Spacetime as a Whole

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Abstract

There is no formal difference between particles and black holes. This formal similarity lies in the intersection of gravity and quantum theory; quantum gravity. Motivated by this similarity, 'wave-black hole duality' is proposed, which requires having a proper energy-momentum tensor of spacetime itself. Such a tensor is then found as a consequence of 'principle of minimum gravitational potential'; a principle that corrects the Schwarzschild metric and predicts extra periods in orbits of the planets. In search of the equation that governs changes of observables of spacetime, a novel Hamiltonian dynamics of a Pseudo-Riemannian manifold based on a vector Hamiltonian is adumbrated. The new Hamiltonian dynamics is then seen to be characterized by a new 'tensor bracket' which enables one to finally find the analogue of Heisenberg equation for a 'tensor observable' of spacetime.

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To approach quantum gravity a useful methodological guiding principle advocated by 't Hooft[1] is that '*If we try to describe something like black holes, their behavior should be understood in the same language as the one we use for other particles;* ***black holes should be treated just like atoms****, [...]*'[1]

Following this method, we would expect a quantum approach to gravity to begin by investigating the status of the first building block of quantum theory of particles, the de Broglie relation, in general relativity (hereafter *GR*). If for the purpose of quantizing gravity a black hole and a quantum particle should be treated formally the same, it would be conceivable to formally generalize $p_\mu = \hbar k_\mu$ to a 2-tensor[2], and postulate the *wave-black hole duality*

$$T_{\mu\nu} = \hbar K_{\mu\nu}, \tag{1}$$

in which $T_{\mu\nu}$ would be the energy-momentum tensor of the black hole, and $K_{\mu\nu}$ its 'wavetensor' with dimensions

$$[\mathbf{K}] = [K_{\mu\nu}] = \text{Length}^{-3} \times \text{Time}^{-1} = \text{Frequency} \times \text{Volume}^{-1} = \text{`Frequency density'}. \tag{2}$$

It is critical not to confuse $T_{\mu\nu}$ which oughts to represent energy of the black hole, i.e. energy-momentum tensor of Schwarzschild spacetime[3] *itself*, and the Dirac delta energy-momentum tensor that acts as the *source* for Schwarzschild solution [2].

We are consequently faced with the (controversial) question of the energy-momentum tensor of the spacetime which is the same as the energy-momentum tensor of the gravitational field according to the Equivalence Principle. As we make no use of any result from the literature on this question, we do not review the previous attempts and refer the interested reader to [3, 4, 5]. It suffices to say that there is still no consensus on a satisfactory *tensor* representing energy-momentum of spacetime *itself*. From Electromagnetism (and previous classical field theories) we know that the energy-momentum tensor of a field is constructed from the field strength squared, which is the 'derivative' of the potential squared. We also know that in GR the notion of potential is replaced by the metric tensor. Altogether therefore we would expect the field strength tensor of gravity to be constructed from

$$Q_{\rho\mu\nu} = \nabla_\rho g_{\mu\nu};$$

which is called *nonmetricity* tensor and has been studied in extensions of general relativity [6]. The obstacle that we now face is that as long as the covariant derivative is taken with respect to the Levi-Civita connection that arises from the *same* metric $g_{\mu\nu}$, this tensor is zero. Had this tensor not been zero, we could readily construct from it a Lagrangian density and consequently energy-momentum tensor of gravitational field. Things would then probably be similar to electromagnetism and there would be a good chance we could quantize the free field Lagrangian, in a manner similar to the way one does for electromagnetism [7, 8].

If we insist on realizing this expectation we need to use *a metric $\tilde{g}_{\mu\nu}$ different from the one that is the solution of the Einstein Field Equations.* To determine this new metric, note

[1] Bold letters by me.

[2] No spatial index is used in this essay. Metric signature $(-,+,+,+)$ is assumed.

[3] As Schwarzschild metric is the prototype of solutions of Einstein Field Equations and black holes, in this essay we take synonymous 'spacetime' and Schwarzschild solution.

that $\tilde{g}_{\mu\nu}$ 'plays the role' of the electromagnetic four-potential, therefore applying Hamilton principle to the action

$$S_J = \frac{1}{2}\int J_{\alpha\beta\gamma}J^{\alpha\beta\gamma}\sqrt{-g}\ d^4x, \tag{3}$$

where

$$J_{\mu\nu\rho} := \frac{1}{2}(\tilde{Q}_{\mu\nu\rho} - \tilde{Q}_{\nu\mu\rho} + \tilde{Q}_{\rho\mu\nu}) = \frac{1}{2}(\nabla_\mu\tilde{g}_{\nu\rho} - \nabla_\nu\tilde{g}_{\mu\rho} + \nabla_\rho\tilde{g}_{\mu\nu}), \tag{4}$$

in which the covariant derivative is the same as in standard GR (Levi-Civita connection arising from the solution of the Einstein Field Equations), determines the new metric[4]. *In principle* therefore we have $\tilde{g}_{\mu\nu}$ determined, which in turn gives the energy-momentum tensor of spacetime via

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta S_J}{\delta g^{\mu\nu}}.$$

Application of Hamilton principle to (3) and a gauge fixing yields

$$\nabla^\mu\nabla_\mu\tilde{g}_{\rho\sigma} = 0, \tag{5}$$

in vacuum. In practice, however, solving such equations is complicated and has been studied by Hadamard[9], DeWitt and Brehme[10], and Friedlander[11]. As we expect, the solutions turn out to be expressed in terms of tensorial generalizations of Green functions (*bi-tensors*). Without attending to solve (5), three key implications can be seen:

- *(5) is linear* in $\tilde{g}_{\mu\nu}$. Owing to linearity of covariant derivatives, (our minimal extension of) *general relativity is linear in a special sense.* This shows that although GR is nonlinear, in a formally 'higher level', it is linear and the linearity may well be employed for quantizing it.

- Unlike the case with the Hilbert action[12], for (3) *the corresponding Euclidean action is bounded below.*

- From the perspective of (3) being a ground for quantization of 'free field gravity', the problem of non-renormalizability of Hilbert action[13, 14] evaporates.

These implications confirm our initial vision that this procedure would be apt to quantizing GR. In view of (3) it is feasible that Hilbert action is *but a means* to the goal of quantization of GR, not its foundation; yet it is indispensable, as the covariant derivative in (5) requires a given metric, determined independently.

While finding a solution to the procedure outlined above[5], poses a momentary practical barrier, we now show that by embarking on a route to put the procedure on a firm *physical* basis, this barrier can be circumvented, facilitating a solution for the case of Schwarzschild spacetime.

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[4]For brevity and without loss of generality, our discourse is limited to vacuum (without source terms).

[5]By 'the procedure' hereafter we mean the one sketched in the beginning of essay; the one that starts by the action (3), and is supposed to yield the energy-momentum tensor of spacetime (itself).

Having in mind that the essence of the above procedure is *having two distinct metrics*[6], let us begin with the historical-logical precursor of GR, where Einstein[7] began: application of special relativity (hereafter *SR*) to Newtonian gravity. For now we adopt a completely classical and Newtonian perspective towards gravity, and consider a massive particle in a gravitational potential ϕ. Conservation of energy requires

Energy of the particle = Energy of the field lost to the particle,

viz.

$$\frac{1}{2}m\|\mathbf{v}\|^2 + m\phi = -m\phi, \tag{6}$$

hence

$$\|\mathbf{v}\|^2 \equiv -4\phi, \tag{7}$$

in which the (weak) equivalence principle is assumed[8]. To apply the second principle of SR[9], from (7) we have

$$\frac{1}{4}\max\|\mathbf{v}\|^2 = -\min\phi,$$

hence

$$\min\phi = -\frac{c^2}{4}.$$

This is a fundamental necessity that GR must locally respect, along with the principles of SR. This *principle of minimum gravitational potential* (hereafter *MGP*) will turn out to be the firm physical ground we were seeking. To see the utility and importance of this principle, apply (7) to the γ factor (of SR) to get[10]

$$\Phi := \frac{1}{\sqrt{1+\frac{4\phi}{c^2}}}; \tag{8}$$

applying MGP now to Newtonian gravity, results in

$$U = -\frac{mc^2}{2}\frac{1}{\sqrt{1+\frac{4\phi}{c^2}}}, \tag{9}$$

for the gravitational potential energy of a particle with mass m. Taylor approximation results in the corrected potential energy

$$U = -\frac{GMm}{r}\left(1+3\frac{GM}{c^2 r}\right), \tag{10}$$

[6]Although (only) similar in words, our approach is fundamentally different from *premetric* and *bimetric* theories. For that reason we only refer the interested reader to [6, 15, 16, 17].

[7]And others, most notably Nordström.

[8]We have set $\phi(\infty) = 0$ everywhere in our discourse. This eliminates the arbitrariness due to addition of a constant to ϕ. Also, higher-order special-relativistic corrections to kinetic energy have no bearing on the argument and are thus dropped.

[9]Known as the *Principle of Constancy of Velocity of Light*.

[10]For brevity, we use a substitution in the γ factor, but clearly it is not necessary for the γ factor to be given to model MGP. One can simply argue that we are looking for the simplest function with a bounded domain. In the elementary functions there are only two such functions: square root, and logarithm. Logarithm is excluded simply because it does not allow for the minimum itself to occur.

which is known –at least– since Wells[18] to account for the Perihelion precession of Mercury[11]. Wells found the new term by comparison with the effective potential of GR and interpreted the result in terms of the notion of effective (field) theories, but he was not able to determine higher-order terms, and instead envisaged of 'performing precise experiments' to find the new terms. We now see that MGP uniquely determines all terms *without reference even to GR*, let alone new experiments.

Before considering status of MGP in GR, first, observe that the first term of (9), i.e. $-mc^2/2$, which results when $r \to \infty$, agrees (expectedly) with the effective potential arising from Schwarzschild geodesics of GR[22],

$$U_{\text{eff}}^{\text{GR}} = m\left(\frac{c^2}{2} - \frac{GM}{r} + \frac{l^2}{2r^2} - \frac{l^2 GM}{r^3 c^2}\right),$$

but we expect this energy (energy of a particle in absence of gravity) to be mc^2, not half of it. We see that 'the 2-factor problem' is *only partially* solved by GR, in that GR gets the observable value right; partially because the effective potential is still half mc^2. So the question of *where has the other half of mc^2 gone?* Calls for an explanation, for which *hypotheses non fingo*[12].

Second, notice that using the definition of gravitational potential, (9) is

$$\tilde{\phi} = -\frac{c^2}{2}\frac{1}{\sqrt{1+\frac{4\phi}{c^2}}},$$

pointing to the *transformational* consequences of MGP. If we are to find such transformations, it is crucial to first find the geometry arising from the Φ factor (8). A simple-minded expectation would be

$$d\epsilon^2 = c^4 dm^2 + 4c^2 dU dm,$$

in which we have generalized the definition of gravitational potential $\phi = U/m$ to a local one

$$\phi := \frac{dU}{dm}.$$

Using the above definition of ϕ and (7), we expect this metric to yield $E = \gamma mc^2$, but it wrongly gives $E = mc^2/\gamma$. We therefore propose

$$c^4 dm^3 = d\epsilon^2\left(dm + \frac{4}{c^2}dU\right), \tag{11}$$

[11] The first occurrence of such correction that I found is in [19], but Wells, as far as I know and could see, was the first to focus on this correction and show it *matches GR* (in its Newtonian limit) for the perihelion precession problem. There are some corrections based on Einstein-Infeld-Hoffmann equations[20, 21]

$$\frac{3G(m_1+m_2)}{rc^2},$$

which, even if we overlook its problem for a *single* particle, and let $m_1 = m_2$, gives a wrong correction. Weinberg[3] considered corrections of higher orders in terms of three coefficients *upon which an a priori relation is projected*, i.e.

$$g_{00} = 1 + 2\alpha\phi + 2(\alpha\gamma - \beta)\phi^2,$$

which need not be the case.

[12] As the results of my investigations in this direction are not yet conclusive enough to be presented.

which does yield $E = \gamma mc^2$ upon the application of (7). Unfortunately however, (11) does not conform to any known geometry as its line element $d\epsilon$ is given *implicitly.* Without proper knowledge of the novel geometry arising from this line element it is not possible to find the transformations. Sadly, the transformations are not yet fully derived to report here.
It is rather astonishing how deep and powerful MGP is. Not only it gives us the effective potential of GR, but also it locally *corrects GR itself*, for the effective potential of GR has only four terms, while the correction from MGP adds infinitely many new terms. On the mathematical physics side, owing to MGP, (11) offers an entire novel domain of research: *geometry of implicit line elements.*

Proceeding to status of MGP in GR, although the conceptual framework of GR is deeper and able of making other predictions that (9) cannot make, according to (7) *violating MGP (locally) is equivalent to (local) violation of the second principle of SR.* As such MGP must be locally respected by GR, which is however evidently not the case, unveiling *a friction between SR and GR.* If we apply MGP to the Schwarzschild metric, a new metric

$$ds'^2 = -\sqrt{1 + \frac{4\phi}{c^2}}c^2dt^2 + \frac{dr^2}{\sqrt{1 + 4\phi/c^2}} + r^2d\Omega^2, \tag{12}$$

is found, to which the Schwarzschild metric is an approximation. Having two metrics which was the essence of our initial procedure, we can now show the relation of MGP and energy-momentum of spacetime. To that end, discard $\mathcal{O}(\phi^3)$ in (12), leaving

$$ds^2 = -\left(1 - \frac{r_s}{r} - \frac{r_s^2}{2r^2}\right)c^2dt^2 + \left(1 - \frac{r_s}{r} - \frac{r_s^2}{2r^2}\right)^{-1}dr^2 + r^2d\Omega^2, \tag{13}$$

which has the same form as the Reissner–Nordström metric,

$$ds^2 = -\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)c^2dt^2 + \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)^{-1}dr^2 + r^2d\Omega^2.$$

Recall that the r^{-2} term in Reissner–Nordström metric is due to the energy-momentum tensor of the electromagnetic field[23],

$$T^{\mathrm{EM}}_{\hat{t}\hat{t}} = \frac{q^2}{2r^4};$$

meaning that the r^{-2} term in (13) results when one solves Einstein Field Equations for a spherically symmetric body with energy-momentum tensor

$$T_{\hat{t}\hat{t}} = -\frac{1}{4\pi G}(\frac{GM}{r^2})^2 = -\frac{1}{4\pi G}(\nabla\phi)^2; \tag{14}$$

but this is (twice) the Newtonian limit of energy density of the gravitational field! This observation suggests the interesting result that *corrections of MGP to Schwarzschild metric account for the energy of the spacetime itself*, i.e. the tensor $\bar{g}_{\mu\nu}$,

$$d\bar{s}^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{r_s^2}{2r^2} + \mathcal{O}(r^{-3})\right)c^2dt^2 + \left(1 - \frac{r_s^2}{2r^2} + \mathcal{O}(r^{-3})\right)^{-1}dr^2 + r^2d\Omega^2, \tag{15}$$

yields the energy-momentum tensor of spacetime when substituted in Einstein Field Equations. The energy-momentum tensor of spacetime $T_{\mu\nu}$ is thus explicitly given by

$$T_{\mu\nu} = \frac{c^4}{16\pi G}\bar{G}_{\mu\nu}. \tag{16}$$

It is important to notice that, first, this equation is not the Einstein Field Equation, differing by a factor of 2; second, $\nabla^\mu T_{\mu\nu} \neq 0$, which will be important later.

Before applying our results to (1), it is important to mention that the corrected metric (12) is not without predictions. The geodesics of (12) are given by

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{c^2\kappa^2}{l^2} - \left(u^2 + \frac{c^2}{l^2}\right)\sqrt{1 - \frac{4GMu}{c^2}}, \tag{17}$$

where $u = 1/r(\varphi)$ and κ, l are constants of integrations. Unlike the geodesic equation of GR which is solved by Weierstrass $\wp$ function[24], (17) is not possible to be solved even in terms of elliptic functions[13]. We can however readily show the mere *existence* of at least two new periods of the solution –assuming its existence–. The importance of new periods is in that as Schwarzschild geodesics are given in terms of elliptic functions which are doubly-periodic[25], and since the Keplerian geodesics are singly-periodic, perihelion precession of Mercury is the result of the second period.
Since Jacobi [25, 26] we know that elliptic functions are the most general multiply periodic functions possible in a single variable. On the other hand, elliptic functions can solve at most second order differential equations, while (17) is of fourth order[14]. This shows that the solution of (17) has at least *four* periods[15].

Having adduced that there is no rational obstacle to soundness of our proposal of wave-black hole duality, it is time to consider *wavefunction of a black hole* (spacetime)

$$\Psi = e^{-\frac{i}{\Omega}K_{\mu\nu}g^{\mu\nu}}, \tag{18}$$

where

$$\Omega = \frac{f_P}{l_P^3} = \frac{1}{t_P l_P^3} = \frac{\text{Planck frequency}}{\text{Planck volume}},$$

is chosen so as to make argument of the exponential dimensionless. This enables us to turn

[13]Algebraically (17) is

$$y^4 = x^5 + x + z,$$

which is called a *tetragonal* curve, while the GR curve is $y^2 = x^3 + x + b$, called an elliptic curve.

[14]Study of this class of equations is still going on; see [27]. Computer programs and numerical approaches exist to solve these equations and find the numerical value of the periods, but it seems almost impossible to me to see the *physical meaning* of new periods without having a clear analytical vision beforehand; it is like giving someone the mere numerical value of the perihelion and expecting them to infer from it what physical phenomena it is signalling. For this reason it is not of physical value to consider solving the equation numerically.

[15]This is only a heuristic argument. A rigorous one would be based on Abelian functions.

metric and energy-momentum tensors into operators (observables)[16]

$$\begin{cases} T_{\mu\nu} \to \hat{T}_{\mu\nu}(f) = i\hbar\Omega\partial_{g^{\mu\nu}} f, \\ g^{\mu\nu} \to \hat{g}^{\mu\nu}(f) = g^{\mu\nu} f, \end{cases} \tag{19}$$

where

$$\partial_{g^{\mu\nu}} := \frac{\partial}{\partial g^{\mu\nu}};$$

implying the uncertainty relation

$$\left[\hat{g}^{\mu\nu}, \hat{T}_{\rho\sigma}\right] = i\hbar\Omega\delta^{\mu}_{(\rho}\delta^{\nu}_{\sigma)}, \tag{20}$$

which is proved similarly to the quantum mechanical case, i.e. for a test function Φ

$$\hat{T}_{\rho\sigma}\hat{g}^{\mu\nu}\Phi = \hat{T}_{\rho\sigma}(g^{\mu\nu}\Phi) = i\hbar\Omega\partial_{g^{\rho\sigma}}(g^{\mu\nu}\Phi) = i\hbar\Omega\delta^{\mu}_{(\rho}\delta^{\nu}_{\sigma)}\Phi + g^{\mu\nu}(i\hbar\Omega\partial_{g^{\rho\sigma}}\Phi) = i\hbar\Omega\delta^{\mu}_{(\rho}\delta^{\nu}_{\sigma)}\Phi + \hat{g}^{\mu\nu}\hat{T}_{\rho\sigma}\Phi.$$

Although slightly similar, (20) is critically different from the usual commutation relation of 'canonical quantum gravity'[13]

$$\left[\hat{h}^{ab}(\mathbf{x}), \hat{p}_{cd}(\mathbf{y})\right] = i\hbar\delta^{a}_{(c}\delta^{b}_{d)}\delta(\mathbf{x}, \mathbf{y}), \tag{21}$$

in which metric and energy-momentum tensor are understood to be *fields*, functions of space, analogous to

$$[\phi^a(\mathbf{x}), \pi_b(\mathbf{y})] = i\hbar\delta^a_b\delta(\mathbf{x}, \mathbf{y});$$

whereas in (20), metric and energy-momentum tensor are understood as *not fields*, but mere 'tensor-coordinates' of a new 'phase space' –which will be described soon–, analogously to

$$[\hat{q}^a, \hat{p}_b] = i\hbar\delta^a_b.$$

The physical significance of this difference is that while (21) means one cannot simultaneously measure precisely the metric and energy-momentum of a *single point of space(time)*, for (20) uncertainty is in measuring simultaneously the metric and energy-momentum of *a particular* black hole in toto. While the constant in (21) is a 'purely quantum-mechanical' one, Ω is a *quantum-gravitational* constant.

The foremost arising question is now that of equation governing changes[17] of observables. Any such equation must be in terms of dynamics *of* spacetime manifold, as we are viewing the manifold of spacetime 'as a whole'; in accord with what (1) and the notion of wavefunction of spacetime indicate. To this purpose Hamiltonian approach is better suited in that it is timeless at the level of canonical coordinates (phase space). We accordingly postulate that *dynamics of spacetime (as a pseudo-Riemannian manifold) is completely determined by its metric $g_{\mu\nu}$ and its energy-momentum tensor $T_{\mu\nu}$*, which are 'independent' of one another,

[16]Technically plane waves are not physically legitimate as they are not normalizable, but this issue can be rigorously avoided in orthodox quantum mechanics; see[28], and similarly here. This therefore is safe as a mere tool to arrive at definitions of operators, and their commutation relations, as we shall see now.

[17]We do not say 'evolution' since conventionally by *evolution*, change *in time* is meant, but the resulting structure is 'timeless'.

thereby requiring *all* dynamics of the manifold to be expressed *solely* in terms of $g_{\mu\nu}$ and $T_{\mu\nu}$. This postulate resembles that of ADM formulation of GR[23, 29], although key differences will be seen. Indeed the coherent structure and relative successes of ADM theory are strong motivations and support for this postulate. An important question now arises as to our expectation from such dynamics, knowing that both 'canonical tensors' are *given* by GR and (15). This implies that the conceptual order of this new sought-after Hamiltonian dynamics is the 'reverse' of the familiar Hamiltonian dynamics: one here should begin with canonical variables and arrive at the Hamiltonian, contrary to the familiar situation in which one begins with the Hamiltonian and arrives at the canonical coordinates (as the solutions of Hamilton equations). This is all naturally anticipated since we do not have even a candidate for the Hamiltonian of a pseudo-Riemannian manifold.

Mathematically the postulate is that to a pseudo-Riemannian manifold $(\mathcal{M}, \mathbf{g})$ of dimension n we associate a $2n^2$-dimensional 'phase space' $\mathcal{P}$ spanned by $(\mathbf{g}, \mathbf{T})$. To find the equations governing dynamics *of* $\mathcal{M}$ we must adopt a brief heuristic approach, in which it is assumed that wisdom of the reader indulges us patiently by their understanding of the inevitable investigative nature of any discourse that aims to go beyond current established knowledge. Our task is essentially finding counterparts of Hamilton equations[18]. Let us ask for the counterpart of the (call it 'first') Hamilton equation

$$\frac{d\mathbf{p}}{dt} = -\frac{\partial \mathcal{H}}{\partial \mathbf{q}}. \tag{22}$$

This equation, when its right-hand-side is let equal to zero, yields the conservation of energy. As the counterpart of conservation of energy in GR is

$$\nabla^\mu T_{\mu\nu} = 0,$$

we look for an equation that yields $\nabla^\mu T_{\mu\nu} = 0$ whenever its one side vanishes. Straightforwardly one is led to

$$\nabla^\mu T_{\mu\nu} = -\partial_{g^{\nu\sigma}} \mathcal{H}^\sigma, \tag{23}$$

in which *vector Hamiltonian* $\mathcal{H}^\sigma$ is yet to be defined, as explained above. Observe that

$$[\mathbf{H}] = [\mathcal{H}^\sigma] = \text{Energy} \times \text{Length}^{-4}, \tag{24}$$

which is energy 'density' *per spacetime*, manifesting that the proposal does not distinguish between space and time *even in its measurement units (dimensions).*

It should be mentioned that for a proper energy-momentum of spacetime (16), the left-hand-side of (23) is not zero. The task of ensuring that the *total* energy of spacetime as a whole is conserved, is undertaken by another equation, to which we shall soon arrive.

Assuming (23), immediately the analogue of the second Hamilton equation is expected to be

$$\nabla^\mu g_{\mu\nu} = \partial_{T^{\nu\sigma}} \mathcal{H}^\sigma,$$

[18] The word *counterpart* is vital. This *cannot* be a 'generalization' of classical Hamiltonian dynamics in any sense. There is no methodologically continuous way to get from *metric*, which knows no coordinates, to coordinates, let alone a total (proper) time derivative. Therefore we cannot expect this new analytical mechanics of manifolds to be a *generalization* proper; it would be a similar but different structure from the symplectic structure; as we shall see.

but the left-hand-side is zero by the metric-compatibility condition of a pseudo-Rimannian manifold[19]. Thus

$$\partial_{T^{\nu\sigma}}\mathcal{H}^{\sigma} = 0. \tag{25}$$

Equations (23) and (25) therefore, *define* the vector Hamiltonian of a pseudo-Riemannian manifold. The notion of a *vectorial* Hamiltonian is not totally alien to quantum gravity and unexpected. In exercising the *Hamiltonian constraints* –which lead to Wheeler-DeWitt equation– one is practically using a vector Hamiltonian $\mathcal{H}^a$ [13, 23]; the 'momentum (spatial) constraint'.

This newly proposed structure now allows us to put the fundamental commutation relation of quantum gravity(20) on a tenable 'classical' origin, by proving the analogue of Liouville's theorem in this new phase space $\mathcal{P}$:

$$\text{div } \mathbf{v}^{\mathbf{H}} = (\partial_{g^{\nu\sigma}}, \partial_{T^{\nu\sigma}}) \begin{pmatrix} \partial_{T^{\nu\sigma}}\mathcal{H}^{\sigma} \\ -\partial_{g^{\nu\sigma}}\mathcal{H}^{\sigma} \end{pmatrix} = \sum_{\nu}\sum_{\sigma} (\partial_{g^{\nu\sigma}}\partial_{T^{\nu\sigma}}\mathcal{H}^{\sigma} - \partial_{T^{\nu\sigma}}\partial_{g^{\nu\sigma}}\mathcal{H}^{\sigma}) = 0. \tag{26}$$

The above equation[20] guides us to define *a bracket* for an arbitrary vector $\mathbf{V}$, and the vector Hamiltonian $\mathbf{H}$, both functions on $\mathcal{P}$, by

$$\{\mathbf{V}, \mathbf{H}\}_{\nu}^{\ \mu} := \sum_{\lambda} \Big((\partial_{g^{\lambda\nu}}\mathcal{H}^{\mu})(\partial_{T^{\lambda\nu}}V^{\mu}) - (\partial_{g^{\lambda\nu}}V^{\mu})(\partial_{T^{\lambda\nu}}\mathcal{H}^{\mu}) \Big). \tag{27}$$

Similarly for an arbitrary 2-tensor $\mathbf{M}$ on $\mathcal{P}$, and the vector Hamiltonian $\mathbf{H}$, we define

$$\{\mathbf{M}, \mathbf{H}\}_{\lambda\mu}^{\ \ \nu} := \sum_{\kappa} \Big((\partial_{g^{\lambda\mu}}M^{\kappa\nu})(\partial_{T^{\lambda\mu}}\mathcal{H}^{\nu}) - (\partial_{g^{\lambda\mu}}\mathcal{H}^{\nu})(\partial_{T^{\lambda\mu}}M^{\kappa\nu}) \Big) \tag{28}$$

This new bracket enables us to write (23) and (25) as

$$\nabla^{\mu}M_{\mu\nu} = \{\mathbf{M}, \mathbf{H}\}_{\nu} \tag{29}$$

in which $\mathbf{M}$ can be metric or energy-momentum tensor[21]. *But we can take $\mathcal{H}_\mu$ as given by* (23) *and* (25)*, and postulate that* (29) *determines changes of any 2-tensor $\boldsymbol{M}$ on $\mathcal{P}$.* For a vector $\mathbf{V}$ this postulate would be

$$\nabla^{\mu}V_{\mu} = \{\mathbf{V}, \mathbf{H}\},$$

implying

$$\nabla^{\mu}\mathcal{H}_{\mu} = \{\mathbf{H}, \mathbf{H}\} = 0, \tag{30}$$

which is the promised conservation of Hamiltonian (conservation of total energy of space-time). This equation determines changes of the vector Hamiltonian itself.

[19] Evidently, should one desire to retain complete formal symmetry and similarity with classical Hamiltonian mechanics, a geometry with non-metricity shall be assumed.

[20] Technically (26) requires a generalization of Clairaut's theorem. To save a huge space, the proof, which requires some theoretical development, will be presented elsewhere.

[21] It is implicitly understood that Einstein Summation Convention does not hold for the index for which Σ is explicitly written. Apart from that, this notation is consistent with *index notation* of tensors, and indices can be contracted according to Einstein summation convention. It is here therefore assumed that contracted indices are not written anymore, i.e. $\{\mathbf{M}, \mathbf{H}\}_{\nu} := \{\mathbf{M}, \mathbf{H}\}_{\nu\theta}^{\ \ \theta}$, and so on.

To sum up, in this new dynamics,
1. One begins by solving Einstein Field Equations to find metric tensor; first canonical tensor,
2. Then the second canonical tensor, energy-momentum tensor, is found via the procedure, or (16),
3. One then finds vector Hamiltonian using (23) and (25),
Finally

- Changes of any tensor on $\mathcal{P}$ would be given by (29),
- Changes of the Hamiltonian itself is given by (30).

Apart from manifest covariance and 'timelessness', this new dynamics of a manifold is distinguished from the ADM formalism in that unlike the conjugate momenta of ADM

$$\pi^{ab} = \frac{\sqrt{h}}{16\pi G}(k^{ab} - kh^{ab}),$$

which are defined from the (induced) metric, the momenta of our proposal are 'independent' from the metric.

The mathematical structure arising from (23),(25) seems to be a promising direction of future inquiry which has not been pursued so far, as far as I know. Recall that the classical Hamiltonian phase space is a *symplectic* manifold, as the cotangent bundle of the configuration space[30, 31], equipped with the differential 2-form

$$\omega = dp_i \wedge dq^i.$$

There is a key difference outright that makes our new structure fundamentally different from a symplectic structure: Metric tensor is not a 2-form, making the new structure not even a *multi-symplectic* structure [32, 33, 34] which employs k-forms[22].

The mission is now accomplished by postulating the correspondence

$$\{\cdot,\cdot\} \longrightarrow \frac{\hbar G^2}{ic^7}\left[\cdot,\cdot\right], \tag{31}$$

in which the constant (inverse Planck Pressure) is necessary for dimensional consistency, following from (24). Applying (31) to (29) yields

$$\nabla^\mu M_{\mu\nu} = i\frac{\hbar G^2}{c^7}\left[\mathbf{H},\mathbf{M}\right]_\nu, \tag{32}$$

where

$$\left[\mathbf{H},\mathbf{M}\right]_\nu = H^\mu M_{\mu\nu} - M_{\mu\nu}H^\mu; \tag{33}$$

for a *tensor observable* $\mathbf{M}$.
As we have worked in the analogue of Heisenberg picture, a key ensuing prospect is to state the dynamics in the analogue of Schrödinger picture.
Page limitation now impels me to finish this essay by pointing another horizon which calls for future research: quantization of free field (5), whose justification and clarification was intricate enough to consume most of this essay.

[22] There are of course many more differences. For example, to witness the symplectic structure of phase space, one needs a base manifold (configuration space) to be defined beforehand. Such structure seems not possible in our case.

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References

[1] G. 't Hooft. The Quantum Black Hole as a Hydrogen Atom: Microstates without strings attached. *arXiv:1605.05119 [gr-qc]*, 2016.

[2] H. Balasin and H. Nachbagauer. The energy-momentum tensor of a black hole, or what curves the schwarzschild geometry? *Classical and Quantum Gravity*, 10(2271), 1993.

[3] S. Weinberg. *Gravitation and Cosmology*. Wiley, 1972.

[4] L. D. Landau and E. M. Lifshitz. *The Classical Theory of Fields*. Addison-Wesley and Pergamon, 1971.

[5] A. I. Nikishov. On energy-momentum tensors of gravitational field. *Physics of Particles and Nuclei*, 32, 2001.

[6] F. W. Hehl et al. Metric-affine gauge theory of gravity: field equations, Noether identities, world spinors, and breaking of dilation invariance. *Physics Reports*, 258(1-2), 1995.

[7] F. Mandl and G. Shaw. *Quantum Field Theory, 2nd Edition*. Wiley, 2010.

[8] M. E. Peskin and D. V. Schroeder. *An Introduction To Quantum Field Theory*. CRC Press, 1995.

[9] J. Hadamrd. *Lectures on Cauchy's Problem in Linear Partial Differential Equations*. Yale University Press, 1923.

[10] B. S. DeWitt and R. W. Brehme. Radiation Damping in a Gravitational Field. *Annals of Physics*, 9(2), 1960.

[11] F.G. Friedlander. *The Wave Equation on a Curved Space-Time (Cambridge Monographs on Mathematical Physics)*. Cambridge University Press, 2010.

[12] G. W. Gibbons et al. Path integrals and the indefiniteness of the gravitational action. *Nuclear Physics B*, 138(1), 1978.

[13] C. Kiefer. *Quantum Gravity*. Oxford University Press, 2007.

[14] Approaches to quantum gravity. In D. Oriti, editor, *Toward a New Understanding of Space, Time and Matter*. Cambridge University Press, 2009.

[15] F. W. Hehl et al. On Kottler's path: Origin and evolution of the premetric program in gravity and in electrodynamics. *International Journal of Modern Physics D*, 25(11), 2016.

[16] F. W. Hehl and Y. Obukhov. *Foundations of Classical Electrodynamics: Charge, Flux, and Metric*. Birkhäuser, Boston, 2003.

[17] J. Beltrán Jiménez et al. Bimetric variational principle for general relativity. *Physical Review D*, 86, 2012.

[18] J. D. Wells. When effective theories predict: the inevitability of Mercury's anomalous perihelion precession. *arXiv:1106.1568 [physics.hist-ph]*, 2011.

[19] Y. Iwasaki. Quantum Theory of Gravitation vs. Classical Theory. *Progress of Theoretical Physics*, 46(5), 1971.

[20] J. F. Donoghue. General relativity as an effective field theory: The leading quantum cerrectiens. *Physical Review D*, 50, 1994.

[21] N. E. J. Bjerrum-Bohr. Quantum gravitational corrections to the nonrelativistic scattering potential of two masses. *Physical Review D*, 67, 2003.

[22] R. M. Wald. *General Relativity*. University of Chicago Press, 1984.

[23] Øyvind Grøn and Sigbjørn Hervik. *Einstein's General Theory of Relativity*. Springer, 2007.

[24] F. T. Hioe and David Kuebel. Characterization of all possible orbits in the Schwarzschild metric revisited. *arXiv:1207.7041 [gr-qc]*, 2012.

[25] A. Markushevich. *Introduction to the Classical Theory of Abelian Functions*. American Mathematical Society, 1992.

[26] C. B. Boyer and U. C. Merzbach. *A History of Mathematics*. Wiley, 1991.

[27] M. England and J. C. Eilbeck. Abelian functions associated with a cyclic tetragonal curve of genus six. *Journal of Physics A: Mathematical and Theoretical*, 42(9), 2009.

[28] B. C. Hall. *Quantum Theory for Mathematicians*. Springer-Verlag New York, 2013.

[29] R. Arnowitt, S. Deser, and C. W. Misner. (Republication of) The dynamics of general relativity. *General Relativity and Gravitation*, 40:1997–2027, 2008.

[30] D. E. Blair. *Riemannian Geometry of Contact and Symplectic Manifolds*. Birkhäuser, Boston, 2010.

[31] G. Rudolph and M. Schmidt. *Differential Geometry and Mathematical Physics*. Springer, Dordrecht, 2013.

[32] F. Cantrijn et al. On the geometry of multisymplectic manifolds. *Journal of the Australian Mathematical Society*, 66(3), 1999.

[33] C. Paufler and H. Römer. De Donder-Weyl equations and multisymplectic geometry. *Reports on Mathematical Physics*, 49(2-3), 2002.

[34] J. Gaset and N. Román-Roy. Multisymplectic unified formalism for Einstein-Hilbert gravity. *Journal of Mathematical Physics*, 59(3), 2018.