

【Proof】

$$A = \begin{pmatrix} 3 & 1 & 4 & 2 & 0 & 3 & 1 & 4 \\ 4 & 2 & 0 & 3 & 1 & 4 & 2 & 0 \\ 0 & 3 & 1 & 4 & 2 & 0 & 3 & 1 \\ 1 & 4 & 2 & 0 & 3 & 1 & 4 & 2 \\ 2 & 0 & 3 & 1 & 4 & 2 & 0 & 3 \\ 3 & 1 & 4 & 2 & 0 & 3 & 1 & 4 \\ 4 & 2 & 0 & 3 & 1 & 4 & 2 & 0 \\ 0 & 3 & 1 & 4 & 2 & 0 & 3 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 \\ 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\ 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 \\ 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 \\ 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} -2 & 1 & -1 & 2 & 0 & -2 & 1 & -1 \\ -1 & 2 & 0 & -2 & 1 & -1 & 2 & 0 \\ 0 & -2 & 1 & -1 & 2 & 0 & -2 & 1 \\ 1 & -1 & 2 & 0 & -2 & 1 & -1 & 2 \\ 2 & 0 & -2 & 1 & -1 & 2 & 0 & -2 \\ -2 & 1 & -1 & 2 & 0 & -2 & 1 & -1 \\ -1 & 2 & 0 & -2 & 1 & -1 & 2 & 0 \\ 0 & -2 & 1 & -1 & 2 & 0 & -2 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 2 & -2 & -1 & 0 & 1 & 2 & -2 \\ -2 & -1 & 0 & 1 & 2 & -2 & -1 & 0 \\ 0 & 1 & 2 & -2 & -1 & 0 & 1 & 2 \\ 2 & -2 & -1 & 0 & 1 & 2 & -2 & -1 \\ -1 & 0 & 1 & 2 & -2 & -1 & 0 & 1 \\ 1 & 2 & -2 & -1 & 0 & 1 & 2 & -2 \\ -2 & -1 & 0 & 1 & 2 & -2 & -1 & 0 \\ 0 & 1 & 2 & -2 & -1 & 0 & 1 & 2 \end{pmatrix}$$

$$AB = BA = BC = CB = DA = AD = \begin{pmatrix} 1 & 4 & 2 & 0 & 3 & 1 & 4 & 2 \\ 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\ 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 \\ 3 & 0 & 2 & 4 & 1 & 3 & 0 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 4 & 2 & 0 & 3 & 1 & 4 & 2 \\ 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\ 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 & 0 & -2 & 1 & -1 & 2 \\ 0 & 1 & 2 & -2 & -1 & 0 & 1 & 2 \\ -1 & -2 & 2 & 1 & 0 & -1 & -2 & 2 \\ -2 & 0 & 2 & -1 & 1 & -2 & 0 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & -1 & 2 & 0 & -2 & 1 & -1 & 2 \\ 0 & 1 & 2 & -2 & -1 & 0 & 1 & 2 \\ -1 & -2 & 2 & 1 & 0 & -1 & -2 & 2 \end{pmatrix}$$

$$CD = DC = \begin{pmatrix} 1 & -1 & 2 & 0 & -2 & 1 & -1 & 2 \\ 0 & 1 & 2 & -2 & -1 & 0 & 1 & 2 \\ -1 & -2 & 2 & 1 & 0 & -1 & -2 & 2 \\ -2 & 0 & 2 & -1 & 1 & -2 & 0 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & -1 & 2 & 0 & -2 & 1 & -1 & 2 \\ 0 & 1 & 2 & -2 & -1 & 0 & 1 & 2 \\ -1 & -2 & 2 & 1 & 0 & -1 & -2 & 2 \end{pmatrix} \quad \therefore AB = BA = BC = CB = CD = DC = DA = AD$$

Therefore, the physical quantity corresponding to A, the physical quantity corresponding to B, the physical quantity corresponding to C, and the physical quantity corresponding to D can be observed simultaneously (both can be known in a single measurement)

The classical adjoint matrices are all 0.

$$\text{Orthogonalize} \begin{pmatrix} -7 & -4 & -1 & 2 & 5 & 8 & 11 & 14 \\ -6 & -3 & 0 & 3 & 6 & 9 & 12 & 15 \\ -5 & -2 & 1 & 4 & 7 & 10 & 13 & 16 \\ -4 & -1 & 2 & 5 & 8 & 11 & 14 & 17 \\ -3 & 0 & 3 & 6 & 9 & 12 & 15 & 18 \\ -2 & 1 & 4 & 7 & 10 & 13 & 16 & 19 \\ -1 & 2 & 5 & 8 & 11 & 14 & 17 & 20 \\ 0 & 3 & 6 & 9 & 12 & 15 & 18 & 21 \end{pmatrix}$$

$$\text{Orthogonalize} \begin{pmatrix} 3 & 1 & 4 & 2 & 5 & 3 & 1 & 4 \\ 4 & 2 & 5 & 3 & 1 & 4 & 2 & 5 \\ 5 & 3 & 1 & 4 & 2 & 5 & 3 & 1 \\ 1 & 4 & 2 & 5 & 3 & 1 & 4 & 2 \\ 2 & 5 & 3 & 1 & 4 & 2 & 5 & 3 \\ 3 & 1 & 4 & 2 & 5 & 3 & 1 & 4 \\ 4 & 2 & 5 & 3 & 1 & 4 & 2 & 5 \\ 5 & 3 & 1 & 4 & 2 & 5 & 3 & 1 \end{pmatrix}$$

outcome

$$\begin{pmatrix} -0.320844 & -0.18334 & -0.0458349 & 0.0916698 & 0.229175 & 0.366679 & 0.504184 & 0.641689 \\ 0.560112 & 0.490098 & 0.420084 & 0.35007 & 0.280056 & 0.210042 & 0.140028 & 0.070014 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0.333333 & 0.111111 & 0.444444 & 0.222222 & 0.555556 & 0.333333 & 0.111111 & 0.444444 \\ 0.224201 & 0.213893 & 0.229355 & 0.219047 & -0.809184 & 0.224201 & 0.213893 & 0.229355 \\ 0.349282 & 0.272298 & -0.495648 & 0.31079 & 0.140663 & 0.349282 & 0.272298 & -0.495648 \\ -0.464605 & 0.414052 & 0.0585771 & 0.452569 & 0.117154 & -0.464605 & 0.414052 & 0.0585771 \\ 0.0276924 & 0.443079 & 0.0276924 & -0.775388 & 0.0553849 & 0.0276924 & 0.443079 & 0.0276924 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

dimension

8

8

extra- dimension

0

0

summand matrix

$$\begin{pmatrix} 1 & 0 & -1 & -2 & -3 & -4 & -5 & -6 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

eigenpolynomial

$$\lambda^8 - 0.169254\lambda^7 - 0.0545545\lambda^6$$

$$\lambda^8 - 0.559531\lambda^7 - 0.0331659\lambda^6 - 0.330446\lambda^5 - 0.21606\lambda^4 + 0.353553\lambda^3$$

eigenvalue

$$\lambda_1 = \left(103365606275727434913541 + \sqrt{92073664384311926058320390392514095807502673881} \right) / 1221429068369631142233944$$

$$\lambda_2 = \left(103365606275727434913541 - \sqrt{92073664384311926058320390392514095807502673881} \right) / 1221429068369631142233944$$

$$\lambda^8 = \lambda^7 = \lambda^6 = \lambda^5 = \lambda^4 = \lambda^3 = 0$$

$$\lambda_1 \approx -0.150528 + 0.858943i \quad \lambda_2 \approx -0.150528 - 0.858943i$$

$$\lambda_3 \approx 0.786528 + 0.184222i \quad \lambda_4 \approx 0.786528 - 0.184222i$$

$$\lambda_5 \approx -0.712469 \quad \lambda^8 = \lambda^7 = \lambda^6 = 0$$

eigenvector

$$v_1 = \left(-495254373196080734760859 + \sqrt{92073664384311926058320390392514095807502673881} \right) / 684137119396352193913600, 1, 0, 0, 0, 0, 0, 0$$

$$v_2 = \left(-495254373196080734760859 - \sqrt{92073664384311926058320390392514095807502673881} \right) / 684137119396352193913600, 1, 0, 0, 0, 0, 0, 0$$

$$v_3 = \left(\frac{6235269209245382619797136}{1039211534874230264472805}, \frac{72744807441191613031835699}{1039211534874230264472805}, 0, 0, 0, 0, 0, 1 \right)$$

$$v_4 = \left(\frac{655334025543686448564678}{131066805108737252539213}, \frac{786400830652423727656550}{131066805108737252539213}, 0, 0, 0, 0, 1, 0 \right)$$

$$v_5 = \left(\frac{5701132961645378565962908}{1425283240411344037518113}, \frac{7126416202056722948608229}{1425283240411344037518113}, 0, 0, 0, 1, 0, 0 \right)$$

$$v_6 = \left(\frac{4285720690704893523946652}{1428573563568297185600759}, \frac{5714294254273190990568036}{1428573563568297185600759}, 0, 0, 1, 0, 0, 0 \right)$$

$$v_7 = \left(\frac{2620243859835970227554164}{1310121929917983854625503}, \frac{3930365789753954441937261}{1310121929917983854625503}, 0, 1, 0, 0, 0, 0 \right)$$

$$v_8 = \left(\frac{1244315466778921215926816}{1244315466778920892972583}, \frac{2488630933557842155035718}{1244315466778920892972583}, 1, 0, 0, 0, 0, 0 \right)$$

$$v_1 \approx (-0.388155 - 0.489906i, 0.0809548 + 0.946574i, -0.0489623 - 0.158437i, 0.29621 - 0.590014i, 1, 0, 0, 0)$$

$$v_2 \approx (-0.388155 + 0.489906i, 0.0809548 - 0.946574i, -0.0489623 + 0.158437i, 0.29621 + 0.590014i, 1, 0, 0, 0)$$

$$v_3 \approx (0.702739 + 1.81025i, -1.20273 + 1.93553i, -0.177366 + 1.16594i, -1.61145 + 0.974721i, 1, 0, 0, 0)$$

$$v_4 \approx (0.702739 - 1.81025i, -1.20273 - 1.93553i, -0.177366 - 1.16594i, -1.61145 - 0.974721i, 1, 0, 0, 0)$$

$$v_5 \approx (7.32419, 3.61052, -20.6752, 2.57662, 1, 0, 0, 0)$$

$$v_6 = (0, 0, -1, 0, 0, 0, 0, 1)$$

$$v_7 = (0, -1, 0, 0, 0, 1, 0, 0)$$

$$v_8 = (-1, 0, 0, 0, 1, 0, 0, 0)$$

Orthogonalize $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$

Orthogonalize $\begin{pmatrix} -1 & -2 & 2 & 1 & -2 & -1 & -2 & 2 \\ 2 & 1 & -2 & -1 & -2 & 2 & 1 & -2 \\ -2 & -1 & -2 & 2 & 1 & -2 & -1 & -2 \\ -2 & 2 & 1 & -2 & -1 & -2 & 2 & 1 \\ 1 & -2 & -1 & -2 & 2 & 1 & -2 & -1 \\ -1 & -2 & 2 & 1 & -2 & -1 & -2 & 2 \\ 2 & 1 & -2 & -1 & -2 & 2 & 1 & -2 \\ -2 & -1 & -2 & 2 & 1 & -2 & -1 & -2 \end{pmatrix}$

outcome

$$\begin{pmatrix} 0.070014 & 0.140028 & 0.210042 & 0.280056 & 0.35007 & 0.420084 & 0.490098 & 0.560112 \\ 0.641689 & 0.504184 & 0.366679 & 0.229175 & 0.0916698 & -0.0458349 & -0.18334 & -0.320844 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -0.208514 & -0.417029 & 0.417029 & 0.208514 & -0.417029 & -0.208514 & -0.417029 & 0.417029 \\ 0.362659 & -0.032969 & -0.219793 & -0.109897 & -0.791257 & 0.362659 & -0.032969 & -0.219793 \\ -0.318701 & -0.230783 & -0.512851 & 0.402955 & -0.0439587 & -0.318701 & -0.230783 & -0.512851 \\ -0.468546 & 0.366429 & 0.00926147 & -0.343396 & -0.417488 & -0.468546 & 0.366429 & 0.00926147 \\ -0.0624064 & -0.370768 & -0.121142 & -0.814955 & 0.154181 & -0.0624064 & -0.370768 & -0.121142 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

dimension

8

8

extra- dimension

0

0

summand matrix

$$\begin{pmatrix} 1 & 0 & -1 & -2 & -3 & -4 & -5 & -6 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

eigenpolynomial

$$\lambda^8 - 0.574198\lambda^7 - 0.0545545\lambda^6$$

$$\lambda^8 - 0.94355\lambda^7 - 0.076792\lambda^6 - 0.342197\lambda^5 - 0.600293\lambda^4 + 0.353553\lambda^3$$

eigenvalue

$$\lambda_1 = \left(17248044055154440916003 + \sqrt{494395351312504405267511492620141096054056169} \right) / 60076972477483579442708$$

$$\lambda_2 = \left(17248044055154440916003 - \sqrt{494395351312504405267511492620141096054056169} \right) / 60076972477483579442708$$

$$\lambda^8 = \lambda^7 = \lambda^6 = \lambda^5 = \lambda^4 = \lambda^3 = 0$$

$$\lambda_1 \approx 0.90373 \quad \lambda_2 \approx -0.880596$$

$$\lambda_3 \approx -0.10325 + 0.757465i \quad \lambda_4 \approx -0.10325 - 0.757465i$$

$$\lambda_5 \approx -0.760185 \quad \lambda^8 = \lambda^7 = \lambda^6 = 0$$

eigenvector

$$v_1 = \left(-65209073258542424058015 + \sqrt{494395351312504405267511492620141096054056169} \right) / 192753646316400418273856, 1, 0, 0, 0, 0, 0, 0$$

$$v_2 = \left(-65209073258542424058015 - \sqrt{494395351312504405267511492620141096054056169} \right) / 192753646316400418273856, 1, 0, 0, 0, 0, 0, 0$$

$$v_3 = \left(\frac{296976506325642689389625}{49496084387607105557793}, \frac{692945181426499533851969}{98992168775214211115586}, 0, 0, 0, 0, 0, 1 \right)$$

$$v_4 = \left(\frac{187989294992639497947995}{37597858998527891986659}, \frac{225587153991167370927304}{37597858998527891986659}, 0, 0, 0, 0, 1, 0 \right)$$

$$v_5 = \left(\frac{1224053168323024984756030}{306013292080756183781451}, \frac{1530066460403781043722368}{306013292080756183781451}, 0, 0, 0, 1, 0, 0 \right)$$

$$v_6 = \left(\frac{966591034601260888222685}{322197011533753605830691}, \frac{1288788046135014458688070}{322197011533753605830691}, 0, 0, 1, 0, 0, 0 \right)$$

$$v_7 = \left(\frac{702655413098298020315560}{351327706549148965519323}, \frac{1053983119647446941196426}{351327706549148965519323}, 0, 1, 0, 0, 0, 0 \right)$$

$$v_8 = \left(\frac{31865320052409009923815}{31865320052409008583351}, \frac{63730640104818017836934}{31865320052409008583351}, 1, 0, 0, 0, 0, 0 \right)$$

$$v_1 \approx (-0.159157, -0.843474, -0.00653499, -0.522842, 1, 0, 0, 0)$$

$$v_2 \approx (0.669272, 0.846187, 0.379009, 0.777168, 1, 0, 0, 0)$$

$$v_3 \approx (-2.1551 + 2.02137i, 1.66501 + 1.47205i, -1.52554 - 1.13985i, -0.0498242 - 1.58452i, 1, 0, 0, 0)$$

$$v_4 \approx (-2.1551 - 2.02137i, 1.66501 - 1.47205i, -1.52554 + 1.13985i, -0.0498242 + 1.58452i, 1, 0, 0, 0)$$

$$v_5 \approx (0.421604, 1.44962, 1.81144, 0.16092, 1, 0, 0, 0)$$

$$v_6 = (0, 0, -1, 0, 0, 0, 0, 1)$$

$$v_7 = (0, -1, 0, 0, 0, 0, 1, 0)$$

$$v_8 = (-1, 0, 0, 0, 0, 1, 0, 0)$$