) \tag{1.6.40}$$

** J-4-2-2-2-1- We suppose that $2 \nmid (k''_1.a) \Longrightarrow 2 \nmid k''_1$ and $2 \nmid a$. As $(A^m)^2 = (2^s)^2.(k''_1.a)$, we call $d^2 = k''_1.a$, then $A^m = 2^s.d \Longrightarrow 2 \mid A^m \Longrightarrow 2 \mid A \Longrightarrow A = 2^i A_1$ with $2 \nmid A_1$ and $i \geq 1$, then: $2^{im}A_1^m = 2^s.d \Longrightarrow s = im$. From the equation (1.6.40), we have $2 \mid (B^n C^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** J-4-2-2-2-1-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j.B_1$, with $j \geq 1$ and $2 \nmid B_1$. The equation (1.6.40) becomes:

$$B_1^n C^l = 2^{2s-jn-2} k''_1 (3b - 4a) = 2^{2im-jn-2} k''_1 (3b - 4a)$$

* We suppose that $2 \nmid (3b - 4a)$:
  - If $2im - jn - 2 \geq 1$, then $2 \mid C^l$, there is no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1.1) is verified.
  - If $2im - jn - 2 \leq 0$, then $2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

* We suppose that $2^\mu \mid (3b - 4a)$, $\mu \geq 1$:
  - If $2im + \mu - jn - 2 \geq 1$, then $2 \mid C^l$, no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1.1) is verified.
  - If $2im + \mu - jn - 2 \leq 0$, then $2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

** J-4-2-2-2-1-2- We suppose that $2 \mid C^l \Longrightarrow 2 \mid C \Longrightarrow C = 2^h.C_1$, with $h \geq 1$ and $2 \nmid C_1$. With the same method used above, we obtain identical results.

** J-4-2-2-2-2- We suppose that $2 \mid (k''_1.a)$:

** J-4-2-2-2-2-1- We suppose that k''_1 and a are coprime:

** J-4-2-2-2-2-1-1- We suppose that $2 \nmid a$ and $2 \mid k''_1 \Longrightarrow k''_1 = 2^{2\mu}.k''^2_2$ and $a = a_1^2$, then the equations (1.6.39-1.6.40) become:

$$A^{2m} = 4^s.2^{2\mu}k''^2_2 a_1^2 \Longrightarrow A^m = 2^{s+\mu}.k''_2.a_1 \tag{1.6.41}$$
$$B^n C^l = 4^{s-1}2^{2\mu}k''^2_2(3b - 4a) = 2^{2s+2\mu-2}k''^2_2(3b - 4a) \tag{1.6.42}$$

The equation (1.6.41) gives $2 \mid A^m \Longrightarrow 2 \mid A \Longrightarrow A = 2^i.A_1$ with $2 \nmid A_1$, $i \geq 1$ and $im = s + \mu$. From the equation (1.6.42), we have $2 \mid (B^n C^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** J-4-2-2-2-2-1-1-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j.B_1$, $2 \nmid B_1$ and $j \geq 1$, then $B_1^n C^l = 2^{2s+2\mu-jn-2} k''^2_2 (3b-4a)$:

* We suppose that $2 \nmid (3b-4a)$:
- If $2im + 2\mu - jn - 2 \geq 1 \Rightarrow 2 \mid C^l$, then there is no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1.1) is verified.
- If $2im + 2\mu - jn - 2 \leq 0 \Rightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

* We suppose that $2^\alpha \mid (3b-4a)$, $\alpha \geq 1$ so that a, b remain coprime:
- If $2im + 2\mu + \alpha - jn - 2 \geq 1 \Rightarrow 2 \mid C^l$, then no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1.1) is verified.
- If $2im + 2\mu + \alpha - jn - 2 \leq 0 \Rightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

** J-4-2-2-2-2-1-1-2- We suppose that $2 \mid C^l \Longrightarrow 2 \mid C \Longrightarrow C = 2^h.C_1$, with $h \geq 1$ and $2 \nmid C_1$. With the same method used above, we obtain identical results.

** J-4-2-2-2-2-1-2- We suppose that $2 \nmid k''_1$ and $2 \mid a \Longrightarrow a = 2^{2\mu}.a_1^2$ and $k''_1 = k''^2_2$, then the equations (1.6.39-1.6.40) become:

$$A^{2m} = 4^s.2^{2\mu}a_1^2 k''^2_2 \Longrightarrow A^m = 2^{s+\mu}.a_1.k''_2. \tag{1.6.43}$$
$$B^n C^l = 4^{s-1}k''^2_2(3b-4a) = 2^{2s-2}k''^2_2(3b-4a) \tag{1.6.44}$$

The equation (1.6.43) gives $2 \mid A^m \Longrightarrow 2 \mid A \Longrightarrow A = 2^i.A_1$ with $2 \nmid A_1$, $i \geq 1$ and $im = s + \mu$. From the equation (1.6.44), we have $2 \mid (B^n C^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** J-4-2-2-2-2-1-2-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j.B_1$, $2 \nmid B_1$ and $j \geq 1$. Then we obtain $B_1^n C^l = 2^{2s-jn-2}k''^2_2(3b-4a)$:

* We suppose that $2 \nmid (3b-4a) \Longrightarrow 2 \nmid b$:
- If $2im - jn - 2 \geq 1 \Rightarrow 2 \mid C^l$, then no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1.1) is verified.
- If $2im - jn - 2 \leq 0 \Rightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

* We suppose that $2^\alpha \mid (3b-4a)$, $\alpha \geq 1$, in this case a, b are not coprime, then the contradiction.

** J-4-2-2-2-2-1-2-2- We suppose that $2 \mid C^l \Longrightarrow 2 \mid C \Longrightarrow C = 2^h.C_1$, with $h \geq 1$ and $2 \nmid C_1$. With the same method used above, we obtain identical results.

** J-4-2-2-2-2-2- We suppose that k''_1 and a are not coprime $2 \mid a$ and $2 \mid k''_1$. Let $a = 2^t.a_1$ and $k''_1 = 2^\mu k''_2$ and $2 \nmid a_1$ and $2 \nmid k''_2$. From (1.6.39), we have $\mu + t = 2\lambda$ and $a_1.k''_2 = \omega^2$. The equations (1.6.39-1.6.40) become:

$$A^{2m} = 4^s k''_1 a = 2^{2s}.2^\mu k''_2.2^t.a_1 = 2^{2s+2\lambda}.\omega^2 \Longrightarrow A^m = 2^{s+\lambda}.\omega \tag{1.6.45}$$
$$B^n C^l = 4^{s-1}2^\mu k''_2(3b-4a) = 2^{2s+\mu-2}k''_2(3b-4a) \tag{1.6.46}$$

From (1.6.45) we have $2 \mid A^m \Longrightarrow 2 \mid A \Longrightarrow A = 2^i A_1$,$i \geq 1$ and $2 \nmid A_1$. From(1.6.46), $2s + \mu - 2 \geq 1$, we deduce that $2 \mid (B^n C^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** J-4-2-2-2-2-2-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j.B_1$, $2 \nmid B_1$ and $j \geq 1$. Then we obtain $B_1^n C^l = 2^{2s+\mu-jn-2}k''^2(3b-4a)$:

* We suppose that $2 \nmid (3b - 4a)$:

- If $2s + \mu - jn - 2 \geq 1 \Rightarrow 2 \mid C^l$, then no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1.1) is verified.
- If $2s + \mu - jn - 2 \leq 0 \Rightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

* We suppose that $2^\alpha \mid (3b - 4a)$, for one value $\alpha \geq 1$. As $2 \mid a$, then $2^\alpha \mid (3b - 4a) \Longrightarrow 2 \mid (3b - 4a) \Longrightarrow 2 \mid (3b) \Longrightarrow 2 \mid b$, then the contradiction with a, b coprime.

** J-4-2-2-2-2-2-2- We suppose that $2 \mid C^l \Longrightarrow 2 \mid C \Longrightarrow C = 2^h.C_1$, with $h \geq 1$ and $2 \nmid C_1$. With the same method used above, we obtain identical results.

** J-4-3- $2 \mid k_1'$ and $2 \mid (3b - 4a)$: then we obtain $2 \mid k_1' \Longrightarrow k_1' = 2^t.k''_1$ with $t \geq 1$ and $2 \nmid k''_1$, $2 \mid (3b - 4a) \Longrightarrow 3b - 4a = 2^\mu.d$ with $\mu \geq 1$ and $2 \nmid d$. We have also $2 \mid b$. If $2 \mid a$, it is a contradition with a, b coprime.

We suppose, in the following, that $2 \nmid a$. The equations (1.6.39-1.6.40) become:

$$A^{2m} = 2^t.k''_1.a = (A^m)^2 \tag{1.6.47}$$

$$B^nC^l = 2^{t-1}k''_1.2^{\mu-1}d = 2^{t+\mu-2}k''_1.d \tag{1.6.48}$$

From (1.6.47), we deduce that the exponent t is even, let $t = 2\lambda$. Then we call $\omega^2 = k''_1.a$, it gives $A^m = 2^\lambda.\omega \Longrightarrow 2 \mid A^m \Longrightarrow 2 \mid A \Longrightarrow A = 2^i.A_1$ with $i \geq 1$ and $2 \nmid A_1$. From (1.6.48), we have $2\lambda + \mu - 2 \geq 1$, then $2 \mid (B^nC^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$:

** J-4-3-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^jB_1$, with $j \geq 1$ and $2 \nmid B_1$. Then we obtain $B_1^nC^l = 2^{2\lambda+\mu-jn-2}.k''_1.d$.
- If $2\lambda + \mu - jn - 2 \geq 1 \Rightarrow 2 \mid C^l \Longrightarrow 2 \mid C$, there is no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1.1) is verified.
- If $2s + t + \mu - jn - 2 \leq 0 \Rightarrow 2 \nmid C$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

** J-4-3-2- We suppose that $2 \mid C^l \Longrightarrow 2 \mid C$. With the same method used above, we obtain identical results.

□

The Main Theorem is proved.

1.7 Examples and Conclusion

1.7.1 Numerical Examples

Example 1:

We consider the example : $6^3 + 3^3 = 3^5$ with $A^m = 6^3$, $B^n = 3^3$ and $C^l = 3^5$. With the notations used in the paper, we obtain:

$$p = 3^6 \times 73, \quad q = 8 \times 3^{11}, \quad \bar{\Delta} = 4 \times 3^{18}(3^7 \times 4^2 - 73^3) < 0$$

$$\rho = \frac{3^8 \times 73\sqrt{73}}{\sqrt{3}}, \quad cos\theta = -\frac{4 \times 3^3 \times \sqrt{3}}{73\sqrt{73}} \tag{1.7.1}$$

As $A^{2m} = \dfrac{4p}{3}.cos^2\dfrac{\theta}{3} \Longrightarrow cos^2\dfrac{\theta}{3} = \dfrac{3A^{2m}}{4p} = \dfrac{3 \times 2^4}{73} = \dfrac{a}{b} \Longrightarrow a = 3 \times 2^4,\ b = 73$; then we obtain:

$$cos\frac{\theta}{3} = \frac{4\sqrt{3}}{\sqrt{73}}, \quad p = 3^6.b \tag{1.7.2}$$

We verify easily the equation (1.7.1) to calculate $cos\theta$ using (1.7.2). For this example, we can use the two conditions from (1.4.9) as $3 \mid a$,$b \mid 4p$ and $3 \mid p$. The cases 1.5.4 and 1.6.3 are respectively used. For the case 1.5.4, it is the case B-2-2-1- that was used and the conjecture (3.1.1) is verified. Concerning the case 1.6.3, it is the case G-2-2-1- that was used and the conjecture (3.1.1) is verified.

Example 2:

The second example is: $7^4+7^3=14^3$. We take $A^m=7^4, B^n=7^3$ and $C^l=14^3$. We obtain $p=57\times 7^6=3\times 19\times 7^6$, $q=8\times 7^{10}$, $\overline{\Delta}=27q^2-4p^3=27\times 4\times 7^{18}(16\times 49-19^3)=-27\times 4\times 7^{18}\times 6075<0$, $\rho=19\times 7^9\times\sqrt{19}$, $cos\theta=-\dfrac{4\times 7}{19\sqrt{19}}$. As $A^{2m}=\dfrac{4p}{3}.cos^2\dfrac{\theta}{3}\Longrightarrow cos^2\dfrac{\theta}{3}=\dfrac{3A^{2m}}{4p}=\dfrac{7^2}{4\times 19}=\dfrac{a}{b}\Longrightarrow a=7^2$, $b=4\times 19$, then $cos\dfrac{\theta}{3}=\dfrac{7}{2\sqrt{19}}$ and we have the two principal conditions $3\mid p$ and $b\mid(4p)$. The calculation of $cos\theta$ from the expression of $cos\dfrac{\theta}{3}$ is confirmed by the value below:

$$cos\theta=cos3(\theta/3)=4cos^3\frac{\theta}{3}-3cos\frac{\theta}{3}=4\left(\frac{7}{2\sqrt{19}}\right)^3-3\frac{7}{2\sqrt{19}}=-\frac{4\times 7}{19\sqrt{19}}$$

Then, we obtain $3\mid p\Rightarrow p=3p'$, $b\mid(4p)$ with $b\neq 2,4$ then $12p'=k_1b=3\times 7^6b$. It concerns the paragraph 1.6.9 of the second hypothesis. As $k_1=3\times 7^6=3k'_1$ with $k'_1=7^6\neq 1$. It is the case J-4-1-2-4-2-2- with the condition $4\mid(3b-4a)$. So we verify :

$$3b-4a=3\times 4\times 19-4\times 7^2=32\Longrightarrow 4\mid(3b-4a)$$

with $A^{2m}=7^8=7^6\times 7^2=k'_1.a$ and k'_1 not a prime, with a and k'_1 not coprime with $\omega=7\nmid\Omega(=2)$. We find that the conjecture (3.1.1) is verified with a common factor equal to 7 (prime and divisor of $k'_1=7^6$).

Example 3:

The third example is: $19^4+38^3=57^3$ with $A^m=19^4$, $B^n=38^3$ and $C^l=57^3$. We obtain $p=19^6\times 577$, $q=8\times 27\times 19^{10}$, $\overline{\Delta}=27q^2-4p^3=4\times 19^{18}(27^3\times 16\times 19^2-577^3)<0$, $\rho=\dfrac{19^9\times 577\sqrt{577}}{3\sqrt{3}}$, $cos\theta=-\dfrac{4\times 3^4\times 19\sqrt{3}}{577\sqrt{577}}$. As $A^{2m}=\dfrac{4p}{3}.cos^2\dfrac{\theta}{3}\Longrightarrow cos^2\dfrac{\theta}{3}=\dfrac{3A^{2m}}{4p}=\dfrac{3\times 19^2}{4\times 577}=\dfrac{a}{b}\Longrightarrow a=3\times 19^2$, $b=4\times 577$, then $cos\dfrac{\theta}{3}=\dfrac{19\sqrt{3}}{2\sqrt{577}}$ and we have the first hypothesis $3\mid a$ and $b\mid(4p)$. Here again, the calculation of $cos\theta$ from the expression of $cos\dfrac{\theta}{3}$ is confirmed by the value below:

$$cos\theta=cos3(\theta/3)=4cos^3\frac{\theta}{3}-3cos\frac{\theta}{3}=4\left(\frac{19\sqrt{3}}{2\sqrt{577}}\right)^3-3\frac{19\sqrt{3}}{2\sqrt{577}}=-\frac{4\times 3^4\times 19\sqrt{3}}{577\sqrt{577}}$$

Then, we obtain $3\mid a\Rightarrow a=3a'=3\times 19^2$, $b\mid(4p)$ with $b\neq 2,4$ and $b=4p'$ with $p=kp'$ soit $p'=577$ and $k=19^6$. This concerns the paragraph 1.5.8 of the first hypothesis. It is the case E-2-2-2-2-1- with $\omega=19$, a',ω not coprime and $\omega=19\nmid(p'-a')=(577-19^2)$ with $s-jn=6-1\times 3=3\geq 1$, and the conjecture (3.1.1) is verified.

1.7.2 Conclusion

The method used to give the proof of the conjecture of Beal has discussed many possibles cases, using elementary number theory and the results of some theorems about Diophantine equations. We have confirmed the method by three numerical examples. In conclusion, we can announce the theorem:

Theorem 1.7.1. *Let A, B, C, m, n, and l be positive natural numbers with $m, n, l > 2$. If :*

$$A^m + B^n = C^l \tag{1.7.3}$$

then A, B, and C have a common factor.

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Chapter 2

Is The Riemann Hypothesis True? Yes It Is

Abstract

In 1859, Georg Friedrich Bernhard Riemann had announced the following conjecture, called Riemann Hypothesis : *The nontrivial roots (zeros) $s = \sigma + it$ of the zeta function, defined by:*

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \quad \Re(s) > 1$$

have real part $\sigma = \dfrac{1}{2}$.

We give a proof that $\sigma = \dfrac{1}{2}$ using an equivalent statement of the Riemann Hypothesis concerning the Dirichlet η function.

Résumé

En 1859, Georg Friedrich Bernhard Riemann avait annoncé la conjecture suivante, dite Hypothèse de Riemann: *Les zéros non triviaux $s = \sigma + it$ de la fonction zeta définie par:*

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ pour } \quad \Re(s) > 1$$

ont comme parties réelles $\sigma = \dfrac{1}{2}$.

On donne une démonstration que $\sigma = \dfrac{1}{2}$ en utilisant une proposition équivalente de l'Hypothèse de Riemann.

2.1 Introduction

In 1859, G.F.B. Riemann had announced the following conjecture [1]:

Conjecture 2.1.1. Let $\zeta(s)$ be the complex function of the complex variable $s=\sigma+it$ defined by the analytic continuation of the function:

$$\zeta_1(s)=\sum_{n=1}^{+\infty}\frac{1}{n^s},\ \text{for}\ \Re(s)=\sigma>1$$

over the whole complex plane, with the exception of $s=1$. Then the nontrivial zeros of $\zeta(s)=0$ are written as :

$$s=\frac{1}{2}+it$$

In this paper, our idea is to start from an equivalent statement of the Riemann Hypothesis, namely the one concerning the Dirichlet η function. The latter is related to Riemann's ζ function where we do not need to manipulate any expression of $\zeta(s)$ in the critical band $0<\Re(s)<1$. In our calculations, we will use the definition of the limit of real sequences. We arrive to give the proof that $\sigma=\dfrac{1}{2}$.

2.1.1 The function ζ.

We denote $s=\sigma+it$ the complex variable of $\mathbb{C}$. For $\Re(s)=\sigma>1$, let ζ_1 be the function defined by :

$$\zeta_1(s)=\sum_{n=1}^{+\infty}\frac{1}{n^s},\ \text{for}\ \ \Re(s)=\sigma>1$$

We know that with the previous definition, the function ζ_1 is an analytical function of s. Denote by $\zeta(s)$ the function obtained by the analytic continuation of $\zeta_1(s)$ to the whole complex plane, minus the point $s=1$, then we recall the following theorem [2]:

Theorem 2.1.2. *The function $\zeta(s)$ satisfies the following :*
1. $\zeta(s)$ has no zero for $\Re(s)>1$;
2. the only pole of $\zeta(s)$ is at $s=1$; it has residue 1 and is simple;
3. $\zeta(s)$ has trivial zeros at $s=-2,-4,\ldots$;
4. the nontrivial zeros lie inside the region $0\le\Re(s)\le1$ (called the critical strip) and are symmetric about both the vertical line $\Re(s)=\dfrac{1}{2}$ and the real axis $\Im(s)=0$.

The vertical line $\Re(s)=\dfrac{1}{2}$ is called the critical line.

The Riemann Hypothesis is formulated as:

Conjecture 2.1.3. (The Riemann Hypothesis,[2]) All nontrivial zeros of $\zeta(s)$ lie on the critical line $\Re(s)=\dfrac{1}{2}$.

In addition to the properties cited by the theorem 2.1.2 above, the function $\zeta(s)$ satisfies the functional relation [2] called also the reflection functional equation for $s\in\mathbb{C}\backslash\{0,1\}$:

$$\zeta(1-s)=2^{1-s}\pi^{-s}cos\frac{s\pi}{2}\Gamma(s)\zeta(s) \tag{2.1.1}$$

where $\Gamma(s)$ is the *gamma function* defined only for $\Re(s)>0$, given by the formula :

$$\Gamma(s)=\int_0^{\infty}e^{-t}t^{s-1}dt,\quad \Re(s)>0$$

So, instead of using the functional given by (2.1.1), we will use the one presented by G.H. Hardy [3] namely Dirichlet's eta function [2]:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)$$

The function eta is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$ [2].

We have also the theorem (see page 16, [3]):

Theorem 2.1.4. *For all $t \in \mathbb{R}$, $\zeta(1+it) \neq 0$.*

So, we take the critical strip as the region defined as $0 < \Re(s) < 1$.

2.1.2 A Equivalent statement to the Riemann Hypothesis.

Among the equivalent statements to the Riemann Hypothesis is that of the Dirichlet function eta which is stated as follows [2]:

Equivalence 2.1.5. The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function :

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \sigma > 1 \tag{2.1.2}$$

that fall in the critical strip $0 < \Re(s) < 1$ lie on the critical line $\Re(s) = \dfrac{1}{2}$.

The series (2.1.2) is convergent, and represents $(1 - 2^{1-s})\zeta(s)$ for $\Re(s) = \sigma > 0$ ([3], pages 20-21). We can rewrite:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \Re(s) = \sigma > 0 \tag{2.1.3}$$

$\eta(s)$ is a complex number, it can be written as :

$$\eta(s) = \rho.e^{i\alpha} \Longrightarrow \rho^2 = \eta(s).\overline{\eta(s)} \tag{2.1.4}$$

and $\eta(s) = 0 \Longleftrightarrow \rho = 0$.

2.2 Preliminaries of the proof

Proof. . We denote $s = \sigma + it$ with $0 < \sigma < 1$. We consider one zero of $\eta(s)$ that falls in critical strip and we write it as $s = \sigma + it$, then we obtain $0 < \sigma < 1$ and $\eta(s) = 0 \Longleftrightarrow (1 - 2^{1-s})\zeta(s) = 0$. We verifies easily the two propositions:

$$\boxed{\textit{s, is one zero of } \eta(s) \textit{ that falls in the critical strip, is also one zero of } \zeta(s)} \tag{2.2.1}$$

Conversely, if s is a zero of $\zeta(s)$ in the critical strip, let $\zeta(s) = 0 \Longrightarrow \eta(s) = (1 - 2^{1-s})\zeta(s) = 0$, then s is also one zero of $\eta(s)$ in the critical strip. We can write:

$$\boxed{\textit{s, is one zero of } \zeta(s) \textit{ that falls in the critical strip, is also one zero of } \eta(s)} \tag{2.2.2}$$

Let us write the function η:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-sLogn} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-(\sigma+it)Logn} =$$

$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn}.e^{-itLogn}$$

$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn}(cos(tLogn) - isin(tLogn))$$

The function η is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$, but not absolutely convergent. Let s be one zero of the function eta, then :

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = 0$$

or:

$$\forall \epsilon' > 0 \quad \exists n_0, \forall N > n_0, \left| \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^s} \right| < \epsilon'$$

We definite the sequence of functions $((\eta_n)_{n\in\mathbb{N}^*}(s))$ as:

$$\eta_n(s) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^s} = \sum_{k=1}^{n} (-1)^{k-1} \frac{cos(tLogk)}{k^\sigma} - i \sum_{k=1}^{n} (-1)^{k-1} \frac{sin(tLogk)}{k^\sigma}$$

with $s = \sigma + it$ and $t \neq 0$.

Let s be one zero of η that lies in the critical strip, then $\eta(s) = 0$, with $0 < \sigma < 1$. It follows that we can write $lim_{n\longrightarrow+\infty}\eta_n(s) = 0 = \eta(s)$. We obtain:

$$lim_{n\longrightarrow+\infty} \sum_{k=1}^{n} (-1)^{k-1} \frac{cos(tLogk)}{k^\sigma} = 0$$

$$lim_{n\longrightarrow+\infty} \sum_{k=1}^{n} (-1)^{k-1} \frac{sin(tLogk)}{k^\sigma} = 0$$

Using the definition of the limit of a sequence, we can write:

$$\forall \epsilon_1 > 0\, \exists n_r, \forall N > n_r, |\Re(\eta(s)_N)| < \epsilon_1 \Longrightarrow \Re(\eta(s)_N)^2 < {\epsilon_1}^2 \tag{2.2.3}$$

$$\forall \epsilon_2 > 0\, \exists n_i, \forall N > n_i, |\Im(\eta(s)_N)| < \epsilon_2 \Longrightarrow \Im(\eta(s)_N)^2 < {\epsilon_2}^2 \tag{2.2.4}$$

Then:

$$0 < \sum_{k=1}^{N} \frac{cos^2(tLogk)}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} \frac{(-1)^{k+k'} cos(tLogk).cos(tLogk')}{k^\sigma k'^\sigma} < \epsilon_1^2$$

$$0 < \sum_{k=1}^{N} \frac{sin^2(tLogk)}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} \frac{(-1)^{k+k'} sin(tLogk).sin(tLogk')}{k^\sigma k'^\sigma} < \epsilon_2^2$$

Taking $\epsilon = \epsilon_1 = \epsilon_2$ and $N > max(n_r, n_i)$, we get by making the sum member to member of the last two inequalities:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma} < 2\epsilon^2 \tag{2.2.5}$$

We can write the above equation as :

$$0 < \rho_N^2 < 2\epsilon^2 \tag{2.2.6}$$

or $\rho(s) = 0$.

2.3 Case $\sigma = \dfrac{1}{2}$.

We suppose that $\sigma = \dfrac{1}{2}$. Let's start by recalling Hardy's theorem (1914) ([2], page 24):

Theorem 2.3.1. *There are infinitely many zeros of $\zeta(s)$ on the critical line.*

From the propositions (2.2.1-2.2.2), it follows the proposition :

Proposition 2.3.2. There are infinitely many zeros of $\eta(s)$ on the critical line.

Let $s_j = \frac{1}{2} + it_j$ one of the zeros of the function $\eta(s)$ on the critical line, so $\eta(s_j) = 0$. The equation (2.2.5) is written for s_j:

$$0 < \sum_{k=1}^{N} \frac{1}{k} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

If $N \longrightarrow +\infty$, the series $\sum_{k=1}^{N} \dfrac{1}{k}$ is divergent and becomes infinite. then:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Hence, we obtain the following result:

$$\boxed{lim_{N \longrightarrow +\infty} \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} = -\infty} \tag{2.3.1}$$

if not, we will have a contradiction with the fact that :

$$lim_{N \longrightarrow +\infty} \sum_{k=1}^{N} (-1)^{k-1} \frac{1}{k^{s_j}} = 0 \Longleftrightarrow \eta(s) \text{ is convergent for } s_j = \frac{1}{2} + it_j$$

2.4 Case $0 < \Re(s) < \dfrac{1}{2}$.

2.4.1 Case where there are zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \dfrac{1}{2}$.

Suppose that there exists $s = \sigma + it$ one zero of $\eta(s)$ or $\eta(s) = 0 \Longrightarrow \rho^2(s) = 0$ with $0 < \sigma < \frac{1}{2} \Longrightarrow s$ lies inside the critical band. We write the equation (2.2.5):

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k^{2\sigma}} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}}$$

But $2\sigma < 1$, it follows that $lim_{N\longrightarrow+\infty}\sum_{k=1}^{N}\frac{1}{k^{2\sigma}}\longrightarrow+\infty$ and then, we obtain :

$$\boxed{\sum_{k,k'=1;k<k'}^{+\infty}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^{\sigma}k'^{\sigma}}=-\infty} \tag{2.4.1}$$

2.5 Case $\frac{1}{2}<\Re(s)<1$.

Let $s=\sigma+it$ be the zero of $\eta(s)$ in $0<\Re(s)<\frac{1}{2}$, object of the previous paragraph. From the proposition (2.2.1), $\zeta(s)=0$. According to point 4 of theorem 2.1.2, the complex number $s'=1-\sigma+it=\sigma'+it'$ with $\sigma'=1-\sigma$, $t'=t$ and $\frac{1}{2}<\sigma'<1$ verifies $\zeta(s')=0$, so s' is also a zero of the function $\zeta(s)$ in the band $\frac{1}{2}<\Re(s)<1$, it follows from the proposition (2.2.2) that $\eta(s')=0\Longrightarrow\rho(s')=0$. By applying (2.2.5), we get:

$$0<\sum_{k=1}^{N}\frac{1}{k^{2\sigma'}}+2\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(t'Log(k/k'))}{k^{\sigma'}k'^{\sigma'}}<2\epsilon^2 \tag{2.5.1}$$

As $0<\sigma<\frac{1}{2}\Longrightarrow 2>2\sigma'=2(1-\sigma)>1$, then the series $\sum_{k=1}^{N}\frac{1}{k^{2\sigma'}}$ is convergent to a positive constant not null $C(\sigma')$. As $1/k^2<1/k^{2\sigma'}$ for all $k>0$, then :

$$0<\zeta(2)=\frac{\pi^2}{6}=\sum_{k=1}^{+\infty}\frac{1}{k^2}<\sum_{k=1}^{+\infty}\frac{1}{k^{2\sigma'}}=C(\sigma')=\zeta_1(2\sigma')=\zeta(2\sigma')$$

From the equation (2.5.1), it follows that :

$$\sum_{k,k'=1;k<k'}^{+\infty}(-1)^{k+k'}\frac{cos(t'Log(k/k'))}{k^{\sigma'}k'^{\sigma'}}=-\frac{C(\sigma')}{2}=-\frac{\zeta(2\sigma')}{2}>-\infty \tag{2.5.2}$$

Case $t=0$

We suppose that $t=0\Longrightarrow t'=0$. The equation (2.5.2) becomes:

$$\sum_{k,k'=1;k<k'}^{+\infty}(-1)^{k+k'}\frac{1}{k^{\sigma'}k'^{\sigma'}}=-\frac{C(\sigma')}{2}=-\frac{\zeta(2\sigma')}{2}>-\infty \tag{2.5.3}$$

Then $s'=\sigma'>1/2$ is a zero of $\eta(s)$, we obtain :

$$\eta(s')=\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^{s'}}=0 \tag{2.5.4}$$

Let us define the sequence S_m as:

$$S_m(s')=\sum_{n=1}^{m}\frac{(-1)^{n-1}}{n^{s'}}=\sum_{n=1}^{m}\frac{(-1)^{n-1}}{n^{\sigma'}}=S_m(\sigma') \tag{2.5.5}$$

From the definition of S_m, we obtain :

$$lim_{m\longrightarrow+\infty}S_m(s')=\eta(s')=\eta(\sigma') \tag{2.5.6}$$

We have also:

$$S_1(\sigma')=1>0 \tag{2.5.7}$$

$$S_2(\sigma')=1-\frac{1}{2^{\sigma'}}>0 \quad because\ 2^{\sigma'}>1 \tag{2.5.8}$$

$$S_3(\sigma')=S_2(\sigma')+\frac{1}{3^{\sigma'}}>0 \tag{2.5.9}$$

We proceed by recurrence, we suppose that $S_m(\sigma') > 0$.

1. $m = 2q \Longrightarrow S_{m+1}(\sigma') = \sum_{n=1}^{m+1} \frac{(-1)^{n-1}}{n^{s'}} = S_m(\sigma') + \frac{(-1)^{m+1-1}}{(m+1)^{\sigma'}}$, it gives:

$$S_{m+1}(\sigma') = S_m(\sigma') + \frac{(-1)^{2q}}{(m+1)^{\sigma'}} = S_m(\sigma') + \frac{1}{(m+1)^{\sigma'}} > 0 \Rightarrow S_{m+1}(\sigma') > 0$$

2. $m = 2q+1$, we can write $S_{m+1}(\sigma')$ as:

$$S_{m+1}(\sigma') = S_{m-1}(\sigma') + \frac{(-1)^{m-1}}{m^{\sigma'}} + \frac{(-1)^{m+1-1}}{(m+1)^{\sigma'}}$$

We have $S_{m-1}(\sigma') > 0$, let $T = \frac{(-1)^{m-1}}{m^{\sigma'}} + \frac{(-1)^m}{(m+1)^{\sigma'}}$, we obtain:

$$T = \frac{(-1)^{2q}}{(2q+1)^{\sigma'}} + \frac{(-1)^{2q+1}}{(2q+2)^{\sigma'}} = \frac{1}{(2q+1)^{\sigma'}} - \frac{1}{(2q+2)^{\sigma'}} > 0 \tag{2.5.10}$$

and $S_{m+1}(\sigma') > 0$.

Then all the terms $S_m(\sigma')$ of the sequence S_m are great then 0, it follows that $lim_{m\longrightarrow+\infty}S_m(s') = \eta(s') = \eta(\sigma') > 0$ and $\eta(\sigma') < +\infty$ because $\Re(s') = \sigma' > 0$ and $\eta(s')$ is convergent. We deduce the contradiction with the hypothesis s' is a zero of $\eta(s)$ and:

$$\boxed{\text{The equation (2.5.3) is false for the case } t' = t = 0.} \tag{2.5.11}$$

Case $t \neq 0$

We suppose that $t \neq 0$. For each $s' = \sigma' + it' = 1 - \sigma + it$ a zero of $\eta(s)$, we have:

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t'Log(k/k'))}{k^{\sigma'}k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty \tag{2.5.12}$$

the left member of the equation (2.5.12) above is finite and depends of σ' and t', but the right member is a function only of σ' equal to $-\zeta(2\sigma')/2$. But for all σ'' so that $2\sigma'' > 1$, we have $\zeta(2\sigma'')$:

$$\zeta(2\sigma'') = \zeta_1(2\sigma'') = \sum_{k=1}^{+\infty} \frac{1}{k^{2\sigma''}} < +\infty$$

It depends only of σ'', then in particular for all σ'' with $2 > 2\sigma'' > 1$, $\zeta(2\sigma'')$ depends only of σ''. Let $\lambda > 0$ be an arbitrary real number very infinitesimal so that $\sigma' + \lambda \in]1/2, 1[$ is not the real part of a zero of $\eta(s)$. We can write to the first order:

$$\zeta(2\sigma' + 2\lambda) = \zeta(2\sigma') + 2\lambda.\zeta'(2\sigma') \tag{2.5.13}$$

$\zeta'(2\sigma')$ is given by:

$$\zeta'(2\sigma') = -\sum_{k=2}^{+\infty} \frac{Logk}{k^{2\sigma'}} > -\infty \tag{2.5.14}$$

because we can choose $\alpha > 0$ so that $\sigma' > 1/2 + \alpha \Longrightarrow 2\sigma' - 2\alpha > 1$ and we obtain:

$$|\zeta'(2\sigma')| \leq \frac{1}{2\alpha} \sum_{k=2}^{+\infty} \frac{Logk^{2\alpha}}{k^{2\alpha}} \frac{1}{k^{2(\sigma'-\alpha)}} \leq \frac{1}{2\alpha} \sum_{k=2}^{+\infty} \frac{1}{k^{2(\sigma'-\alpha)}} < +\infty$$

Numerically, the left member of the equation (2.5.13) is independent of t', the preponderant term of the right member $\zeta(2\sigma')$ depends of t' using the equation (2.5.12), then the contradiction and we conclude that the result giving by the equation (2.5.12) is false.

$$\boxed{\text{It follows that the equation (2.5.12) is false for the cases } t' \neq 0.} \tag{2.5.15}$$

From (2.5.11-2.5.15), we conclude that the function $\eta(s)$ has no zeros for all $s' = \sigma' + it'$ with $\sigma' \in]1/2, 1[$, it follows that the case of the paragraph (2.4) above concerning the case $0 < \Re(s) < \dfrac{1}{2}$ is false. Then, the function $\eta(s)$ has all its zeros on the critical line $\sigma = \dfrac{1}{2}$. From the equivalent statement (3.1.1), it follows that **the Riemann hypothesis is verified**. □

From the calculations above, we can verify easily the following known proposition:

Proposition 2.5.1. For all $s = \sigma$ real with $0 < \sigma < 1$, $\eta(s) > 0$ and $\zeta(s) < 0$.

2.6 Conclusion.

In summary: for our proofs, we made use of Dirichlet's $\eta(s)$ function:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad s = \sigma + it$$

on the critical band $0 < \Re(s) < 1$, in obtaining:

- $\eta(s)$ vanishes for $0 < \sigma = \Re(s) = \dfrac{1}{2}$;
- $\eta(s)$ does not vanish for $0 < \sigma = \Re(s) < \dfrac{1}{2}$ and $\dfrac{1}{2} < \sigma = \Re(s) < 1$.

Consequently, all the zeros of $\eta(s)$ inside the critical band $0 < \Re(s) < 1$ are on the critical line $\Re(s) = \dfrac{1}{2}$. Applying the equivalent proposition to the Riemann Hypothesis (3.1.1), we conclude that **the Riemann hypothesis is verified** and all the nontrivial zeros of the function $\zeta(s)$ lie on the critical line $\Re(s) = \dfrac{1}{2}$. The proof of the Riemann Hypothesis is thus completed.

We therefore announce the important theorem as follows:

Theorem 2.6.1. *The Riemann Hypothesis is true:*
All nontrivial zeros of the function $\zeta(s)$ with $s = \sigma + it$ lie on the vertical line $\Re(s) = \dfrac{1}{2}$.

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Chapter 3

Is The Conjecture $c < rad^{1.63}(abc)$ True?

Abstract

In this paper, we consider the abc conjecture, we will give the proof that the conjecture $c < rad^{1.63}(abc)$ is true. It constitutes the key to resolve the abc conjecture.

Résumé:

Dans cet article, nous considérons la conjecture abc. Nous donnons la preuve de la conjecture $c < rad^{1.63}(abc)$ qui constitue la clé pour résoudre la conjecture abc.

3.1 Introduction and notations

Let a be a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{3.1.1}$$

We denote:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{3.1.2}$$

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the abc conjecture is given below:

Conjecture 3.1.1. (***abc*** **Conjecture**): For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{3.1.3}$$

where K is a constant depending only of ϵ.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [2]. It concerned the best example given by E. Reyssat [2]:

$$2 + 3^{10}.109 = 23^5 \Longrightarrow c < rad^{1.629912}(abc) \tag{3.1.4}$$

A conjecture was proposed that $c < rad^2(abc)$ [3]. In 2012, A. Nitaj [4] proposed the following conjecture:

Conjecture 3.1.2. Let a, b, c be positive integers relatively prime with $c = a + b$, then:

$$c < rad^{1.63}(abc) \tag{3.1.5}$$
$$abc < rad^{4.42}(abc) \tag{3.1.6}$$

In this paper, we will give the proof of the conjecture given by (3.1.5) that constitutes the key to obtain the proof of the abc conjecture using classical methods with the help of some theorems from the field of the number theory.

3.2 The Proof of the conjecture $c < rad^{1.63}(abc)$, case $c = a + b$

Let a, b, c be positive integers, relatively prime, with $c = a + b$, $b < a$ and $R = rad(abc)$, $c = \prod_{j'=1}^{j'=J'} c_{j'}^{\beta_{j'}}$, $\beta_{j'} \geq 1$, $c_{j'} \geq 2$ prime integers.

In the following, we will give the proof of the conjecture $c < rad^{1.63}(abc)$.

Proof. :
I- We suppose that $c < rad(abc)$, then we obtain:

$$c < rad(abc) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$$

and the condition (3.1.5) is satisfied.

II- We suppose that $c = rad(abc)$, then a, b, c are not coprime, case to reject.

III- In the following, we suppose that $c > rad(abc)$ and a, b and c are not all prime numbers.

$$c = \mu_c rad(c) = a + b = \mu_a rad(a) + \mu_b rad(b) \overset{?}{<} rad^{1.63}(abc) \tag{3.2.1}$$

III-1- We suppose $\mu_a \leq rad^{0.63}(a)$. We obtain :

$$c = a + b < 2a \leq 2rad^{1.63}(a) < rad^{1.63}(abc) \Longrightarrow c < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$$

Then (3.2.1) is satisfied.

III-2- We suppose $\mu_c \leq rad^{0.63}(c)$. We obtain :

$$c = \mu_c rad(c) \leq rad^{1.63}(c) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$$

and the condition (3.2.1) is satisfied.

III-3- We suppose $\mu_c > rad^{0.63}(c)$ and $\mu_a > rad^{0.63}(a)$.

III-3-1- Case : $rad^{0.63}(c) < \mu_c \leq rad^{1.63}(c)$ and $rad^{0.63}(a) < \mu_a \leq rad^{1.63}(a)$.
We can write:

$$\left.\begin{array}{l} \mu_c \leq rad^{1.63}(c) \Longrightarrow c \leq rad^{2.63}(c) \\ \\ \mu_a \leq rad^{1.63}(a) \Longrightarrow a \leq rad^{2.63}(a) \end{array}\right\} \Longrightarrow ac \leq rad^{2.63}(ac) \Longrightarrow a^2 < ac \leq rad^{2.63}(ac)$$
$$\Longrightarrow a < rad^{1.315}(ac) \Longrightarrow c < 2a < 2rad^{1.315}(ac) < rad^{1.63}(abc)$$
$$\Longrightarrow \boxed{c = a + b < R^{1.63}}$$

III-3-2- Case : $\mu_c > rad^{1.63}(c)$ or $\mu_a > rad^{1.63}(a)$

III-3-2-1- We suppose that $\mu_c > rad^{1.63}(c)$ and $\mu_a \leq rad^2(a)$:

III-3-2-1-1- Case $rad(a) < rad(c)$:
In this case $a = \mu_a.rad(a) \leq rad^3(a) \leq rad^{1.63}(a)rad^{1.37}(a) < rad^{1.63}(a).rad^{1.37}(c) \Longrightarrow c < 2a < 2rad^{1.63}(a).rad^{1.37}(c) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$.

III-3-2-1-2- Case $rad(c) < rad(a) < rad^{\frac{1.63}{1.37}}(c)$: As $a \leq rad^{1.63}(a).rad^{1.37}(a) < rad^{1.63}(a).rad^{1.63}(c) \Longrightarrow c < 2a < 2rad^{1.63}(a).rad^{1.63}(c) < R^{1.63} \Longrightarrow \boxed{c < R^{1.63}}$.

III-3-2-1-3- Case $rad^{\frac{1.63}{1.37}}(c) < rad(a)$:

III-3-2-1-3-1- We suppose $c \leq rad^{3.26}(c)$, we obtain:

$$c \leq rad^{3.26}(c) \Longrightarrow c \leq rad^{1.63}(c).rad^{1.63}(c) \Longrightarrow$$
$$c < rad^{1.63}(c).rad^{1.37}(a) < rad^{1.63}(c).rad^{1.63}(a).rad^{1.63}(b) = R^{1.63} \Longrightarrow \boxed{c < R^{1.63}}$$

III-3-2-1-3-2- We suppose $c > rad^{3.26}(c) \Longrightarrow \mu_c > rad^{2.26}(c)$.

III-3-2-1-3-2-1- We consider the case $\mu_a = rad^2(a) \Longrightarrow a = rad^3(a)$. Then, we obtain that $X = rad(a)$ is a solution in positive integers of the equation:

$$X^3 + 1 = c - b + 1 = c' \tag{3.2.2}$$

But it is the case $c' = 1 + a$.

III-3-2-1-3-2-1-1- We suppose that $c' = rad^n(c')$ with $n \geq 4$, we obtain the equation:

$$rad^n(c') - rad^3(a) = 1 \tag{3.2.3}$$

But the solutions of the equation (3.2.3) are [5] :($rad(c') = 3, n = 2, rad(a) = +2$), it follows the contradiction with $n \geq 4$ and the case $c' = rad^n(c'), n \geq 4$ is to reject.

III-3-2-1-3-2-1-2- In the following, we will study the cases $\mu_{c'} = A.rad^n(c')$ with $rad(c') \nmid A, n \geq 0$. The above equation (3.2.2) can be written as :

$$(X+1)(X^2 - X + 1) = c' \tag{3.2.4}$$

Let δ any divisor of c', then:

$$X + 1 = \delta \tag{3.2.5}$$
$$X^2 - X + 1 = \frac{c'}{\delta} = c'' = \delta^2 - 3X \tag{3.2.6}$$

We recall that $rad(a) > rad^{\frac{1.63}{1.37}}(c)$.

III-3-2-1-3-2-1-2-1- We suppose $\delta = l.rad(c')$. We have $\delta = l.rad(c') < c' = \mu_{c'}.rad(c') \Longrightarrow l < \mu_{c'}$. As δ is a divisor of c', then l is a divisor of $\mu_{c'}$, we write $\mu_{c'} = l.m$. From $\mu_{c'} = l(\delta^2 - 3X)$, we obtain:

$$m = l^2 rad^2(c') - 3rad(a) \Longrightarrow 3rad(a) = l^2 rad^2(c') - m$$

A- Case $3|m \Longrightarrow m = 3m', m' > 1$: As $\mu_{c'} = ml = 3m'l \Longrightarrow 3|rad(c')$ and $(rad(c'), m')$ not coprime. We obtain:

$$rad(a) = l^2 rad(c').\frac{rad(c')}{3} - m'$$

It follows that a, c' are not coprime, then the contradiction.

B - Case $m = 3 \Longrightarrow \mu_{c'} = 3l \Longrightarrow c' = 3lrad(c') = 3\delta = \delta(\delta^2 - 3X) \Longrightarrow \delta^2 = 3(1+X) = 3\delta \Longrightarrow \delta = lrad(c') = 3 \Longrightarrow c' = 3\delta = 9 = a+1 \Longrightarrow a = 8 \Longrightarrow c \leq 15$, then it is a trivial case.

III-3-2-1-3-2-1-2-2- We suppose $\delta = l.rad^2(c'), l \geq 2$. If $n = 0$ then $\mu_{c'} = A$ and from the equation above (3.2.6):

$$c'' = \frac{c'}{\delta} = \frac{\mu_{c'}.rad(c')}{lrad^2(c')} = \frac{A.rad(c')}{lrad^2(c')} = \frac{A}{lrad(c')} \Rightarrow rad(c')|A$$

It follows the contradiction with the hypothesis above $rad(c') \nmid A$.

III-3-2-1-3-2-1-2-3- In the following, we suppose that $n > 0$.

If $lrad(c') \nmid \mu_{c'}$ then the case is to reject. We suppose $lrad(c')|\mu_{c'} \Longrightarrow \mu_{c'} = m.lrad(c')$, then $\dfrac{c'}{\delta} = m = \delta^2 - 3rad(a)$.

C - Case $m = 1 = c'/\delta \Longrightarrow \delta^2 - 3rad(a) = 1 \Longrightarrow (\delta-1)(\delta+1) = 3rad(a) = rad(a)(\delta+1) \Longrightarrow \delta = 2 = l.rad^2(c')$, then the contradiction.

D - Case $m = 3$, we obtain $3(1 + rad(a)) = \delta^2 = 3\delta \Longrightarrow \delta = 3 = lrad^2(c')$. Then the contradiction.

E - Case $m \neq 1, 3$, we obtain: $3rad(a) = l^2rad^4(c') - m \Longrightarrow rad(a)$ and $rad(c')$ are not coprime. Then the contradiction.

III-3-2-1-3-2-1-2-4- We suppose $\delta = l.rad^n(c'), l \geq 2$ with $n \geq 3$. From $c' = \mu_{c'}.rad(c') = lrad^n(c')(\delta^2 - 3rad(a))$, we denote $m = \delta^2 - 3rad(a) = \delta^2 - 3X$.

F - As seen above (paragraphs C,D), the cases $m = 1$ and $m = 3$ give contradictions, it follows the reject of these cases.

G - Case $m \neq 1, 3$. Let q be a prime that divides m, it follows $q|\mu'_c \Longrightarrow q = c'_{j'_0} \Longrightarrow c'_{j'_0}|\delta^2 \Longrightarrow c'_{j'_0}|3rad(a)$. Then $rad(a)$ and $rad(c')$ are not coprime. It follows the contradiction.

III-3-2-1-3-2-1-2-5- We suppose $\delta = \prod_{j \in J_1} c_j'^{\beta_j}$, $\beta_j \geq 1$ with at least one $j_0 \in J_1$ with $\beta_{j_0} \geq 2$, $rad(c') \nmid \delta$. We can write:

$$\delta = \mu_\delta.rad(\delta), \quad rad(c') = m.rad(\delta), \quad m > 1, \quad (m, \mu_\delta) = 1 \tag{3.2.7}$$

Then, we obtain:

$$c' = \mu_{c'}.rad(c') = \mu_{c'}.m.rad(\delta) = \delta(\delta^2 - 3X) = \mu_\delta.rad(\delta)(\delta^2 - 3X) \Longrightarrow$$
$$m.\mu_{c'} = \mu_\delta(\delta^2 - 3X) \tag{3.2.8}$$

- We suppose $\mu_{c'} = \mu_\delta \Longrightarrow m = \delta^2 - 3X = (\mu_{c'}.rad(\delta))^2 - 3X$. As $\delta < \delta^2 - 3X \Longrightarrow m > \delta \Longrightarrow rad(c') > m > \mu_{c'}.rad(\delta) > rad^3(c')$ because $\mu_{c'} > rad^{2.26}(c')$, it follows $rad(c') > rad^2(c')$. Then the contradiction.

- We suppose $\mu_{c'} < \mu_\delta$. As $rad(a) = \mu_\delta rad(\delta) - 1$, we obtain:

$$rad(a) > \mu_{c'}.rad(\delta) - 1 > 0 \Longrightarrow rad(ac') > c'.rad(\delta) - rad(c') > 0 \Longrightarrow$$
$$c' > rad(ac') > c'.rad(\delta) - rad(c') > 0 \Longrightarrow 1 > rad(\delta) - \frac{rad(c')}{c'} > 0, \quad rad(\delta) \geq 2$$
$$\Longrightarrow \text{The contradiction} \tag{3.2.9}$$

- We suppose $\mu_{c'} > \mu_\delta$. In this case, from the equation (3.2.8) and as $(m,\mu_\delta) = 1$, it follows we can write:

$$\mu_{c'} = \mu_1.\mu_2, \quad \mu_1,\mu_2 > 1 \tag{3.2.10}$$
$$c' = \mu_{c'} rad(c') = \mu_1.\mu_2.rad(\delta).m = \delta.(\delta^2 - 3X) \tag{3.2.11}$$
$$\text{so that} \quad m.\mu_1 = \delta^2 - 3X, \quad \mu_2 = \mu_\delta \Longrightarrow \delta = \mu_2.rad(\delta) \tag{3.2.12}$$

**1- We suppose $(\mu_1,\mu_2) \neq 1$, then $\exists c'_{j_0}$ so that $c'_{j_0}|\mu_1$ and $c'_{j_0}|\mu_2$. But $\mu_\delta = \mu_2 \Rightarrow c'^2_{j_0}|\delta$. From $3X = \delta^2 - m\mu_1 \Longrightarrow c'_{j_0}|3X \Longrightarrow c'_{j_0}|X$ or $c'_{j_0} = 3$.

- If $c'_{j_0}|X$, it follows the contradiction with $(c',a) = 1$.

- If $c'_{j_0} = 3$. We have $m\mu_1 = \delta^2 - 3X = \delta^2 - 3(\delta - 1) \Longrightarrow \delta^2 - 3\delta + 3 - m.\mu_1 = 0$. As $3|\mu_1 \Longrightarrow \mu_1 = 3^k\mu'_1, 3 \nmid \mu'_1, k \geq 1$, we obtain:

$$\delta^2 - 3\delta + 3(1 - 3^{k-1}m\mu'_1) = 0 \tag{3.2.13}$$

**1-1- We consider the case $k > 1 \Longrightarrow 3 \nmid (1 - 3^{k-1}m\mu'_1)$. Let us recall the Eisenstein criterion [6]:

> **Theorem 3.2.1.** ***(Eisenstein Criterion)*** *Let $f = a_0 + \cdots + a_nX^n$ be a polynomial $\in \mathbb{Z}[X]$. We suppose that $\exists\, p$ a prime number so that $p \nmid a_n$, $p|a_i$, $(0 \leq i \leq n-1)$, and $p^2 \nmid a_0$, then f is irreducible in $\mathbb{Q}$.*

We apply Eisenstein criterion to the polynomial $R(Z)$ given by:

$$R(Z) = Z^2 - 3Z + 3(1 - 3^{k-1}m\mu'_1) \tag{3.2.14}$$

then:

- $3 \nmid 1$, - $3|(-3)$,- $3|3(1 - 3^{k-1}m\mu'_1)$, and - $3^2 \nmid 3(1 - 3^{k-1}m\mu'_1)$.

It follows that the polynomial $R(Z)$ is irreducible in $\mathbb{Q}$, then, the contradiction with $R(\delta) = 0$.

**1-2- We consider the case $k = 1$, then $\mu_1 = 3\mu'_1$ and $(\mu'_1, 3) = 1$, we obtain:

$$\delta^2 - 3\delta + 3(1 - m\mu'_1) = 0 \tag{3.2.15}$$

**1-2-1- We consider that $3 \nmid (1 - m.\mu'_1)$, we apply the same Eisenstein criterion to the polynomial $R'(Z)$ given by:

$$R'(Z) = Z^2 - 3Z + 3(1 - m\mu'_1)$$

and we find a contradiction with $R'(\delta) = 0$.

**1-2-2- We consider that $3|(1 - m.\mu'_1) \Longrightarrow m\mu'_1 - 1 = 3^i.h$, $i \geq 1$, $3 \nmid h, h \in \mathbb{N}^*$. δ is an integer root of the polynomial $R'(Z)$:

$$R'(Z) = Z^2 - 3Z + 3(1 - m\mu'_1) = 0 \Rightarrow \text{ the discriminant of } R'(Z) \text{ is } :\Delta = 3^2 + 3^{i+1} \times 4.h \tag{3.2.16}$$

As the root δ is an integer, it follows that $\Delta = l^2 > 0$ with l a positive integer. We obtain:

$$\Delta = 3^2(1 + 3^{i-1} \times 4h) = l^2 \tag{3.2.17}$$
$$\Longrightarrow 1 + 3^{i-1} \times 4h = q^2 > 1, q \in \mathbb{N}^* \tag{3.2.18}$$

We can write the equation (3.2.15) as :

$$\delta(\delta - 3) = 3^{i+1}.h \Longrightarrow 3^3\mu'_1\frac{rad(\delta)}{3}.\left(\mu'_1 rad(\delta) - 1\right) = 3^{i+1}.h \Longrightarrow \tag{3.2.19}$$
$$\mu'_1\frac{rad(\delta)}{3}.\left(\mu'_1 rad(\delta) - 1\right) = h \tag{3.2.20}$$

We obtain $i = 2$ and $q^2 = 1 + 12h = 1 + 4\mu'_1 rad(\delta)(\mu'_1 rad(\delta) - 1)$. Then, q satisfies :

$$q^2 - 1 = 12h \Rightarrow \frac{(q-1)}{2}.\frac{(q+1)}{2} = 3h = (\mu'_1 rad(\delta) - 1).\mu'_1 rad(\delta) \Rightarrow \tag{3.2.21}$$
$$q - 1 = 2\mu'_1 rad(\delta) - 2 \tag{3.2.22}$$
$$q + 1 = 2\mu'_1 rad(\delta) \tag{3.2.23}$$

It follows that $(q = x, 1 = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{3.2.24}$$

with $N = 12h > 0$. Let $Q(N)$ be the number of the solutions of (3.2.24) and $\tau(N)$ is the number of suitable factorization of N, then we announce the following result concerning the solutions of the Diophantine equation (3.2.24) (see theorem 27.3 in [7]):
- If $N \equiv 2(\bmod 4)$, then $Q(N) = 0$.
- If $N \equiv 1$ or $N \equiv 3(\bmod 4)$, then $Q(N) = [\tau(N)/2]$.
- If $N \equiv 0(\bmod 4)$, then $Q(N) = [\tau(N/4)/2]$.
$[x]$ is the integral part of x for which $[x] \leq x < [x] + 1$.

Let (α', m'), $\alpha', m' \in \mathbb{N}^*$ be another pair, solution of the equation (3.2.24), then $\alpha'^2 - m'^2 = x^2 - y^2 = N = 12h$, but $q = x$ and $1 = y$ satisfy the equation (3.2.23) given by $x + y = 2\mu'_1 rad(\delta)$, it follows α', m' verify also $\alpha' + m' = 2\mu'_1 rad(\delta)$, that gives $\alpha' - m' = 2(\mu'_1 rad(\delta) - 1)$, then $\alpha' = x = q = 2\mu'_1 rad(\delta)$ and $m' = y = 1$. So, we have given the proof of the uniqueness of the solutions of the equation (3.2.24) with the condition $x + y = 2\mu'_1 rad(\delta)$. As $N = 12h = 4\mu'_1 rad(\delta).\left(\mu'_1 rad(\delta) - 1\right) \Longrightarrow N \equiv 0(\bmod 4) \Longrightarrow Q(N) = [\tau(N/4)/2] = [\tau(3h)/2]$, the expression of $3h = \mu'_1.rad(\delta).\left(\mu'_1 rad(\delta) - 1\right)$, then $Q(N) = [\tau(3h)/2] > 1$. But $Q(N) = 1$, then the contradiction and the case $3|(1 - m.\mu'_1)$ is to reject.

**2- We suppose that $(\mu_1, \mu_2) = 1$.

From the equation $m\mu_1 = \delta^2 - 3X = \delta^2 - 3(\delta - 1)$, we obtain that δ is a root of the following polynomial :

$$R(Z) = Z^2 - 3Z + 3 - m.\mu_1 = 0 \tag{3.2.25}$$

The discriminant of $R(Z)$ is:

$$\Delta = 9 - 4(3 - m.\mu_1) = 4m.\mu_1 - 3 = q^2 \quad \text{with } q \in \mathbb{N}^* \quad \text{as } \delta \in \mathbb{N}^* \tag{3.2.26}$$

- We suppose that $2|m\mu_1 \Longrightarrow c'$ is even. Then $q^2 \equiv 5(\bmod 8)$, it gives a contradiction because a square is $\equiv 0, 1$ or $4(\bmod 8)$.

- We suppose c' an odd integer, then a is even. It follows $a = rad^3(a) \equiv 0(\bmod 8) \Longrightarrow c' \equiv 1(\bmod 8)$. As $c' = \delta^2 - 3X.\delta$, we obtain $\delta^2 - 3X.\delta \equiv 1(\bmod 8)$. If $\delta^2 \equiv 1(\bmod 8) \Longrightarrow -3X.\delta \equiv 0(\bmod 8) \Longrightarrow 8|X.\delta \Longrightarrow 4|\delta \Longrightarrow c'$ is even. Then, the contradiction. If $\delta^2 \equiv 4(\bmod 8) \Longrightarrow \delta \equiv 2(\bmod 8)$ or $\delta \equiv 6(\bmod 8)$. In the two cases, we obtain $2|\delta$. Then, the contradiction with c' an odd integer.

It follows that the case $c > rad^{3.26}(c)$ and $a = rad^3(a)$ is impossible.

III-3-2-1-3-2-2- We suppose $c > rad^{3.26}(c)$ and large and $\mu_a < rad^2(a)$. Then $c = rad^3(c) + h, h > rad^3(c)$, h a positive integer and we can write $a + l = rad^3(a)$, $l > 0$. Then we obtain :

$$rad^3(c) + h = rad^3(a) - l + b \Longrightarrow rad^3(a) - rad^3(c) = h + l - b > 0 \tag{3.2.27}$$

as $rad(a) > rad^{\frac{1.63}{1.37}}(c)$. We obtain the equation:

$$rad^3(a) - rad^3(c) = h + l - b = m > 0 \tag{3.2.28}$$

Let $X = rad(a) - rad(c)$, then X is an integer root of the polynomial $H(X)$ defined as:

$$H(X) = X^3 + 3rad(ac)X - m = 0 \tag{3.2.29}$$

To resolve the above equation, we denote $X = u + v$, It follows that u^3, v^3 are the roots of the polynomial $G(t)$ given by:

$$G(t) = t^2 - mt - rad^3(ac) = 0 \tag{3.2.30}$$

The discriminant of $G(t)$ is $\Delta = m^2 + 4rad^3(ac) = \alpha^2, \quad \alpha > 0$. The two real roots of (3.2.30) are:

$$t_1 = u^3 = \frac{m+\alpha}{2}, \quad t_2 = v^3 = \frac{m-\alpha}{2} \tag{3.2.31}$$

As $m = rad^3(a) - rad^3(c) > 0$, we obtain that $\alpha = rad^3(a) + rad^3(c) > 0$, then from the expression of the discriminant Δ, it follows that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{3.2.32}$$

with $N = 4rad^3(ac) > 0$. From the expression of Δ above, we remark that α and m verify the following equations:

$$x + y = 2u^3 = 2rad^3(a) \tag{3.2.33}$$
$$x - y = -2v^3 = 2rad^3(c) \tag{3.2.34}$$
$$\text{then} \quad x^2 - y^2 = N = 4rad^3(a).rad^3(c) \tag{3.2.35}$$

As (α, m) is a couple of solutions of the Diophantine equation (3.2.32) and $\alpha > m$, then $\exists\, d, d'$ positive integers with $d > d'$ and $N = d.d'$ so that :

$$d + d' = 2\alpha \tag{3.2.36}$$
$$d - d' = 2m \tag{3.2.37}$$

III-3-2-1-3-2-2-1- Now, we consider for example, the case $d = 4rad^3(a)$ and $d' = rad^3(c) \Longrightarrow d > d'$.We rewrite the equations (3.2.36-3.2.37):

$$4rad^3(a) + rad^3(c) = 2(rad^3(a) + rad^3(c)) \Longrightarrow 2rad^3(a) = rad^3(c)) \tag{3.2.38}$$
$$4rad^3(a) - rad^3(c) = 2(rad^3(a) - rad^3(c)) \Longrightarrow 2rad^3(a) = -rad^3(c)) \tag{3.2.39}$$

Then the contradiction.

III-3-2-1-3-2-2-2- we consider the case $d = 4rad^3(c)rad^3(a)$ and $d' = 1 \Longrightarrow d > d'$.We rewrite the equations (3.2.36-3.2.37):

$$4rad^3(c)rad^3(a) + 1 = 2(rad^3(c) + rad^3(a)) \Longrightarrow 2rad^3(c) = 1 \tag{3.2.40}$$
$$4rad^3(c)rad^3(a) - 1 = 2(rad^3(c) - rad^3(a)) \Longrightarrow 2rad^3(c) = -1 \tag{3.2.41}$$

Then the contradiction.

III-3-2-1-3-2-2-3- Let c_1 be the first factor of $rad(c)$. we consider the case $d = 4c_1 rad^3(a)$ and $d' = \frac{rad^3(c)}{c_1} \Longrightarrow d > d'$. We rewrite the equation (3.2.36):

$$4c_1 rad^3(a) + \frac{rad^3(c)}{c_1} = 2(rad^3(a) + rad^3(c)) \Longrightarrow \tag{3.2.42}$$
$$2rad^3(a)(2c_1 - 1) = \frac{rad^3(c)}{c_1}(2c_1 - 1) \Longrightarrow 2rad^3(a) = rad^2(c).\frac{rad(c)}{c_1} \tag{3.2.43}$$

$c_1 = 2$ or not, there is a contradiction.

The others cases of the expressions of d and d' not coprime so that $N = d.d'$ give also contradictions.

Let $Q(N)$ be the number of the solutions of (3.2.32), as $N \equiv 0(\bmod 4)$, then $Q(N) = [\tau(N/4)/2]$. From the study of some cases above, we obtain that $Q(N) < [(\tau(N)/4)/2]$. It follows the contradiction.

Then the cases $\mu_a \leq rad^2(a)$ and $c > rad^{3.26}(c)$ are impossible.

III-3-2-2 We suppose that $rad^{1.63}(c) < \mu_c \leq rad^2(c)$ and $\mu_a > rad^{1.63}(a)$:

III-3-2-2-1- Case $rad(c) < rad(a)$: As $c \leq rad^3(c) = rad^{1.63}(c).rad^{1.37}(c) \Longrightarrow c < rad^{1.63}(c).rad^{1.37}(a) < rad^{1.63}(ac) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$.

III-3-2-2-2- Case $rad(a) < rad(c) < rad^{\frac{1.63}{1.37}}(a)$:
As $c \leq rad^3(c) \leq rad^{1.63}(c).rad^{1.37}(c) \Longrightarrow c < rad^{1.63}(c).rad^{1.63}(a) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$.

III-3-2-2-3- Case $rad^{\frac{1.63}{1.37}}(a) < rad(c)$:

III-3-2-2-3-1- We suppose $rad^{1.63}(a) < \mu_a \leq rad^{2.26}(a) \Longrightarrow a \leq rad^{1.63}(a).rad^{1.63}(a) \Longrightarrow a < rad^{1.63}(a).rad^{1.37}(c) \Longrightarrow c = a + b < 2a < 2rad^{1.63}(a).rad^{1.63}(c) < rad^{1.63}(abc) \Longrightarrow c < R^{1.63} \Longrightarrow \boxed{c < R^{1.63}}$.

III-3-2-2-3-2- We suppose $\mu_a > rad^{2.26}(a)$ and $\mu_c \leq rad^2(c)$. Using the same method as it was explicated in the paragraphs **III-3-2-1-3-2-** (permuting a, c), we arrive at a contradiction (see the appendix). It follows that the case $\mu_c = rad^2(c)$ and $\mu_a > rad^{2.26}(a)$ is impossible.

III-3-2-2-3-2-2- We suppose $a > rad^{3.26}(a)$ and large and $\mu_c < rad^2(c)$. Then $a = rad^3(a) + h, h > rad^3(a)$, h a positive integer and we can write $c + l = rad^3(c)$, $l > 0$. Then we obtain :

$$rad^3(c) - rad^3(a) = h + l + b > 0 \tag{3.2.44}$$

as $rad(c) > rad^{\frac{1.63}{1.37}}(a)$. Let $X = rad(c) - rad(a)$, then X is an integer root of the polynomial $H(X)$ defined as:

$$H(X) = X^3 + 3rad(ac)X - m = 0 \tag{3.2.45}$$

To resolve the above equation, we denote $X = u + v$, It follows that u^3, v^3 are the roots of the polynomial $G(t)$ given by:

$$G(t) = t^2 - mt - rad^3(ac) = 0 \tag{3.2.46}$$

The discriminant of $G(t)$ is $\Delta = m^2 + 4rad^3(ac) = \alpha^2, \quad \alpha > 0$. The two real roots of (3.2.46) are:

$$t_1 = u^3 = \frac{m + \alpha}{2}, \quad t_2 = v^3 = \frac{m - \alpha}{2} \tag{3.2.47}$$

As $m = rad^3(c) - rad^3(a) > 0$, we obtain that $\alpha = rad^3(a) + rad^3(c) > 0$, then from the expression of the discriminant Δ, it follows that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{3.2.48}$$

with $N = 4rad^3(ac) > 0$. It is the same case (permuting a and c) as the case above **III-3-2-1-3-2-2-** and we obtain contradictions.
Then the cases $\mu_c \leq rad^2(c)$ and $a > rad^{3.26}(a)$ are impossible.

III-3-3- Case $\mu_a > rad^{1.63}(a)$ and $\mu_c > rad^{1.63}(c)$: Taking into account the cases studied above, it remains to see the following two cases:

- $\mu_c > rad^2(c)$ and $\mu_a > rad^{1.63}(a)$,
- $\mu_a > rad^2(a)$ and $\mu_c > rad^{1.63}(c)$.

III-3-3-1- We suppose $\mu_c > rad^2(c)$ and $\mu_a > rad^{1.63}(a) \Longrightarrow c > rad^3(c)$ and $a > rad^{2.63}(a)$. We can write $c = rad^3(c) + h$ and $a = rad^3(a) + l$ with h a positive integer and $l \in \mathbb{Z}$.

III-3-3-1-1- We suppose $rad(c) < rad(a)$. We obtain the equation:

$$rad^3(a) - rad^3(c) = h - l - b = m > 0 \tag{3.2.49}$$

Let $X = rad(a) - rad(c)$, from the above equation, X is a real root of the polynomial:

$$H(X) = X^3 + 3rad(ac)X - m = 0 \tag{3.2.50}$$

As above, to resolve (3.2.50), we denote $X = u + v$, It follows that u^3, v^3 are the roots of the polynomial $G(t)$ given by :

$$G(t) = t^2 - mt - rad^3(ac) = 0 \tag{3.2.51}$$

The discriminant of $G(t)$ is:

$$\Delta = m^2 + 4rad^3(ac) = \alpha^2, \quad \alpha > 0 \tag{3.2.52}$$

The two real roots of (3.2.51) are:

$$t_1 = u^3 = \frac{m+\alpha}{2}, \quad t_2 = v^3 = \frac{m-\alpha}{2} \tag{3.2.53}$$

As $m = rad^3(a) - rad^3(c) > 0$, we obtain that $\alpha = rad^3(a) + rad^3(c) > 0$, then from the equation (3.2.52), it follows that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{3.2.54}$$

with $N = 4rad^3(ac) > 0$. From the equations (3.2.53), we remark that α and m verify the following equations:

$$x + y = 2u^3 = 2rad^3(a) \tag{3.2.55}$$

$$x - y = -2v^3 = 2rad^3(c) \tag{3.2.56}$$

$$\text{then} \quad x^2 - y^2 = N = 4rad^3(a).rad^3(c) \tag{3.2.57}$$

Let $Q(N)$ be the number of the solutions of (3.2.54) and $\tau(N)$ is the number of suitable factorization of N, and using the same method as in the paragraph **III-3-2-2-3-2-2-** above, we obtain a contradiction.

III-3-3-1-2- We suppose $rad(a) < rad(c)$. We obtain the equation:

$$rad^3(c) - rad^3(a) = b + l - h = m > 0 \tag{3.2.58}$$

Let X be the variable $X = rad(c) - rad(a)$, we use the similar calculations as in the paragraph above **III-3-3-1-1-**, we find a contradiction.

It follows that the case $\mu_c > rad^2(c)$ and $\mu_a > rad^{1.63}(a)$ is impossible.

III-3-3-2- We suppose $\mu_a > rad^2(a)$ and $\mu_c > rad^{1.63}(c)$, we obtain $a > rad^3(a)$ and $c > rad^{2.63}(c)$. We can write $a = rad^3(a) + h$ and $c = rad^3(c) + l$ with h a positive integer and $l \in \mathbb{Z}$.

The calculations are similar to those in the case **III-3-3-1-**. We obtain a contradiction.

It follows that the case $\mu_c > rad^{1.63}(c)$ and $\mu_a > rad^2(a)$ is impossible. □

We can state the following important theorem:

Theorem 3.2.2. *Let a, b, c positive integers relatively prime with $c = a + b$, then $c < rad^{1.63}(abc)$.*

From the theorem above, we can announce also:

Corollary 3.2.2.1. Let a, b, c positive integers relatively prime with $c = a + b$, then the conjecture $c < rad^2(abc)$ is true.

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Appendix

III-3-2-2-3-2- We suppose $\mu_a > rad^{2.26}(a)$ and $\mu_c \leq rad^2(c)$

III-3-2-2-3-2-1- We consider the case $\mu_c = rad^2(c) \Longrightarrow c = rad^3(c)$. Then, we obtain that $Y = rad(c)$ is a solution in positive integers of the equation:

$$Y^3 + 1 = a + b + 1 = c' \tag{3.2.59}$$

But it is the case $c' = 1 + c$.

III-3-2-2-3-2-1-1- We suppose that $c' = rad^n(c')$ with $n \geq 4$, we obtain the equation:

$$rad^n(c') - rad^3(c) = 1 \tag{3.2.60}$$

But the solutions of the equation (3.2.60) are [5] :$(rad(c') = 3, n = 2, rad(c) = +2)$, it follows the contradiction with $n \geq 4$ and the case $c' = rad^n(c'), n \geq 4$ is to reject.

III-3-2-2-3-2-1-2-In the following, we will study the cases $\mu_{c'} = A.rad^n(c')$ with $rad(c') \nmid A, n \geq 0$. The above equation (3.2.59) can be written as :

$$(Y + 1)(Y^2 - Y + 1) = c' \tag{3.2.61}$$

Let δ any divisor of c', then:

$$Y + 1 = \delta \tag{3.2.62}$$

$$Y^2 - Y + 1 = \frac{c'}{\delta} = c'' = \delta^2 - 3Y \tag{3.2.63}$$

We recall that $rad(c) > rad^{\frac{1.63}{1.37}}(a)$.

III-3-2-2-3-2-1-2-1- We suppose $\delta = l.rad(c')$. We have $\delta = l.rad(c') < c' = \mu'_c.rad(c') \Longrightarrow l < \mu'_c$. As δ is a divisor of c', then l is a divisor of μ'_c, we write $\mu'_c = l.m$. From $\mu'_c = l(\delta^2 - 3Y)$, we obtain:

$$m = l^2 rad^2(c') - 3rad(c) \Longrightarrow 3rad(c) = l^2 rad^2(c') - m$$

A- Case $3|m \Longrightarrow m = 3m'$, $m' > 1$: As $\mu'_c = ml = 3m'l \Longrightarrow 3|rad(c')$ and $(rad(c'), m')$ not coprime. We obtain:

$$rad(c) = l^2 rad(c').\frac{rad(c')}{3} - m'$$

It follows that c,c′ are not coprime, then the contradiction.

B - Case $m = 3 \Longrightarrow \mu'_c = 3l \Longrightarrow c' = 3lrad(c') = 3\delta = \delta(\delta^2 - 3Y) \Longrightarrow \delta^2 = 3(1+Y) = 3\delta \Longrightarrow \delta = lrad(c') = 3 \Rightarrow c' = 3\delta = 9 = c+1 \Rightarrow c = 8$, then it is a trivial case.

III-3-2-2-3-2-1-2-2- We suppose $\delta = l.rad^2(c'), l \geq 2$. If $n = 0$ then $\mu_{c'} = A$ and from the equation above (3.2.63):

$$c'' = \frac{c'}{\delta} = \frac{\mu_{c'}.rad(c')}{lrad^2(c')} = \frac{A.rad(c')}{lrad^2(c')} = \frac{A}{lrad(c')} \Rightarrow rad(c')|A$$

It follows the contradiction with the hypothesis above $rad(c') \nmid A$.

III-3-2-2-3-2-1-2-3- In the following, we suppose that $n > 0$.

If $lrad(c') \nmid \mu_{c'}$ then the case is to reject. We suppose $lrad(c')|\mu_{c'} \Longrightarrow \mu_{c'} = m.lrad(c')$, then $\dfrac{c'}{\delta} = m = \delta^2 - 3rad(c)$.

C′ - Case $m = 1 = c'/\delta \Longrightarrow \delta^2 - 3rad(c) = 1 \Longrightarrow (\delta-1)(\delta+1) = 3rad(c) = rad(c)(\delta+1) \Longrightarrow \delta = 2 = l.rad^2(c')$, then the contradiction.

D′ - Case $m = 3$, we obtain $3(1 + rad(c)) = \delta^2 = 3\delta \Longrightarrow \delta = 3 = lrad^2(c')$. Then the contradiction.

E′ - Case $m \neq 1, 3$, we obtain: $3rad(c) = l^2rad^4(c') - m \Longrightarrow rad(c)$ and $rad(c')$ are not coprime. Then the contradiction.

III-3-2-2-3-2-1-2-4- We suppose $\delta = l.rad^n(c'), l \geq 2$ with $n \geq 3$. From $c' = \mu_{c'}.rad(c') = lrad^n(c')(\delta^2 - 3rad(c))$, we denote $m = \delta^2 - 3rad(c) = \delta^2 - 3Y$.

F′ - As seen above (paragraphs C′,D′), the cases $m = 1$ and $m = 3$ give contradictions, it follows the reject of these cases.

G′ - Case $m \neq 1, 3$. Let q be a prime that divides m, it follows $q|\mu'_c \Longrightarrow q = c'_{j'_0} \Longrightarrow c'_{j'_0}|\delta^2 \Longrightarrow c'_{j'_0}|3rad(c)$. Then $rad(c)$ and $rad(c')$ are not coprime. It follows the contradiction.

III-3-2-2-3-2-1-2-5- We suppose $\delta = \prod_{j \in J_1} c_j'^{\beta_j}$, $\beta_j \geq 1$ with at least one $j_0 \in J_1$ with $\beta_{j_0} \geq 2$, $rad(c') \nmid \delta$. We can write:

$$\delta = \mu_\delta.rad(\delta), \quad rad(c') = m.rad(\delta), \quad m > 1, \quad (m, \mu_\delta) = 1 \tag{3.2.64}$$

Then, we obtain:

$$c' = \mu_{c'}.rad(c') = \mu_{c'}.m.rad(\delta) = \delta(\delta^2 - 3Y) = \mu_\delta.rad(\delta)(\delta^2 - 3Y) \Longrightarrow$$
$$m.\mu_{c'} = \mu_\delta(\delta^2 - 3Y) \tag{3.2.65}$$

- We suppose $\mu_{c'} = \mu_\delta \Longrightarrow m = \delta^2 - 3Y = (\mu_{c'}.rad(\delta))^2 - 3Y$. As $\delta < \delta^2 - 3Y \Longrightarrow m > \delta \Longrightarrow rad(c') > m > \mu_{c'}.rad(\delta) > rad^3(c')$ because $\mu_{c'} > rad^{2.26}(c')$, it follows $rad(c') > rad^2(c')$. Then the contradiction.

- We suppose $\mu_{c'} < \mu_\delta$. As $rad(c) = \mu_\delta rad(\delta) - 1$, we obtain:

$$rad(c) > \mu_{c'}.rad(\delta) - 1 > 0 \Longrightarrow rad(cc') > c'.rad(\delta) - rad(c') > 0 \Longrightarrow$$
$$c' > rad(cc') > c'.rad(\delta) - rad(c') > 0 \Longrightarrow 1 > rad(\delta) - \frac{rad(c')}{c'} > 0, \quad rad(\delta) \geq 2$$
$$\Longrightarrow \text{The contradiction} \tag{3.2.66}$$

- We suppose $\mu_{c'} > \mu_\delta$. In this case, from the equation (3.2.65) and as $(m, \mu_\delta) = 1$, it follows we can write:

$$\mu_{c'} = \mu_1.\mu_2, \quad \mu_1, \mu_2 > 1 \tag{3.2.67}$$
$$c' = \mu_{c'}rad(c') = \mu_1.\mu_2.rad(\delta).m = \delta.(\delta^2 - 3Y) \tag{3.2.68}$$
$$\text{so that} \quad m.\mu_1 = \delta^2 - 3Y, \quad \mu_2 = \mu_\delta \Longrightarrow \delta = \mu_2.rad(\delta) \tag{3.2.69}$$

**1- We suppose $(\mu_1, \mu_2) \neq 1$, then $\exists c'_{j_0}$ so that $c'_{j_0}|\mu_1$ and $c'_{j_0}|\mu_2$. But $\mu_\delta = \mu_2 \Rightarrow c'^2_{j_0}|\delta$. From $3Y = \delta^2 - m\mu_1 \Longrightarrow c'_{j_0}|3Y \Longrightarrow c'_{j_0}|Y$ or $c'_{j_0} = 3$.

- If $c'_{j_0}|Y$, it follows the contradiction with $(c', c) = 1$.
- If $c'_{j_0} = 3$. We have $m\mu_1 = \delta^2 - 3Y = \delta^2 - 3(\delta - 1) \Longrightarrow \delta^2 - 3\delta + 3 - m.\mu_1 = 0$. As $3|\mu_1 \Longrightarrow \mu_1 = 3^k\mu'_1, 3 \nmid \mu'_1, k \geq 1$, we obtain:

$$\delta^2 - 3\delta + 3(1 - 3^{k-1}m\mu'_1) = 0 \tag{3.2.70}$$

**1-1- We consider the case $k > 1 \Longrightarrow 3 \nmid (1 - 3^{k-1}m\mu'_1)$. We apply Eisenstein criterion [6] to the polynomial $R(Z)$ given by:

$$R(Z) = Z^2 - 3Z + 3(1 - 3^{k-1}m\mu'_1) \tag{3.2.71}$$

then:

- $3 \nmid 1$, - $3|(-3)$,- $3|3(1 - 3^{k-1}m\mu'_1)$, and - $3^2 \nmid 3(1 - 3^{k-1}m\mu'_1)$.

It follows that the polynomial $R(Z)$ is irreducible in $\mathbb{Q}$, then, the contradiction with $R(\delta) = 0$.

**1-2- We consider the case $k = 1$, then $\mu_1 = 3\mu'_1$ and $(\mu'_1, 3) = 1$, we obtain:

$$\delta^2 - 3\delta + 3(1 - m\mu'_1) = 0 \tag{3.2.72}$$

* If $3 \nmid (1 - m.\mu'_1)$, we apply the same Eisenstein criterion to the polynomial $R'(Z)$ given by:

$$R'(Z) = Z^2 - 3Z + 3(1 - m\mu'_1)$$

and we find a contradiction with $R'(\delta) = 0$.

**1-2-2- We consider that $3|(1 - m.\mu'_1) \Longrightarrow m\mu'_1 - 1 = 3^i.h$, $i \geq 1$, $3 \nmid h, h \in \mathbb{N}^*$. δ is an integer root of the polynomial $R'(Z)$:

$$R'(Z) = Z^2 - 3Z + 3(1 - m\mu'_1) = 0 \Rightarrow \text{ the discriminant of } R'(Z) \text{ is :}$$
$$\Delta = 3^2 + 3^{i+1} \times 4.h \tag{3.2.73}$$

As the root δ is an integer, it follows that $\Delta = l^2 > 0$ with l a positive integer. We obtain:

$$\Delta = 3^2(1 + 3^{i-1} \times 4h) = l^2 \tag{3.2.74}$$
$$\Longrightarrow 1 + 3^{i-1} \times 4h = q^2 > 1, q \in \mathbb{N}^* \tag{3.2.75}$$

We can write the equation (3.2.72) as :

$$\delta(\delta - 3) = 3^{i+1}.h \Longrightarrow 3^3\mu'_1\frac{rad(\delta)}{3}.\left(\mu'_1 rad(\delta) - 1\right) = 3^{i+1}.h \Longrightarrow \tag{3.2.76}$$
$$\mu'_1\frac{rad(\delta)}{3}.\left(\mu'_1 rad(\delta) - 1\right) = h \tag{3.2.77}$$

We obtain $i = 2$ and $q^2 = 1 + 12h = 1 + 4\mu'_1 rad(\delta)(\mu'_1 rad(\delta) - 1)$. Then, q satisfies :

$$q^2 - 1 = 12h \Rightarrow \tfrac{(q-1)}{2}.\tfrac{(q+1)}{2} = 3h = (\mu'_1 rad(\delta) - 1).\mu'_1 rad(\delta) \Rightarrow \tag{3.2.78}$$
$$q - 1 = 2\mu'_1 rad(\delta) - 2 \tag{3.2.79}$$
$$q + 1 = 2\mu'_1 rad(\delta) \tag{3.2.80}$$

It follows that $(q = x, 1 = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{3.2.81}$$

with $N = 12h > 0$. Let $Q(N)$ be the number of the solutions of (3.2.81) and $\tau(N)$ is the number of suitable factorization of N, then we announce the following result concerning the solutions of the Diophantine equation (3.2.81) (see theorem 27.3 in [7]):

- If $N \equiv 2(\bmod 4)$, then $Q(N) = 0$.
- If $N \equiv 1$ or $N \equiv 3(\bmod 4)$, then $Q(N) = [\tau(N)/2]$.
- If $N \equiv 0(\bmod 4)$, then $Q(N) = [\tau(N/4)/2]$.

Let (α', m'), $\alpha', m' \in \mathbb{N}^*$ be another pair, solution of the equation (3.2.81), then $\alpha'^2 - m'^2 = x^2 - y^2 = N = 12h$, but $q = x$ and $1 = y$ satisfy the equation (3.2.80) given by $x + y = 2\mu_1' rad(\delta)$, it follows α', m' verify also $\alpha' + m' = 2\mu_1' rad(\delta)$, that gives $\alpha' - m' = 2(\mu_1' rad(\delta) - 1)$, then $\alpha' = x = q = 2\mu_1' rad(\delta)$ and $m' = y = 1$. So, we have given the proof of the uniqueness of the solutions of the equation (3.2.81) with the condition $x + y = 2\mu_1' rad(\delta)$. As $N = 12h \equiv 0(\bmod 4) \Longrightarrow Q(N) = [\tau(N/4)/2] = [\tau(3h)/2]$, the expression of $3h = \mu_1'.rad(\delta).\left(\mu_1' rad(\delta) - 1\right)$, then $Q(N) = [\tau(3h)/2] > 1$. But $Q(N) = 1$, then the contradiction and the case $3|(1 - m.\mu_1')$ is to reject.

** We suppose that $(\mu_1, \mu_2) = 1$.

From the equation $m\mu_1 = \delta^2 - 3X = \delta^2 - 3(\delta - 1)$, we obtain that δ is a root of the following polynomial :

$$R(Z) = Z^2 - 3Z + 3 - m.\mu_1 = 0 \tag{3.2.82}$$

The discriminant of $R(Z)$ is:

$$\Delta = 9 - 4(3 - m.\mu_1) = 4m.\mu_1 - 3 = q^2 \quad \text{with } q \in \mathbb{N}^* \quad \text{as } \delta \in \mathbb{N}^* \tag{3.2.83}$$

- We suppose that $2|m\mu_1 \Longrightarrow c'$ is even. Then $q^2 \equiv 5(\bmod 8)$, it gives a contradiction because a square is $\equiv 0, 1$ or $4(\bmod 8)$.

- We suppose c' an odd integer, then c is even. It follows $c = rad^3(c) \equiv 0(\bmod 8) \Longrightarrow c' \equiv 1(\bmod 8)$. As $c' = \delta^2 - 3Y.\delta$, we obtain $\delta^2 - 3Y.\delta \equiv 1(\bmod 8)$. If $\delta^2 \equiv 1(\bmod 8) \Longrightarrow -3Y.\delta \equiv 0(\bmod 8) \Longrightarrow 8|Y.\delta \Longrightarrow 4|\delta \Longrightarrow c'$ is even. Then, the contradiction. If $\delta^2 \equiv 4(\bmod 8) \Longrightarrow \delta \equiv 2(\bmod 8)$ or $\delta \equiv 6(\bmod 8)$. In the two cases, we obtain $2|\delta$. Then, the contradiction with c' an odd integer.

It follows that the case $\mu_a > rad^{2.26}(a)$ and $\mu_c = rad^2(c)$ is impossible.

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Chapter 4

Is The *abc* Conjecture True?

Abstract

In this paper, we consider the *abc* conjecture. As the conjecture $c < rad^2(abc)$ is true, then we give the proof of the *abc* conjecture for $\epsilon \geq 1$ and for the case $\epsilon \in]0,1[$, we consider that the *abc* conjecture is false, from the proof, we arrive in a contradiction.

Résumé

Dans cet article, nous considérons la conjecture *abc*. Comme la conjecture $c < rad^2(abc)$ est vraie, nous donnons la preuve que la conjecture *abc* est vraie pour $\epsilon \geq 1$ et pour les cas $\epsilon \in]0,1[$, supposant que la conjecture est fausse nous arrivons à une contradiction.

4.1 Introduction and notations

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{4.1.1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{4.1.2}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [4]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 4.1.1. (***abc* Conjecture**): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{4.1.3}$$

where K is a constant depending only of ϵ.

The idea to try to write a paper about this conjecture was born after the publication in September 2018, of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. The difficulty to find a proof of the *abc* conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$. So, I will give a simple proof that can be understood by undergraduate students.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [4]. A conjecture was proposed that $c < rad^2(abc)$ [3]. It is the key to resolve the *abc* conjecture. In my paper, as the conjecture $c < rad^2(abc)$ holds (chapter 3), I propose an elementary proof of the *abc* conjecture.

4.2 The Proof of the *abc* conjecture

Proof. We note $R = rad(abc)$ in the case $c = a + b$ or $R = rad(ac)$ in the case $c = a + 1$.

4.2.1 Case : $\epsilon \geq 1$

As $c < R^2$ is true, we have $\forall \epsilon \geq 1$:

$$c < R^2 \leq R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad with\ K(\epsilon) = e,\ \epsilon \geq 1 \tag{4.2.1}$$

Then the *abc* conjecture is true.

4.2.2 Case: $\epsilon < 1$

From the statement of the *abc* conjecture 4.1.1, we want to give a proof that $c < K(\epsilon)R^{1+\epsilon} \Longrightarrow LogK(\epsilon) + (1 + \epsilon)LogR - Logc > 0$.

For our proof, we proceed by contradiction of the abc conjecture. We suppose that the *abc* conjecture is false:

$$\begin{aligned} \exists \epsilon_0 \in]0,1[, \forall K(\epsilon) > 0, \quad \exists c_0 = a_0 + b_0; \quad a_0, b_0, c_0 \text{ coprime so that} \\ c_0 > K(\epsilon_0)R_0^{1+\epsilon_0} \text{ and } \forall \epsilon \in]0,1[,\ c_0 > K(\epsilon)R_0^{1+\epsilon} \end{aligned} \tag{4.2.2}$$

We choose the constant $K(\epsilon) = e^{\frac{1}{\epsilon^2}}$. Let :

$$Y_{c_0}(\epsilon) = \frac{1}{\epsilon^2} + (1 + \epsilon)LogR_0 - Logc_0, \epsilon \in]0,1[\tag{4.2.3}$$

From the above explications, if we will obtain $\forall \epsilon \in]0,1[, Y_{c_0}(\epsilon) > 0 \Longrightarrow c_0 < K(\epsilon)R_0^{1+\epsilon} \Longrightarrow c_0 < K(\epsilon_0)R_0^{1+\epsilon_0}$, then the contradiction with (4.2.2).

About the function Y_{c_0}, we have:

$$\begin{aligned} lim_{\epsilon \longrightarrow 1} Y_{c_0}(\epsilon) = 1 + Log(R_0^2/c_0) = \lambda > 0 \\ lim_{\epsilon \longrightarrow 0} Y_{c_0}(\epsilon) = +\infty \end{aligned}$$

The function $Y_{c_0}(\epsilon)$ has a derivative for $\forall \epsilon \in]0,1[$, we obtain:

$$Y'_{c_0}(\epsilon) = -\frac{2}{\epsilon^3} + LogR_0 = \frac{\epsilon^3 LogR_0 - 2}{\epsilon^3} \tag{4.2.4}$$

$Y'_{c_0}(\epsilon) = 0 \Longrightarrow \epsilon = \epsilon' = \sqrt[3]{\dfrac{2}{LogR_0}} \in]0,1[$ for $R_0 \geq 8$.

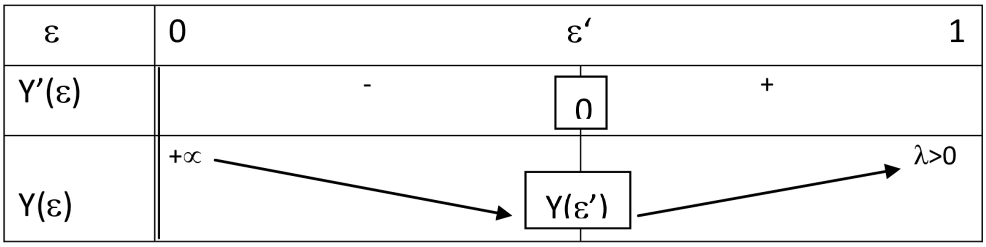

Figure 4.1: Table of variations

Discussion from the table (Fig.: 4.1):

- If $Y_{c_0}(\epsilon') \geq 0$, it follows that $\forall \epsilon \in]0,1[, Y_{c_0}(\epsilon) \geq 0$, then the contradiction with $Y_{c_0}(\epsilon_0) < 0 \Longrightarrow c_0 > K(\epsilon_0)R_0^{1+\epsilon_0}$ and the supposition that the *abc* conjecture is false can not hold. Hence the *abc* conjecture is true for $\epsilon \in]0,1[$.

- If $Y_{c_0}(\epsilon') < 0 \Longrightarrow \exists\ 0 < \epsilon_1 < \epsilon' < \epsilon_2 < 1$, so that $Y_{c_0}(\epsilon_1) = Y_{c_0}(\epsilon_2) = 0$. Then we obtain $c_0 = K(\epsilon_1)R_0^{1+\epsilon_1} = K(\epsilon_2)R_0^{1+\epsilon_2}$. We recall the following definition:

Definition 4.2.1. The number ξ is called algebraic number if there is at least one polynomial:

$$l(x) = l_0 + l_1 x + \cdots + a_m x^m, \quad a_m \neq 0 \tag{4.2.5}$$

with integral coefficients such that $l(\xi) = 0$, and it is called transcendental if no such polynomial exists.

We consider the equality :

$$c_0 = K(\epsilon_1)R_0^{1+\epsilon_1} \Longrightarrow \frac{c_0}{R_0} = \frac{\mu_{c_0}}{rad(a_0b_0)} = e^{\dfrac{1}{\epsilon_1^2}} R_0^{\epsilon_1} \tag{4.2.6}$$

i) - We suppose that $\epsilon_1 = \beta_1$ is an algebraic number then $\beta_0 = 1/\epsilon_1^2$ and $\alpha_1 = R_0$ are also algebraic numbers. We obtain:

$$\frac{c_0}{R_0} = \frac{\mu_{c_0}}{rad(a_0b_0)} = e^{\dfrac{1}{\epsilon_1^2}} R_0^{\epsilon_1} = e^{\beta_0}.\alpha_1^{\beta_1} \tag{4.2.7}$$

From the theorem (see theorem 3, page 196 in [1]):

Theorem 4.2.1. *$e^{\beta_0}\alpha_1^{\beta_1}\ldots\alpha_n^{\beta_n}$ is transcendental for any nonzero algebraic numbers $\alpha_1, \ldots, \alpha_n, \beta_0, \ldots, \beta_n$.*

we deduce that the right member $e^{\beta_0}.\alpha_1^{\beta_1}$ of (4.2.7) is transcendental, but the term $\dfrac{\mu_{c_0}}{rad(a_0b_0)}$ is an algebraic number, then the contradiction and the case $Y_{c_0}(\epsilon') < 0$ is impossible. It follows $Y_{c_0}(\epsilon') \geq 0$ then the *abc* conjecture is true.

ii) - We suppose that ϵ_1 is transcendental, then $1/(\epsilon_1^2), e^{1/(\epsilon_1^2)}$ and $R_0^{\epsilon_1} = e^{\epsilon_1 Log R_0}$ are also transcendental, we obtain that c_0/R_0 is transcendental, then the contradiction with c_0/R_0 an algebraic number. It follows that $Y_{c_0}(\epsilon') \geq 0$ and the *abc* conjecture is true.

Then the proof of the *abc* conjecture is finished. As $c < R^2$ is true, we obtain that $\forall\epsilon > 0$, $\exists K(\epsilon) > 0$, if $c = a + b$ with a, b, c positive integers relatively coprime, then :

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{4.2.8}$$

and the constant $K(\epsilon)$ depends only of ϵ.

Q.E.D

Ouf, end of the mystery!

□

4.3 Conclusion

As $c < R^2$ is true, we have given an elementary proof of the *abc* conjecture. We can announce the important theorem:

Theorem 4.3.1. *The abc conjecture is true:*
For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then:

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{4.3.1}$$

where K is a constant depending of ϵ.

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