ζ-Padé SRWS theory with high dimensional approximation

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Abstract

In my previous paper about Statistical Random Walk Summation(SRWS) theory[1], I proposed a new expansion of typical critical Green function for the Anderson transition in the Orthogonal class. In this paper, I perform an approximate summation for the series of the typical critical Green function. Padé approximant is used to take a summation. The perturbation series of the critical exponent ν of localization length from upper critical dimension is obtained. The dimensional dependence of the critical exponent is directly related with Riemann ζ function. Thus, the number theory and the critical phenomena of the Anderson transition is connected. Therefore I call this method as ζ-Padé SRWS theory. Existence of lower critical dimension is understood as the infinite existence of prime numbers. Besides it, analogy with statistical mechanics also becomes clear.

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1 Introduction

The Anderson transition is a disorder driven quantum phase transition exhibiting critical phenomena. The critical exponent is thought to depend only on fundamental properties of the system such as dimensionality and symmetry. In previous paper, I propose the new theoretical framework to understand the Anderson transition in the Orthogonal class. The key points is power series expansion of the typical critical Green function and its approximate summation method. In this paper, I use rough approximation for power series and take a summation directly. Then, typical critical Green function is obtained closed form. We can extract the critical exponent ν. The explicit expression of the dimensional dependence of ν tells us the number theory is deeply related with the critical phenomena. Analogy with statistical mechanics is also obtained in same discussion.

2 Anderson transition and SRWS Green function

In this section, I review starting equation used later in this paper. The relevant dimensionality is explicitly given by a spectral dimension of a lattice in SRWS.

The typical critical Green function of SRWS theory is given by,

$$G_{\text{typical}} \simeq z^{|x-y|+1}\sum_{n=0}^{\infty} c_n z^n \tag{1}$$

$$c_n = \sum_{t=|x-y|}^{|x-y|+n} A_{xy}(t)\left(\frac{2}{W}\right)^{t+1}\binom{n+|x-y|}{t} \tag{2}$$

$$A_x y(t) = \begin{cases} k^t\left(\frac{t^2}{l}\right)^{-dt/2l} & (t<l) \\ k^t t^{-d/2} \quad (t \geq l) \end{cases} \tag{3}$$

$$l = \frac{|x-y|^2}{4D} \tag{4}$$

3 the critical exponent obtained from SRWS with approximations

The treatment of binomial coefficient is difficult. So, I roughly average the summation in c_n by using following formula.

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n \tag{5}$$

Besides it, I approximate by using only $t \geq l$ terms. Therefore, I approximate c_n as

$$\begin{aligned} c_n &= \frac{1}{W} 2^{|x-y|+n+1} \sum_{t=|x-y|}^{|x-y|+n} t^{-d/2} \left(\frac{2k}{W}\right)^t \\ &= -\frac{1}{W} 4^{|x-y|+n+1} \left(\frac{k}{W}\right)^{|x-y|+n+1} \Phi(2k/W, d/2, |x-y|+n+1) \end{aligned} \tag{6}$$

Here, function Φ is Lerch transcendent. By taking $d \to \infty$ and $W \to \infty$ limit,

$$\begin{aligned} G_{\text{typical}} &\simeq \frac{1}{W} \sum_{n=0}^{\infty} \left(\frac{4zk}{W}\right)^{|x-y|+n+1} \Phi(2k/W, d/2, n+|x-y|+1) \\ &\simeq -\frac{1}{W} \sum_{n=0}^{\infty} \left(\frac{4zk}{W}\right)^{|x-y|+n+1} (n+|x-y|+1)^{-d/2} \\ &= \frac{1}{W} \left(\frac{4zk}{W}\right)^{|x-y|+1} \Phi(4zk/W, d/2, |x-y|+1) \end{aligned} \tag{7}$$

However final summation is wrong limit, Padé approximant $[N/(N+|x-y)]$ limit should be taken to obtain $O(z)z \to \infty$ behaviour.

Let $z = \omega/1-\omega$, and take $\omega \to 1$ limit,

$$\begin{aligned} G_{\text{typical}} &\simeq \frac{1}{W} \left(\frac{4zk}{W}\right) \sum_{n=0}^{\infty} \left(\frac{4k}{W}(\omega + \omega^2 + \cdots)\right)^{|x-y|+n} (n+|x-y|+1)^{-d/2} \\ &\simeq \frac{1}{W} \left(\frac{4zk}{W}\right) \sum_{n=0}^{\infty} \left(\frac{4k}{W}(\omega + \omega^2 + \cdots + \omega^{C+1})\right)^{|x-y|+n} (n+|x-y|+1)^{-d/2} \\ &\simeq \frac{1}{W} \left(\frac{4zk}{W}\right) \sum_{n=0}^{\infty} \left(\frac{4k}{W}(1 + \omega + \cdots + \omega^{C})\right)^{|x-y|+n} (n+|x-y|+1)^{-d/2} \\ &\simeq \frac{1}{W} \left(\frac{4zk}{W}\right) \sum_{n=0}^{\infty} \left(\frac{4Ck}{W}\right)^{|x-y|+n} (n+|x-y|+1)^{-d/2} \\ &\simeq \frac{z}{W} \sum_{n=0}^{\infty} \left(\frac{4Ck}{W}\right)^{|x-y|+n} (n+|x-y|+1)^{-d/2} \\ &\simeq \frac{z}{W} \left(\frac{4Ck}{W}\right)^{|x-y|+1} \Phi(a, d/2, |x-y|+1) \end{aligned} \tag{8}$$

Here,

$$a = 4Ck/W \tag{9}$$

We expect in $d \to \infty$ limit,

$$G_{\text{typical}} \propto \exp(-distance A a^{\nu}) \tag{10}$$

$$distance = \frac{|x-y|^2}{4D} \tag{11}$$

Therefore,localization length ξ becomes

$$\xi = Aa^{\nu} = -\frac{\ln(G_{\text{typical}})}{dist} \propto A(1 + \nu(a-1)) \tag{12}$$

I expand a^{ν} around $a = 1$ because this expansion point gives correct lower critical dimension. This fact becomes clear later. By taking $|x-y| \to \infty$ limit,

$$\begin{aligned} \xi &= -\frac{\ln(G_{\text{typical}})}{dist} \\ &\simeq -\frac{1}{distance} \ln \Phi(a, d/2, |x-y|+1) \end{aligned} \tag{13}$$

This expression is similar to the relationship between Helmholtz free energy and partition function. Analogy is between $distance, \Phi(a, d/2, |x-y|+1)$ and inverse temperature,partition function. Especially, the fact that analogy with partition function becomes Lerch transcendent is surprising result. This suggests, each kind of ζ function may be related with the certain critical phenomena.

Then,using following formula,

$$(n + |x-y|)^{-d/2} = \sum_{k=0}^{\infty} \binom{-d/2}{k} a^k n^{-d/2} - k \tag{14}$$

We obtain,

$$\begin{aligned} \xi &\simeq -\frac{1}{distance} \ln\{(1 + 2^{-d/2}a + 3^{-d/2}a^2 + \cdots) - d/2(1 + 2^{-d/2-1}a + \cdots)\} \\ &\simeq -\frac{1}{distance} \ln\left(\frac{\mathrm{Li}_{d/2}(a)}{a} - \frac{d}{2}\frac{\mathrm{Li}_{d/2+1}(a)}{a}|x-y| + \frac{d}{2}\frac{d+2}{2}\frac{\mathrm{Li}_{d/2+2}(a)}{a}|x-y|^2 - \cdots\right) \end{aligned} \tag{15}$$

Take a limit $a \to 0$ only for first term such that $\frac{\mathrm{Li}_{d/2}(a)}{a} \sim 1$, we obtain,

$$\xi = \frac{1}{distance} \ln\left(1 - \frac{d}{2}\frac{\mathrm{Li}_{d/2+1}(a)}{a}|x-y| + \frac{d}{2}\frac{d+2}{2}\frac{\mathrm{Li}_{d/2+2}(a)}{a}|x-y|^2 - \cdots\right)^{-1} \tag{16}$$

Here, calculate series at $|x-y| \to \infty$,

$$\left(1 - \frac{d}{2}\frac{\mathrm{Li}_{d/2+1}(a)}{a}|x-y| + \frac{d}{2}\frac{d+2}{2}\frac{\mathrm{Li}_{d/2+2}(a)}{a}|x-y|^2\right)^{-1} = 1 + \sum_{k=1}^{4} a_k/|x-y|^k \tag{17}$$

$$\begin{aligned} \ln\left(1 - \frac{d}{2}\frac{\mathrm{Li}_{d/2+1}(a)}{a}|x-y| + \frac{d}{2}\frac{d+2}{2}\frac{\mathrm{Li}_{d/2+2}(a)}{a}|x-y|^2\right)^{-1} &\simeq \sum_{k=1}^{4} a_k/|x-y|^k \\ &= [1/3](|x-y|^{-1}) \end{aligned} \tag{18}$$

Taking $|x-y| \to \infty$ limit,

$$\begin{aligned} \xi &= 4D\frac{\mathrm{Li}^4_{d/2+1}(a)d^4}{4a^3\mathrm{Li}_{d/2+2}(a)d(d/2+1)} \\ &\simeq 4D\frac{\mathrm{Li}_{d/2+1}(a)d^4}{4\mathrm{Li}_{d/2+2}(a)d(d/2+1)} \end{aligned} \tag{19}$$

Then, expanding around $a = 1$, we obtain,

$$Aa^{\nu} \simeq 4Dd^3\zeta(d/2+1)/(2(d+2)\zeta(d/2+2))+d^3(a-1)(\zeta(d/2+2)\zeta(d/2)-\zeta(d/2+1)^2)/(2(d+2)\zeta(d/2+2)^2) \tag{20}$$

Here, at the limit $W \to \infty$,

$$D \propto W^{-1/2} \propto a^{1/2} \tag{21}$$

Therefore, we obtain,

$$\nu \simeq \frac{1}{2} + \frac{\zeta(d/2)\zeta(d/2+2) - \zeta(d/2+1)^2}{\zeta(d/2+1)\zeta(d/2+2)} \tag{22}$$

ν diverges at $d = 2$ because $\zeta(d/2)$ diverges. In other words, the critical exponent diverges at the lower critical dimension because infinitely many prime numbers exist.

4 The estimated value of the critical exponent ν

Expansion series from lower critical dimension is obtained by S.Hikami[2],

$$\nu \sim \frac{1}{d-2} - \frac{9\zeta(3)}{4}(d-2)^2 + \frac{27}{16}\zeta(4)(d-2)^3 \tag{23}$$

For integer dimensions, the estimated value of the critical exponents are listed in Table.1

d	$Eq.$(22)	$Eq.$(23)	numerical estimate
3	1.257	0.122	1.571 ± .004[3]
4	0.758	4.293	1.156 ± .014[4]
5	0.622	25.3	0.969 ± .015[4]
6	0.567	73.87	0.78 ± .06[5]

Table 1: Estimated critical exponents for the orthogonal symmetry class for $d = 3, 4, 5$ and 6 obtained from Eq.(22).

For non-integer dimensions, the estimated value of the critical exponents are listed in Table.2

Within perturbation method, these estimated values are best values without summation method.

5 Conclusion

I take an approximate summation of typical critical Green function in SRWS theory for the Anderson transition in the orthogonal symmetry class. Obtained perturbation series from upper critical dimension is better than previous study which is expansion from lower critical dimension. I confirmed that quite rough approximation gives not too bad estimate of the critical exponent ν. Therefore, we may expect that if correct summation

d	$Eq.$(22)	$Eq.$(23)	Refs.[6, 7, 8]
2.22	5.418	4.434	4.402 ± .18
2.226	5.273	4.308	2.82 ± .05
2.32	3.716	2.908	2.59 ± .19
2.33	3.603	2.801	2.92 ± .14
2.365	3.258	2.468	2.27 ± .06
2.41	2.902	2.11	2.50 ± .21
2.54	2.215	1.351	2.24 ± .31

Table 2: Same as for Table 1 but for fractals with spectral dimension $2 < d < 3$. Values of critical exponents in Ref. [6] were provided by M. Schreiber.

method is given we might get exact value of the critical exponent ν. The next step is to build an analytical theory of Anderson transition based on this new theoretical framework. Another way to further study is treating several kinds of ζ function as a partition function and study the critical phenomena of them.

References

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