Signature invariance in a neutron Dirac equation with an external magnetic field

Han Geurdes

GDS Applied Math BV Doetinchem The Netherlands

han.geurdes@gmail.com

ABSTRACT

In this paper the Dirac equation for a neutron in an external magnetic field is studied. The use of Dirac's equation for a neutron is somewhat controversial as far as the author is aware of. Nevertheless, the present study could be an interesting theoretical excercise in metric signature invariance. More in particular it is argued that in a specific external magnetic field and with non-vanishing time dependence of the 4×1 wave function vector, parity transformation invariance and time reversal invariance is not possible. This implies signature non-invariance.

INTRODUCTION

Let us assume a neutron mass, m_0 and, c=ħ=1. For a neutron in an external magnetic field [1], the m in the free particle Dirac equation is written as,

$m_H = m_0 + \mu_N^{'} \begin{pmatrix} \vec{\sigma} \cdot \vec{H} & 0 \\ 0 & \vec{\sigma} \cdot \vec{H} \end{pmatrix}$. The $\vec{H}$ is the magnetic field vector. The $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$,

is the triplet [2] of Pauli matrices and $\mu_N^{'}$ is associated to the anomalous magnetic moment of the neutron.

The main condition we look into is to vary the external magnetic field $\vec{H}$ such that finally

$m_0 = -\mu_N^{'} \frac{\bar{\psi} \left(\vec{\sigma} \cdot \vec{H} \right) \psi}{\bar{\psi} \psi}$ and arrange it so that $\vec{H}(-t, -\vec{x}) = \vec{H}(t, \vec{x})$.

In the present paper we will call the metric $g^{\alpha,\beta}$ West coast when signature is (+,-,-,-). The metric is East coast when the signature is (-,+,+,+). Viz. [3]. No other conventions are implied.

PRELIMIARIES & DERIVATION

Let us start with the following Dirac equation, inspired by [1, eq 1.4], with m_H.

(1) $$i\gamma_0 \frac{\partial}{\partial x^0}\psi - i\gamma_k \frac{\partial}{\partial x_k}\psi = m_H \psi.$$

Use is made of 2×2 matrices σ_k i.e. the Pauli matrices [2]

(2) $$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Here, the ψ is a 4×1 complex vector. The γ_0, γ_k $(k = 1,2,3)$ together with, γ_5, are [4]

(3) $$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note, $\{\gamma_\alpha, \gamma_\beta\} = 2g_{\alpha,\beta}$, with signature (+,-,-,-). This is a West coast theory.

With the use of the operators $P_R = \frac{1}{2}(1+\gamma_5)$ and $P_L = \frac{1}{2}(1-\gamma_5)$ via $\psi_R = P_R\psi$ and $\psi_L = P_L\psi$ and $\psi = \psi_R + \psi_L$, together with, $\gamma_5\psi = \psi_R - \psi_L$, the following relations are obtained

$(4a)$ $$E\psi_R + i(\vec{\sigma}\cdot\vec{p})\psi_R = m_H\gamma_0\psi_L$$

$(4b)$ $$E\psi_L - i(\vec{\sigma}\cdot\vec{p})\psi_L = m_H\gamma_0\psi_R$$

Here, $E = i\frac{\partial}{\partial x^0} = i\frac{\partial}{\partial t}$, and $\vec{p} = -i\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$. Equation (4) is the $(\vec{p}, E)$ representation [2] of equation (1). Note $\gamma_5 m_H = m_H\gamma_5$ and $\gamma_0 m_H = m_H\gamma_0$, while , $\gamma_0\sigma_k = \sigma_k\gamma_0$.

Further, a helicity operator $\hat{h}$ is defined by $(\vec{\sigma}\cdot\vec{p})/p$ and $p = \|\vec{p}\|$, the euclidean norm.

With, $\gamma_5\sigma_k = \sigma_k\gamma_5$, and $\gamma_k = \gamma_0\gamma_5\sigma_k$. Hence, adding (4a) and (4b),

(5) $$\frac{E}{p}\psi + \hat{h}\gamma_5\psi = \frac{m_H}{p}\gamma_0\psi$$

Therefore, if $\psi = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}$, and both ψ_u and ψ_d are 2×1, from (5) and (3) we see that

(6) $$\hat{h}\ \psi_d = \frac{1}{p}(m_H - E)\psi_u$$

$$\hat{h}\ \psi_u = -\frac{1}{p}(m_H + E)\psi_d$$

Let us, subsequently, define the 2×1 vector product, $a \otimes b = \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \end{pmatrix}$. From equation (6) the

$(7a)$ $$\psi_d \otimes \hat{h}\ \psi_d = \frac{1}{p}(m_H - E)\psi_d \otimes \psi_u$$

$(7b)$ $$\psi_u \otimes \hat{h}\ \psi_u = -\frac{1}{p}(m_H + E)\psi_u \otimes \psi_d$$

follow. Obviously, $\psi_d \otimes \psi_u = \psi_u \otimes \psi_d$.

And so summing equation (7a) and (7b) gives

(8) $$\psi_d \otimes \hat{h}\ \psi_d + \psi_u \otimes \hat{h}\psi_u \ = -\frac{2E}{p}\psi_u \otimes \psi_d$$

Similarly to the previous we also find

(9) $$\psi_u \otimes \hat{h}\ \psi_d + \psi_d \otimes \hat{h}\ \psi_u = \frac{1}{p}(m_H - E)\psi_u \otimes \psi_u - \frac{1}{p}(m_H + E)\psi_d \otimes \psi_d$$

Suppose now that, $\varphi_+ = \psi_u + \psi_d$. Then, summing (8) and (9),

(10) $$\varphi_+ \otimes \hat{h}\varphi_+ = \frac{1}{p}\psi_u \otimes \left((m_H - E)\psi_u - E\psi_d\right) - \frac{1}{p}\psi_d \otimes \left((m_H + E)\psi_d + E\psi_u\right)$$

If, $\varphi_- = \psi_u - \psi_d$, and having, $\varphi_+ \otimes \varphi_- = (\psi_u + \psi_d) \otimes (\psi_u - \psi_d) = (\psi_u \otimes \psi_u) - (\psi_d \otimes \psi_d)$, equation (10) turns into

(11) $$\varphi_+ \otimes \hat{h}\varphi_+ = -\frac{E}{p}\varphi_+ \otimes \varphi_+ + \frac{m_H}{p}\varphi_+ \otimes \varphi_-$$

Furthermore, the equation for φ_- is obtained subtracting (9) from (8)

(12) $$\varphi_- \otimes \hat{h}\varphi_- = \psi_u \otimes \left(-\frac{1}{p}(m_H - E)\psi_u - \frac{E}{p}\psi_d\right) + \psi_d \otimes \left(\frac{1}{p}(m_H + E)\psi_d - \frac{E}{p}\psi_u\right)$$

Using φ_+ the following equation for $\varphi_- \otimes \hat{h}\varphi_-$ arises

(13) $$\varphi_- \otimes \hat{h}\varphi_- = \frac{E}{p}\varphi_- \otimes \varphi_- - \frac{m_H}{p}\varphi_- \otimes \varphi_+$$

Looking at equations (11) and (13), it is possible to obtain two equations; one for φ_+ and one for φ_-

(14) $$\hat{h}\varphi_+ = -\frac{E}{p}\varphi_+ + \frac{m_H}{p}\varphi_-$$

$$\hat{h}\varphi_- = \frac{E}{p}\varphi_- - \frac{m_H}{p}\varphi_+$$

Subsequently, let us define $\Phi = \begin{pmatrix}\varphi_+ \\ \varphi_-\end{pmatrix}$ and obtain from equation (14)

(15) $$\hat{h}\Phi = -\gamma_0 \frac{E}{p}\Phi + \frac{m_H}{p}\gamma_0\gamma_5\Phi$$

If we recall that $\gamma_k = \gamma_0\gamma_5\sigma_k$ for $k = 1,2,3$, then an interesting new form of Dirac equation is found by multiplying left and right hand of equation (15) with $\gamma_0\gamma_5 p$

Going from the $(\vec{p},E)$ representation back to the $(\vec{x},t)$, the following differential equation is found

(16) $$i\gamma_5 \frac{\partial}{\partial x^0}\Phi + i\gamma_k \frac{\partial}{\partial x_k}\Phi = m_H \Phi$$

For completeness, $\Phi = \begin{pmatrix} \psi_u + \psi_d \\ \psi_u - \psi_d \end{pmatrix} = \begin{pmatrix} \psi_u \\ -\psi_d \end{pmatrix} + \begin{pmatrix} \psi_d \\ \psi_u \end{pmatrix} = (\gamma_0 + \gamma_5)\psi$, is a transformation of,

$\psi = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}$, from the Dirac equation in (1). Let us write $t = x^0$ and assume that Φ is symmetric in time reversal, and parity i.e. $\Phi(-t,-\vec{x})=\Phi(t,\vec{x})$. In effect this means ψ is $\psi(-t,-\vec{x}) = \psi(t,\vec{x})$.

If we multiply $\Phi=(\gamma_0 + \gamma_5)\psi$ with $(\gamma_0 + \gamma_5)$, the result is 2ψ. The factor 2 drops off from equation (16). Multiplication left and right hand (16) with γ_5 gives

(17) $$i\gamma_5 \frac{\partial}{\partial t}\gamma_5\Phi - i\gamma_k \frac{\partial}{\partial x_k}\gamma_5\Phi = m_H \gamma_5\Phi$$

with $\gamma_5\gamma_k = \gamma_5\gamma_0\gamma_5\sigma_k = -\gamma_0\gamma_5\sigma_k\gamma_5 = -\gamma_k\gamma_5$. Then, multiplication with γ_0

(18) $$-i\gamma_5 \frac{\partial}{\partial t}\gamma_0\Phi - i\gamma_k \frac{\partial}{\partial x_k}\gamma_0\Phi = m_H \gamma_0\Phi$$

Therefore, when $t \rightarrow -t$ and $x_k \rightarrow -x_k$ in (18) and $\psi(-t,-\vec{x}) = \psi(t,\vec{x})$, the sum of equations (17) and (18) gives

(19) $$i\gamma_5 \frac{\partial}{\partial t}\psi + i\gamma_k \frac{\partial}{\partial x_k}\psi = m_H \psi$$

RESULT & DISCUSSION

The interesting result is that when equation (1) and (19) are summed, the following expression for t derivation arises

(20) $$i\left(\gamma_0 + \gamma_5\right)\frac{\partial}{\partial t}\psi = 2m_H \psi$$

If we then employ the condition $\left(\exists_{\vec{H}\in\mathbb{R}^3}\right) m_0 = -\mu_N' \dfrac{\bar{\psi}\left(\vec{\sigma}\cdot\vec{H}\right)\psi}{\bar{\psi}\psi}$, and

$\vec{H}(-t,-\vec{x}) = \vec{H}(t,\vec{x})$ and employ $\psi(-t,-\vec{x}) = \psi(t,\vec{x}) = \psi$ in equation (1) we find

(21) $$\mathrm{i}\bar{\psi}\gamma_0 \frac{\partial}{\partial t}\psi = 0$$

The condition gives, $\bar{\psi} m_H \psi = 0$.

With $\bar{\psi} = \psi^\dagger\gamma_0$, it follows from (21) in terms of ψ_u and ψ_d and with the over-dot for $\frac{\partial}{\partial t}$ abbreviation that, $\psi_u^\dagger \dot{\psi}_u + \psi_d^\dagger \dot{\psi}_d = 0$. With (20) and $\bar{\psi} m_H \psi = 0$ we also find the equation $\psi_u^\dagger \dot{\psi}_d - \psi_d^\dagger \dot{\psi}_u = 0$.

Let us assume for simplicity that

(23a) $$\psi_u(t,\vec{x}) = \begin{pmatrix} \phi(t,\vec{x}) \\ 0 \end{pmatrix} \text{ and } \psi_d(t,\vec{x}) = \begin{pmatrix} \chi(t,\vec{x}) \\ 0 \end{pmatrix}$$

(23*b*) $$\frac{\partial}{\partial t}\psi_u \neq 0 \text{ and } \frac{\partial}{\partial t}\psi_d \neq 0$$

Then the condition $\psi_u^\dagger \psi_u \neq \psi_d^\dagger \psi_d$ warrants $\bar{\psi}\psi \neq 0$ and this implies $\phi^* \phi \neq \chi^* \chi$. Hence, $\phi \neq \chi$. We can therefore assume additionally, $\left\{\phi^*\right\}^2 + \left\{\chi^*\right\}^2 \neq 0$. It is now possible to derive

(24) $$\phi^* \dot{\phi} + \chi^* \dot{\chi} = 0,$$

from $\psi_u^\dagger \dot{\psi}_u + \psi_d^\dagger \dot{\psi}_d = 0,$ and

(25) $$-\chi^* \dot{\phi} + \phi^* \dot{\chi} = 0$$

from $\psi_u^\dagger \dot{\psi}_d - \psi_d^\dagger \dot{\psi}_u = 0$. From (23) and (24) we can obtain $\left(\left\{\phi^*\right\}^2 + \left\{\chi^*\right\}^2\right)\dot{\chi} = 0$. Because, $\left\{\phi^*\right\}^2 + \left\{\chi^*\right\}^2 \neq 0,$ it follows that $\dot{\chi} = 0$ and in turn $\dot{\phi} = 0$. But if we insist that $\frac{\partial}{\partial t}\psi_u \neq 0$ and $\frac{\partial}{\partial t}\psi_d \neq 0$, in (23b), then the equation (20), is not possible under $\bar{\psi} m_H \psi$=0.

With the matrix form of P and T [2, eq 2.18] we found

(26) $$\psi_{East}(t,\vec{x}) = \psi(-t,-\vec{x}) \neq \psi(t,\vec{x}) = \psi_{West}(t,\vec{x}).$$

The PT transformations are interpreted as a (+,-,-,-) $\rightarrow$ (-,+,+,+) change.

Further, we have $\bar{\psi}\psi = |\phi|^2 - |\chi|^2$. With $(0,0,H_3)$ and (23a) it follows that the term $H_3\bar{\psi}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}\psi = H_3\left(|\phi|^2 - |\chi|^2\right)$. Therefore we can have the simpler form $m_0 = -\mu_N^{'} H_3$.

The $\psi_{East}(t,\vec{x})$ lives on the East coast where, in our terminology, only the signature of the metric tensor is -1 times the one of the West coast. Hence, when in experiment at a certain point in time, the experimenter sets the external magnetic field to $(0,0,H_3)$, in the lab frame, then, $m_H = 0$ and $m_0 = -\mu_N^{'} H_3 \neq 0$. in (1) and (19).

This entails that (1) and (19) both look as though the change (+,-,-,-) $\rightarrow$ (-,+,+,+) would not matter in the parlour game of [3]. However, it is demonstrated here that it does. Change the magnetic conditions and ψ_{West} is not equal ψ_{East}, viz. equation (26).

The found non-invariance is interpreted as: two different neutrons that are not interchangeable under $(+,-,-,-) \rightarrow (-,+,+,+)$ and (23), despite the fact that $m_H = 0$ gives a free particle Dirac equation such as in [3, eq 3 and comment]. If one claims that ψ_{East} is just a different representation of the same neutron we are back at Einstein's criticism [5, page 179-180] which claims that there is a 1-1 relation between particle (state) and wave function.Therefore, there could be more associated to the metric signature than meets the eye.

DECLARATIONS

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