Regarding Geometrization of MOND

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Abstract

One of the candidates for a resolution of the problem of dark matter is the Modified Newtonian Dynamics, which modifies the Newtonian gravity so as to fit the data. One of the key open problems of this theory which can have important empirical consequences is that of its geometrization. In this note I argue that this problem has a simple solution: metric tensor in MOND is not the gravitational potential *itself*.

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1 Introduction

One of the paramount attacks on MOND is that it cannot account for *gravitational lensing* in a satisfactory theoretical framework. According to the established understanding a theoretical investigation of the phenomenon of gravitational lensing requires geometrization of

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gravity so as to find the null geodesics (of light). All of the existing attempts of geometrizing MOND[1] defeat the purpose by introducing at least one extra unknown field. Such proposals are methodologically discontinuous, meaning that there is not a clear firm logical procedure for transition from MOND (as a modified theory of gravity) to geometrodynamics.

'MOND' can be understood in two different senses. The one originally proposed by Milgrom modifies Newton's Second Law, hence we shall call it *Modified Inertia*. MOND as modified inertia violates the principle of conservation of energy hence Bekenstein and Milgrom[2] formulated it as a modified theory of gravity.
MOND as modified inertia is less popular nowadays. Nevertheless we do not rule out the possibility and adopt a pragmatist approach: we consider all possibilities having in mind the aim of geometrization. The approach that is more promising will be pursued.

2 Regarding Kepler to Newton transition

Before attending to geometrization of MOND there is a question to consider: it is sometimes said that the current theory of MOND is similar to Kepler's laws before Newton 'explained' them. Therefore it is expected that a 'deeper' theory would account for MOND and that theory might as well geometrize it. This is especially supported by the numerical so-far-coincidence

$$a_0 \approx \frac{cH_0}{2\pi}, \tag{1}$$

meaning that a_0 is not really a constant. Recall that the constant in Kepler's Third law also was not really a universal constant

$$\frac{a^3}{T^2} \approx \frac{GM}{4\pi^2}.$$

It is therefore quite conceivable –and even expected– that a 'deeper' theory exists that would explain this 'coincidence', among other things.

How do we know that it is right to approach geometrization without regard to such expected theory? Theoretically as far as the current semi-Riemannian differential geometry is concerned no trace of such theory exists, as we shall see. Going beyond the current conventional 'pure metric' theory immediately faces arbitrariness from all directions, even when I will point to a possible geometric implementation of MOND it would still be hard to see such deeper theory has any role to play. Methodologically no clear procedure is yet at hand. A methodological semi-jump seems to be needed.

Thus I suspect that the question of geometrization of MOND, and its underlying theory are separate[1].

[1]There can of course be two levels of geometrization. The underlying theory itself might be possible to geometrize. I am talking about the 'current level'.

3 Modified Gravity or Modified Inertia?

3.1 Modified Inertia

MOND as Modified Inertia begins with the modification of Newton's Second Law

$$\mathbf{F} = m\mathbf{a}\mu(\|\mathbf{a}\|/a_0), \tag{2}$$

where μ is the *interpolating function*. To geometrize this theory, we try to find the geodesic equation from the above equation using coordinate transformation similar to what one does in the unmodified case; see. As the function μ is a scalar it will not change under coordinate transformations, thus it will not give us anything interesting. If we want to insist on this approach yielding something new and interesting, we must instead modify the momentum

$$p^\mu = \frac{dx^\mu}{d\tau} f(a). \tag{3}$$

After a coordinate transformation

$$p^{\mu'} = f \frac{dx^\nu}{d\tau} \frac{\partial X^\mu}{\partial x^\nu}, \tag{4}$$

therefore

$$\frac{d}{d\tau} p^{\mu'} = \frac{df}{da} \frac{da}{d\tau} \frac{dx^\nu}{d\tau} \frac{\partial X^\mu}{\partial x^\nu} + f \frac{d^2 x^\nu}{d\tau^2} \frac{\partial X^\mu}{\partial x^\nu} + f \frac{dx^\nu}{d\tau} \frac{dx^\alpha}{d\tau} \frac{\partial^2 X^\mu}{\partial x^\nu \partial x^\alpha},$$

yielding

$$\frac{d^2 x^\lambda}{d\tau^2} = -\frac{dx^\nu}{d\tau} \frac{dx^\alpha}{d\tau} \left(\frac{\partial^2 X^\mu}{\partial x^\nu \partial x^\alpha} \frac{\partial x^\lambda}{\partial X^\mu} \right) - \frac{dx^\lambda}{d\tau} \frac{d \log f}{da} j, \tag{5}$$

which is written using the Christoffel symbol

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\mu_{\alpha\nu} \frac{dx^\nu}{d\tau} \frac{dx^\alpha}{d\tau} + \frac{dx^\lambda}{d\tau} j \frac{d \log f}{da} = 0, \tag{6}$$

where j is the of the jerk vector. This whole approach is not at all aesthetically appealing to me.

3.2 Modified Gravity

Bekenstein-Milgrom theory (MOND as a modified theory of gravity) has the following non-linear Poisson equation

$$\nabla \cdot [f(\nabla\phi)\nabla\phi] = 4\pi G\rho \tag{7}$$

which results in

$$\mu\mathbf{g} = \mathbf{a}. \tag{8}$$

To make any progress in finding the relativistic generalization of this equation, according to the Spinoza Principle this equation must be brought under the form

$$\nabla^2 \varphi = 4\pi G\rho. \tag{9}$$

This is readily done if we define φ by

$$\nabla\varphi := f(\nabla\phi)\nabla\phi. \tag{10}$$

The fundamental solution ϕ of the corresponding Laplace equation for (7) is then given implicitly by

$$\boxed{\frac{GM}{r^2}\hat{\mathbf{r}} = f(\nabla\phi)\nabla\phi} \tag{11}$$

This equation ought to yield the ϕ that is supposed to determine the Schwarzschild metric. Recall that General Relativity *itself*, that is, GR *before* making contact with the Newtonian potential via the weak-field limit 'does not care' about the *explicit* form of the potential: the vacuum solution of the Einstein Field Equations for a spherically-symmetric body with mass M is simply

$$ds^2 = -\left(1 + \frac{2\phi(r)}{c^2}\right)c^2dt^2 + \left(1 + \frac{2\phi(r)}{c^2}\right)^{-1}dr^2 + r^2d\Omega^2, \tag{12}$$

and *that is all as far as GR is concerned.*
General Relativity leaves ϕ undetermined[2], *so there is no restriction from GR's side on ϕ as far as it is a radial function. It can well be the fundamental solution of the MONDian Laplace equation instead.* The tendency from acquaintance to think of

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1}dr^2 + r^2d\Omega^2.$$

as *the* solution of Einstein Field Equations is the root of all misunderstandings about geometrization of MOND.

With the gravitational potential ϕ given by (11) and the metric (12), the geometrization of MOND is done. In fact 'geometrization' is indeed too heavy a word to use for what needs to be done. There is little that is needed to be done: only the passage to Newtonian gravity (which is now MONDian gravity instead) is changed.
To see this, compare (8) with the geodesic equation for $\gamma \approx 1$

$$\Gamma^{\mu}_{\hat{t}\hat{t}} = -\frac{1}{2}g^{\mu\lambda}\partial_\lambda g_{\hat{t}\hat{t}} = f(\|\nabla\phi\|)g^{\mu} = f(\|\nabla\phi\|)\partial^{\mu}\phi, \tag{13}$$

therefore instead of the usual identification of the $\hat{t}\hat{t}$-component of metric (perturbation) with the Newtonian gravitational potential, we must have

$$f(\|\nabla\phi\|)\partial_i\phi = \frac{c^2}{2}\frac{\partial h_{00}}{\partial x^i}, \tag{14}$$

meaning that instead of the usual correspondence of standard GR

$$h_{00}^{\mathrm{GR}} = \frac{2}{c^2}\int \partial_i\phi \; dx^i = \frac{2}{c^2}\phi + C, \tag{15}$$

[2]Just look at any textbook of GR.

for MOND we have

$$h_{00}^{\mathrm{MOND}} = \frac{2}{c^2} \int f \partial_i \phi \; dx^i. \tag{16}$$

People like however to insist on (15). Insisting on (15) results in a 'no-go theorem' which purports that *no purely metric-based, relativistic formulation of MOND (as a modified theory of gravity) whose energy functional is stable (in the sense of being quadratic in perturbations) can be consistent with the observed amount of gravitational lensing from galaxies*[3].
With (16) however, this theorem is simply not applicable and one can proceed to calculate the gravitational lensing. The reason for the no-go theorem boils down to the fact that with insisting on (15) MOND corrections to general relativity can be removed, in the weak field limit, by a conformal transformation

$$g_{\mu\nu}(x) \longrightarrow \Omega^2(x) g_{\mu\nu}(x).$$

With 16 however, we have the transformation

$$\partial_\lambda g_{\mu\nu}(x) \longrightarrow \mu\big(\partial_\alpha g_{\beta\gamma}\big)(x) \partial_\lambda g_{\mu\nu}(x) \tag{17}$$

instead, which is evidently *not a conformal transformation.*

It is not that I do not like to create a new geometry for MOND; I will show a path in a moment. It is simply the fact that reality does not care about our wishes[3]: if we want to test the most serious phenomenological objection to MOND, which is gravitational lensing, there is no other way which avoids arbitrariness: for gravitational lensing –its is said that– we need a metric and a purely metric theory simply cannot yield (13) *unless* the identification is (16) is made, *instead* of the usual assumption that in the weak-field limit the metric tensor corresponds *directly* to the MONDian gravitational potential *itself.*

Let us now investigate the path which insists on a new geometry. A simple-minded reading of (16) can be proposing a different connexion. The first and only idea comes to mind is

$$\tilde{\Gamma} = f \Gamma,$$

but connexion times a scalar has a different transformation rule hence there is no guarantee that the geodesic equation will remain the same. We thus observe that a new connexion is necessary but not sufficient. A serious problem now is that we are immediately facing arbitrariness: as there is no clear method on how to proceed here, everything we do would be trial and error. We need a reliable systematic procedure (method) to find a connexion that allows for (16). Such a procedure is very hard to see[4].
The most appealing possibility is that we look for the geometry that is underlying the following modified geodesic equation,

$$\boxed{\ddot{x}^\mu + f(\ddot{x}^\nu) \Gamma^\mu_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma = 0} \tag{18}$$

No existing theory of differential geometry however can yield this equation[5].

A study is under progress to calculate the gravitational lensing effect of MOND and its comparison to GR and the experimental data.

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[3]I have written more about this 'slavish lust' here.

[4]See one of my failed attempts here.

[5]As an amusement I occasionally think on developing such a geometry, but apart from difficulty, for clear reasons presented I do not think this geometry is needed to begin with.

References

[1] F. Benoît and Stacy S. McGaugh. Modified Newtonian Dynamics (MOND): Observational Phenomenology and Relativistic Extensions. *Living Reviews in Relativity*, 15(1), 2012.

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[3] M.E. Soussa and R.P. Woodard. A generic problem with purely metric formulations of MOND. *Physics Letters B*, 578:253–258, 2004.