Quantity Theory
and
The Continuum Hypothesis

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Abstract

How far away is the nearest galaxy to the Milky Way?
How long is the distance from Boston to New York?
How deep is the Marianas Trench?
How high is the tallest building?
How many marbles are in that bag?

The above questions all require measuring quantities to arrive at an answer. Measuring the number of light-years from the Milky Way to the other galaxies in the local group will reveal which is closest. Measuring the number of miles from Boston to New York will reveal the distance between them. The Marianas Trench will contain a measurable number of feet that will be its depth. The tallest building will contain a measurable number of feet that will be its height. Counting the marbles will yield the quantity of marbles contained in the bag.

While the units of measure of quantities and methods of obtaining measurements vary, all measurable quantities have one thing in common; they can all be expressed as real numbers.

Introduction

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The above questions all require measuring quantities to arrive at an answer. Measuring the number of light-years from the Milky Way to the other galaxies in the local group will reveal which is closest. Measuring the number of miles from Boston to New York will reveal the distance between them. The Marianas Trench will contain a measurable number of feet that will be its depth. The tallest building will contain a measurable number of feet that will be its height. Counting the marbles will yield the quantity of marbles contained in the bag.

The quantities considered above are of finite magnitude. They can all be calculated in a finite number of steps conducted over a finite interval of time. We call the process of calculating a quantity, measuring. While the units of measure of quantities and methods of obtaining measurements vary, all measurable quantities have one thing in common; they can all be expressed as real numbers.

Quantity and Infinity, Measuring, Counting

Quantity

Definition 1 - A **quantity**, **q**, is a discrete magnitude.

How many… how long/short… how high/low… what is the result of performing addition/subtraction/multiplication/division… are all questions that can have quantities for answers. Quantities are derived from measurements. Length, weight, number of, height and duration are all examples of measurable quantities. While the methods of measuring quantities and the units of measure of quantities can differ, the result of conducting the measurement is always a discrete magnitude that is expressed as a real number.

Axiom 1 - A quantity **q**, is expressed as a real number.

Definition 2 - A **quantitative descriptor** is a non-numerical word or phrase used to characterize a quantity.

That crowd is large/small… that person is tall/short… the snow is deep… are examples of quantitative descriptors. When used in reference to quantities, quantitative descriptors are based on estimates derived from casual observations. They describe a quantity by comparing it to a generally accepted standard of what is considered a normal quantity of what is being observed. For instance, observing a 6'-6" professional basketball player walking down a city street, most people would describe her as tall. However, that same player might be described as just average when observed among a group of other players.

To convert a quantitative descriptor to a quantity it is necessary to successfully measure, in some way, the object(s) being observed in order to calculate a quantity.

Infinity

Definition 3 – Infinity, ∞, is unbound, limitless extension.

Infinity embodies the notion that some things go on indefinitely, without a foreseeable end. Infinite things have no boundary, can't be "gathered in" and held in isolation. There's no surrounding the infinite. Infinity cannot be the answer to questions such as "how many?", "how high?", "how long?"... Those questions require the calculation of a quantity. Infinity cannot be calculated because,

Axiom 2 - ∞ is not a real number.

Theorem 1 - ∞ is not a quantity.

Proof - By Axiom 1, a quantity is expressed as a real number and by Axiom 2, ∞ is not a real number. It follows that ∞ is not a quantity and this proves Theorem 1. □

Infinity is a quantitative descriptor. When saying that there are infinitely many natural numbers, we are describing the quantity of natural numbers, not assigning the quantity a value derived from measuring the number of natural numbers.

Theorem 1 lends credence what is intuitively obvious; that numbers which can't be quantified cannot be used to determine quantities. Consider, every use of $\boldsymbol{\pi}$ in an equation is the employment of a placeholder, $\boldsymbol{\pi}$, for an approximation of the value of the ratio of a circle's circumference to its diameter. $\boldsymbol{\pi}$, being of incalculable length can't be measured and therefore has no quantifiable value. The same holds true for all numbers that can't be quantified. When the calculation is performed, the symbol is replaced with an approximation of the value the symbol represents.

Definition 4 - A **numerical quantitative descriptor** is a number, the value of which can only be approximated.

$$\pi, e, \sqrt{2}$$

are all examples of numerical quantitative descriptors. Before calculations using these numbers can be completed, approximations of their values must be substituted for the symbols representing the numbers in the equations. No matter to what degree the value of the numerical quantitative descriptor is calculated, the actual number used in the calculation will not be the numerical quantitative descriptor.

Corollary 1 - Numerical quantitative descriptors cannot be used to determine quantities.

Proof - Definition 1 states that a quantity is a discrete magnitude. Determining a quantity involves calculating a discrete finite value. Since numerical quantitative descriptors have no quantifiable value, they cannot produce a finite value if used in a calculation. Therefore, numerical quantitative descriptors cannot be used to calculate quantities, and this completes the proof. □

Measuring

Definition 5 - **Measuring** is a process that yields a quantity after a finite number of steps are performed in a finite amount of time.

If M is a method of measuring, t_i and t_j an interval of time that the steps, s_m thru s_n of measuring are performed and x the object being measured then Definition 5 is expressed as:

$$M_{s_m}^{s_n}(x)_{t_i}^{t_j}$$

Measuring requires time to carry out. Time is spent choosing a method of measurement, preparing to carry out the measurement, performing the measurement and recording the result. Once completed, the process yields a definite result (expressed as a real number) after a finite number of steps have been performed in a finite amount of time.

If **q** is the quantity sought by measuring x and the measurement yields a real number, then:

$$q(x) \leftarrow M_{s_m}^{s_n}(x)_{t_i}^{t_j}$$

which is read as, the quantity q(x) is a result of measuring x in s_n - s_m steps over a finite time interval t_j - t_i.

Example 1 – What is the distance D, between point A = (1, 1) and point B = (2, 1) along a traceable path P? Though the example is trivial it illustrates the application of Definition 5.

Given a starting point A, an end point B, a traceable path, P between the two, an interval of time, (t_i, t_j) and an appropriate method of measuring, M, that is performed in a finite number of steps, (s_m, s_n); letting P = x and D(A, B) = q(x) the formula indicating that the distance D, between A and B is the result of measuring P is given by:

$$\mathbf{D(A, B) \leftarrow M_{s_m}^{s_n}(P)_{t_i}^{t_j}}$$

In Example 1, since the path P, between the points A and B is traceable then there exists a method of measurement which yields the quantity D(A, B), the distance separating points A and B along the defined path, P.

The notations representing the steps taken and the time interval are there to indicate that both quantities must be finite. While the actual values are not relevant to the result, all four parameters must stand for actual finite values.

Example 2 - Consider the line Q described by the following function:

$$\boldsymbol{f(x) = 1 : x \geq 0}$$

Q begins at point (0,1) and extends horizontally one unit up from the x-axis indefinitely. What is the total length L, of Q? Can some process of measuring Q satisfy

$$\mathbf{M_{s_m}^{s_n}(Q)_{t_i}^{t_j}}$$

in a finite number of steps s_n - s_m over a finite interval of time $t_j – t_i$? If so, the measurement will yield a quantity that will equal L, of Q so that

$$\mathbf{L(Q) \leftarrow M_{s_m}^{s_n}(Q)_{t_i}^{t_j}}$$

L cannot be infinity since L has to be expressed as a quantity and infinity, by Theorem 1, is not a quantity. Q has a starting point A at (0, 1) but no end point B. While it's possible to specify t_i, t_j cannot be specified since there's no endpoint on Q. So, while Q can be measured anywhere along its length, the overall length of Q cannot be determined. Every possible measurement of Q will yield a result that is a line segment of Q that is of finite length. At no point will Q transition from finite to infinite by any measurement.

The conclusion follows that the length of Q is incalculable. No process of measuring as defined in Definition 5 can be performed that will yield the length of Q expressed as a real number. Therefore,

$$L(Q) \not\leftarrow M_{s_m}^{s_n}(Q)_{t_i}^{t_j}$$

Counting

Definition 6 - **Counting** is a form of measuring.

To measure the length of a line segment of Q, count the number of units of measurement there are from the starting point to the end point along the path traced out by Q. To measure the size of a herd of cattle, count the number of individual cows in the herd.

Consider the function:

$$\mathbf{f(x) = x + 1 : x = 0, 1, 2, 3, \ldots}$$

Theorem 2 – The range of $f(x)$ is not countable.

In order to prove Theorem 2, we will construct a list of the range of $f(x)$ one item at a time and show that the list so constructed cannot be a complete listing of the range of $f(x)$.

Proof - To measure the range of $f(x)$ count the number of objects in the range of $f(x)$ by listing all the values in the range of $f(x)$ one after another. To prove Theorem 2, it is necessary to show that such a list cannot be constructed. Using the diagonal method, a number, Z, will be constructed that will never appear in the list of the range of $f(x)$. That is to say, any measure of the range of $f(x)$ will not contain the number Z or,

$$Z \not\leftarrow M_{s_m}^{s_n}(f(x))_{t_i}^{t_j}$$

Below is a demonstration employing the diagonal method showing how the range of $f(x)$ cannot be listed in its entirety.

$f(x)$

	1	4	9	6	7	5	2	3	4	6	0	4	8	9	6
		2	3	4	5	6	8	2	1	4	6				
			3	4	5	4	5	4	8	2	6				
				4	5	5	5	8	2						
					5										
		2	5	4	8	6	5	7							
					9	9	7	9	6	3	6	6	6	6	6
			2	2	2	2	2	8							
								7	9	3	5	8	9	6	1
										1	0	1	0	1	1
						2	5	8	6	4	5	1			
									5	5	5	1	2	9	8
											1	1	1	3	1
														1	4
								…							
Z	3	6	5	1	2	3	1	4	7	2	4	5	6	7	…

The above list of random numbers from $f(x)$ is constructed so that one digit of each number in every row falls along the diagonal indicated by the highlighted cells. The number Z is constructed by selecting a digit that is not equal to the digit in the highlighted cell in the same column. Every number in the list will differ from Z by at least one digit. No matter how many numbers are entered into the list, Z will never be one of them. Therefore, since Z will never appear in the list, the list will never be complete demonstrating that $f(x)$ is uncountable and this completes the proof. □

Corollary 2 – The length of Z is finite and countable for all possible measurements.

Proof - The length of Z is a quantity. By Axiom 2, infinity is not a quantity since infinity cannot be expressed as a real number. Therefore, the length of Z cannot be infinite. Furthermore, the length of Z measured in any number of steps, s_n - s_m over any time interval $t_j - t_i$ will always yield a quantity satisfying,

$$L(Z) \leftarrow M_{s_m}^{s_n}(Z)_{t_i}^{t_j}$$

Therefore, the length of Z is finite and countable, and this completes the proof □

Corollary 3 - At no point will the length of Z transition from finite to infinite by any measurement.

Proof - Corollary 3 is proved by induction as follows:

$$f(x) = x + 1 : x = 0, 1, 2, 3, \ldots$$

1 4 9 6 7 5 2 3 4 6 0 4 8 9 6

Z_1 3

$$Z_1 \nleftarrow M_{s_m}^{s_n}(f(x))_{t_i}^{t_j}$$

…

1 4 9 6 7 5 2 3 4 6 0 4 8 9 6

2 3 4 5 6 8 2 1 4 6

3 4 5 4 5 4 8 2 6

4 5 5 5 8 2

5

2 5 4 8 6 5 7

Z_n 3 6 5 1 2 3

$$Z_n \nleftarrow M_{s_m}^{s_n}(f(x))_{t_i}^{t_j}$$

1 4 9 6 7 5 2 3 4 6 0 4 8 9 6

2 3 4 5 6 8 2 1 4 6

3 4 5 4 5 4 8 2 6

4 5 5 5 8 2

5

2 5 4 8 6 5 7

9 9 7 9 6 3 6 6 6 6 6

Z_{n+1} 3 6 5 1 2 3 1

$$Z_{n+1} \nleftarrow M_{s_m}^{s_n}(f(x))_{t_i}^{t_j}$$

Z is of finite length for Z_1, Z_n, and Z_{n+1}. By induction we have demonstrated that any measurement of the length of Z will be finite, and this completes the proof. □

Quantity and Collection, Specification, Well-defined, Set

Quantity Recap

Thus far, we have established that:

1. A quantity is a discrete magnitude (Definition 1).
2. Quantities are expressed as real numbers (Axiom 1).
3. A quantitative descriptor characterizes a quantity (Definition 2).
4. Infinity is unbound, limitless extension (Definition 3).
5. Infinity is not a real number (Axiom 2).
6. Infinity is not a quantity (Theorem 1).
7. A numerical quantitative descriptor is a number, the value of which can only be approximated (Definition 4).
8. Numerical quantitative descriptors cannot be used to determine quantities (Corollary 1).
9. Measuring is a finite process (Definition 5).
10. Counting is a form of measuring (Definition 6).
11. The range of $f(x) = x + 1 : x = 0, 1, 2, 3, \ldots$ is not countable (Theorem 2).
12. The length of Z is finite and countable (Corollary 2).
13. The length of Z will be finite for all measurements of Q (Corollary 3).

Collection

Definition 7 - A **collection** is a grouping of tangible or intangible objects**.**

A collection can consist of any number of objects without regard to characteristics of the individual items in the collection. A box full of marbles of different sizes, colors and designs is a collection of marbles (tangible). A list of ideas without regard to subject, viewpoint and truth, or lack of same constitutes a collection of ideas (intangible).

Specification

Definition 8 - A **specification** is one or more rules for the construction of a collection.

Using the above examples, “Assemble a box full of marbles.” constitutes a specification for constructing a collection of marbles and “Come up with 10 ideas.” constitutes a specification for constructing a collection of ideas.

Well-defined

Definition 9 - A **well-defined** specification contains rules that will prevent ambiguities in the construction of a collection.

Assemble a box of marbles consisting of one red marble, one blue marble and one green marble constitutes a well-defined specification for assembling a collection of three marbles, one red, one blue and one green.

Assemble a list of ideas related to Cantor's Continuum Hypothesis is a well-defined specification for creating a collection of ideas related to Cantor's Continuum Hypothesis.

Theorem 3 - A collection created from a well-defined specification will contain only unique objects.

Proof - By Definition 7, a collection is a grouping of objects. Definition 9 states that a well-defined specification for a collection produces no ambiguities. Therefore, a collection created from a well-defined specification will contain only unique objects and this completes the proof. □

Set

Definition 10 - A **Set** is a well-defined collection.

Let $f(x) = x + 1 : x = 0, 1, 2, 3, \ldots$ be the specification defining the set of natural numbers, $\mathbb{N}$ such that $\mathbb{N}$ = the range of $f(x)$. Therefore:

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

Theorem 4 - $\mathbb{N}$ is uncountable.

Georg Cantor used the diagonal method to prove that the real numbers are uncountable by showing that a one-to-one correspondence cannot be established between the real numbers and the natural numbers. He assumed that the natural numbers are countable, that a complete list of $\mathbb{N}$ can be written down one element after another with no omissions. But,

Proof – Theorem 2 proved that the range of $f(x) = x + 1 : x = 0, 1, 2, 3, \ldots$ is uncountable. Since $\mathbb{N}$ is defined as the range of $f(x)$ in Definition 10 then $\mathbb{N}$ **= {1, 2, 3, …}** and by substitution conforms to the proof offered in Theorem 2 completing the proof of Theorem 4. □

Quantity and The Continuum Hypothesis

The Continuum Hypothesis

The Continuum Hypothesis is herein restated as a theorem.

Theorem 5 - There is no set of real numbers with a cardinality β, such that $\aleph_0 < \beta < \aleph_1$

Proof - Let

$$\mathbf{S = \{x : 0 < x < 1 \text{ and } x \text{ is binary}\}}$$

and

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

Using the arrangement rule from Theorem 2's proof, the diagram below shows how the numbers Z and Y are derived from the diagonals highlighted in each list. Notice that infinitely long strings of digits are not necessary in either list. It is evident that Z and Y won't appear in their respective lists no matter how long each list grows. Any measurements taken of both lists will always confirm that neither Z nor Y will be contained in the list.

$\mathbb{N}$ = **{1, 2, 3, …}**

1
2
3
4
5
6
7
8
9
10
…
… 6 1 5 2 8 7 6 9 5 3 Z

S = {x : 0 < x < 1 and x is binary}

. 1 0 0
. 1 0 1 1 1 1
. 1
. 0 1 0 0 0 1
. 1 0 1 0
. 1 0 1 0 0 0 0 1 1
. 0 1 0 1 0 0 0 0 1
. 1 1 1 1 0 1 0 1
. 0 0 1 0 1 0 0 1 1 0 1
. 0 1 0 1 1

Y . 0 1 0 0 1 0 1 0 1 0 1 …

For all measurements of $\mathbb{N}$ and S we have,

$$Z_n \nleftarrow M_{s_m}^{s_n}(\mathbb{N})_{t_i}^{t_j}$$

and

$$Y_n \nleftarrow M_{s_m}^{s_n}(S)_{t_i}^{t_j}$$

The values of Z and Y will vary depending on how far into the list we count the number of elements of $\mathbb{N}$ and S, but the fact remains that the current values of Z and Y will never appear in either list. Additionally, by Corollary 3, Z is guaranteed to be of finite length over any desired number of measurements of the list so there's no need to postulate infinite integers.

Since lists of both sets, $\mathbb{N}$ and S, produce numbers that will never appear in their respective lists, both sets must be of the same size, or cardinality. That means that $\aleph_0$ and $\aleph_1$ must be equal. Therefore, no subset of the continuum can have a cardinality $\aleph_0 < \beta < \aleph_1$ and this completes the proof. □