A New Look at Black Holes via Thermal Dimensions and the Complex Coordinates/Temperature Vectors Correspondence

Carlos Castro Perelman
Ronin Institute, 127 Haddon Place, Montclair, N.J. 07043.
Center for Theoretical Studies of Physical Systems
Clark Atlanta University, Atlanta, Georgia. 30314.
perelmanc@hotmail.com

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Abstract

It is shown how the crucial **active diffs** symmetry of General Relativity allows to shift the radial location $r = 2GM$ of the horizon associated with the Schwarzschild metric to the $r = 0^+$ location of a *diffeomorphic* metric. In doing so, one ends up with a spacetime void surrounding the singularity at $r = 0$. In order to explore the "interior" region of this void we introduce complex radial coordinates whose imaginary components have a direct link to the inverse Hawking temperature, and which furnish a path that provides access to interior region. In addition, we show that the black hole entropy $\frac{A}{4}$ (in Planck units) is equal to the *area* of a rectangular strip in the *complex* radial-coordinate plane associated to this above path. The gist of the physical interpretation behind this construction is that there is an emergence of thermal dimensions which unfolds as one plunges into the interior void region via the use of complex coordinates. And whose imaginary components capture the span of the thermal dimensions. The filling of the void leads to an *emergent* internal/thermal dimension via the imaginary part β_r of the complex radial variable $\mathbf{r} = r + i\beta_r$.

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A considerable progress in recent years has been made in understanding the quantum aspects of black holes and the Hawking evaporation process. This progress involved the role of islands, replica wormholes, holography, the Page curve, saddle points in the gravitational path integral, fine-grained von Neumann entropy, quantum information, complexity, ..., see [1] for a recent review and a vast number of references. One of the main motivations is that black holes provide a window into the microscopic structure of spacetime in quantum gravity. Recently, the quantum information contained in Hawking radiation has been calculated, verifying a key aspect of the consistency of black hole evaporation with quantum mechanical unitarity. This calculation relied crucially on recent progress in understanding the emergence of bulk spacetime from a boundary holographic description [1].

In this short note we shall take a completely different look based on the key role of active diffeomorphisms and a complex coordinates/four-temperature vector connection introduced below. The static spherically symmetric (SSS) *vacuum* solution of Einstein's equations [2] was found by Schwarzschild [3] and is more widely known in terms of the solution provided by Hilbert [5] as

$$(ds)^2 \;=\; (1 - \frac{2GM}{r})\;(dt)^2 \;-\; (1 - \frac{2GM}{r})^{-1}\;(dr)^2 \;-\; r^2\;(d\Omega)^2 \qquad (1)$$

where the solid angle infinitesimal element is $(d\Omega)^2 = (d\theta)^2 + sin^2(\theta)(d\phi)^2$.

The higher-dimensional extension of the metric can be obtained by simply replacing $(d\Omega)^2 \to (d\Omega_{D-2})^2$ (the $D-2$-dim solid angle) and $1 - \frac{2GM}{r} \to 1 - (\frac{r_h}{r})^{D-3}$ where r_h is the horizon radius expressed in terms of M and the gravitational coupling G_D in D dimensions whose units are $(length)^{D-2}$.

The solution (1) is defined modulo diffeomorphisms. All diffeomorphic metrics to (1) are physically equivalent. There are two types of diffeomorphisms. The *passive* ones where the spacetime points remain $fixed$ but there is a change of coordinates $x^\mu \to x'^\mu = f^\mu(x^\nu)$. A typical example are the Kruskal-Szekers change of coordinates $U(r,t), V(r,t)$ giving a maximal extension of the Schwarzschild metric into the interior region of the black hole. And the *active* diffs where the spacetime points are physically *displaced* while leaving the coordinates fixed $x^\mu = x'^\mu$. In view of this, let us perform an *active* diffs where one activates an actual *outwards radial* displacement $r \to \rho(r) \geq r$ of the spacetime points, so the metric (1) becomes

$$(ds)^2 \;=\; (1 - \frac{2GM}{\rho(r)})\;(dt)^2 \;-\; (1 - \frac{2GM}{\rho(r)})^{-1}\;(d\rho)^2 \;-\; \rho^2(r)\;(d\Omega)^2. \qquad (2)$$

An *inwards* radial displacement $\rho(r) < r$ is called in the mathematical literature a "deformation retract".

Note that one has *not* relabeled the radial variable r by giving it another name and calling it "ρ", because $\rho(r)$ is itself a function of r. Furthermore, one has *not* performed a radial reparametrization $r' = \rho(r)$ because an *active* diffs is *not* the same as a *passive* diffs (a coordinate transformation). The metric (2) assumes the *same* values as the original metric (1) but at $different$ radial *locations* due to the active physical *displacements* of the spacetime points. The metric solution (2) does *not* violate Birkhoff's theorem because it is obtained from the Hilbert-Schwarzschild metric via an active diffs. It is well known that the *extended* Schwarzschild metric solution for $r < 0$ with $M > 0$, corresponds to a solution in the region $r > 0$ with $M < 0$. Negative masses are associated with repulsive gravity. For this reason, the domain of values of r will be chosen to span the whole real axis $-\infty \leq r \leq \infty$.

The temporal component of the metric (2) leads to modifications of the Newtonian potential at distances of the order of $2GM$. One recovers the Newtonian potential at large distances compared to $2GM$ because in the regime when $r >> 2GM$ one has $\rho(r) \sim r$ such that $\frac{GM}{\rho(r)} \sim \frac{GM}{r}$. The graph of the function $\rho(r)$ is asymptotic to the graph of r.

Due to the spherical symmetry one must have that $\rho(r = 0) = 0$, since the location of the physical point-mass must retain the position of being the geometrical *center* of spherical symmetry. One cannot have a situation with $\rho(r = 0) = 2GM$ [4] because this would imply that the center point $r = 0$ is displaced to an $infinity$ of points which comprise an spherical shell of radius $R = 2GM$. This map $\rho(r = 0) = 2GM$ would be infinite-valued and not one-to-one. Another reason why $\rho(r = 0) = 0$ is because a point mass must have zero area and zero volume simultaneously. In [6] we argued the possibility for a geometrical entity to have a non-zero area while having a zero volume simultaneously. This can occur with $fractal$ surfaces which are space-filling. In this case the area of the fractal surface is infinite but the volume is zero. It is worth exploring this $fractal$ horizon scenario in a theory of Quantum Gravity.

Let us define the active diffs by the map $r \to \rho(r)$ and such that $|\rho(r)| \geq |r|$ as follows

$$\rho(r = 0) = 0; \quad \rho(r) = \frac{r}{1 - e^{-r/2GM}}, \quad r > 0; \quad \rho(r) = \frac{r}{1 - e^{r/2GM}}, \quad r < 0, \tag{3}$$

One has $\rho(-r) = -\rho(r)$ and in this way one extends the solutions to the $r < 0$ region. A negative r sounds strange but one must not forget that $r = \sqrt{x^2 + y^2 + z^2}$ and there is always a $\pm$ sign in front of every square root. Since $\rho(r)$ is antisymmetric in r it must vanish at $r = 0$, which is consistent with the fact the the center of symmetry $\rho(r = 0) = 0$ must remain fixed as stated previously.

From eq-(3) one infers that $\rho(r = 0^{\pm}) = lim_{\epsilon \to 0} \rho(r = \pm\epsilon) = \pm 2GM$ while $\rho(r = 0) = 0$ (a point mass must have zero area and zero volume). The metric (2) has a horizon at $\rho = 2GM$ which corresponds to $r = 0^+$, and there is a singularity at $\rho = 0$ which corresponds to $r = 0$. There is a *discontinuity* of $\rho(r)$ at $r = 0$. Because a point mass is an infinitely compact source there is nothing *wrong* with the possibility of having a *discontinuity* of the metric at the location of the singularity $r = 0$. In this extreme case, when the the location of the horizon merges with the singularity, there is a null-line singularity at $r = 0$ and a null-surface at $r = 0^+$. This may sound quite paradoxically but it is a consequence of the metric *discontinuity* at $r = 0$, the location of the point mass (singularity).

The Kretschmann invariant is $K \sim R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau} \sim \frac{(2GM)^2}{\rho(r)^6}$. It diverges at $r = 0$ but it is finite at $r = 0^{\pm}$ due to the discontinuity of the metric at $r = 0$ resulting from $\rho(0) = 0; \rho(0^{\pm}) = \pm 2GM$. In [6] we argued why this key fact may have important consequences for the resolution of the fire wall problem and the complementarity controversy in black holes [7].

In 1975, Stephen Hawking and Jacob Bekenstein showed how the black holes should slowly radiate away energy, which poses a problem. From the no hair theorem, one would expect the Hawking radiation to be completely independent of the material entering the black hole. Nevertheless, if the material entering the black hole were a pure quantum state, the transformation of that state into the mixed state of Hawking radiation would destroy information about the original quantum state. This violates Liouville's theorem and presents a physical paradox. Hawking remained convinced that the equations of black hole thermodynamics together with the no-hair theorem led to the conclusion that quantum information may be destroyed. This is the so-called Black Hole Information Paradox.

An *heuristic* explanation by Hawking and described by Penrose is the following. Virtual particle-antiparticle pairs are constantly being created out of the vacuum but then annihilated in a very short time. But very near the horizon of a black hole, it's possible for one particle to fall in before the annihilation can happen, in which case the other one escapes as Hawking radiation. For this, the virtual particles both become *real*, and energy conservation demands that the ingoing particles have *negative* energy. This they can do because the Killing vector κ becomes *spacelike* inside the horizon. If κ^a is spacelike the conserved energy $p_a \kappa^a$ (being assessed from infinity) can be negative, where p_a is the particle's four momentum.

However the solutions described in this work allow for the horizon to be displaced arbitraily close to the singularity, and in the limiting case when the horizon and (null) singularity merge, the interior region disappears and such that there is no longer room for the ingoing particle to go and acquire a negative energy. The virtual particles in this case would be annihilated at the (null) singularity. Thus it is plausible to avoid the Hawking emission process in this scenario.

The main purpose of this work is two-fold. Firstly, it is to implement an analytical continuation of the metric (2), and the active diffs $\rho(r)$, via the introduction of *complex* coordinates, which in turn, lead to *complex* metrics. The analytical continuation will allow us to explore the *interior void* region given by $0 < \rho < 2GM$. In other words, the analytical continuation via complex coordinates and a complex metric, will allow us to study the *interior* of the black hole which was inaccessible to an observer equipped only with real coordinates. Namely, what is a void empty region from the perspective of real coordinates, it is not void nor empty from the perspective of complex coordinates.

The discontinuity at $r = 0$ of the map $\rho(r)$ in (3) can be envisioned as a removal of the point $r = 0$ (the singularity) from R^3, leading to the *punctured* space $R^3 - \{0\}$. In the ρ-picture, the map of $R^3 - \{0\}$ leads to a region that is *not* simply connected, and given by the *exterior* of a spherical *void* surrounding the singularity at $\rho = 0$ of radius $\rho = 2GM$. The removal of a point naturally leads to a Topology change which has been exploited by some authors by removing the black hole interior via an Antipodal identification of the points on a sphere and associated with a RP^3 projective space [8].

Secondly, we will show that by identifying the *imaginary* components of the complex coordinates with the inverse $(T^{-1})^\mu = \frac{T^\mu}{T_\nu T^\nu} = \beta^\mu$ of the four-temperature vector $T^\mu = (T^0, T^1, T^2, T^3)$, we will obtain in a nice geometrical manner the expression for the Hawking temperature $T_H = (8\pi GM)^{-1}$ in units $\hbar = c = k_B = 1$. In addition, we also find the exact expression for the black hole entropy in terms of the area of a rectangulat strip in the *complex*-ρ plane.

We define the *complex* radial coordinates by

$$\mathbf{r} \equiv r + i\beta_r, \quad \rho(\mathbf{r}) \ = \ \rho(r + i\beta_r) \ \equiv \ \gamma(r, \beta_r) \ + \ i\beta_\rho(r, \beta_r) \tag{4}$$

The imaginary components of the coordinates are postulated to have a one-to-one correspondence with the inverse of the four-temperature vector components.

As usual we must have that $\rho(\mathbf{r} = 0 + i0) = 0 + i0 = \mathbf{0}$. And for $\mathbf{r} \neq 0 + i0$, we define the map as

$$\rho(r + i\beta_r) \ = \ \rho(Re^{i\alpha}) \ = \ \frac{Re^{i\alpha}}{1 \ - \ e^{-Re^{i\alpha}/2GM}}, \quad R \equiv \sqrt{r^2 + \beta_r^2}, \quad tan(\alpha) = \frac{\beta_r}{r} \tag{5}$$

one infers that when $R = \epsilon$, *all* the points on the infinitesimal circle of radius $\epsilon = \sqrt{r^2 + \beta_r^2}$ will be mapped to $2GM$ in the limit $\epsilon \to 0$ when the circle shrinks to zero. Once again, there is a discontinuity at the origin since one must have $\rho(\mathbf{r} = 0 + i0) = \mathbf{0}$. From the above definition (5) one learns that

$$\rho(0 + i2\pi GM) \ = \ i\pi GM, \quad \rho(0 + i4\pi GM) \ = \ i\infty \tag{6}$$

$$\rho(0 - i2\pi GM) \ = \ - \ i\pi GM, \quad \rho(0 - i4\pi GM) \ = \ - \ i\infty \tag{7}$$

A path along the positive imaginary radial axis from $\mathbf{r} = i0^+$ to $\mathbf{r} = i2\pi GM$ is mapped to a path in the *complex*-ρ plane starting at $\rho = 2GM + i0$ and ending at $\rho = 0 + i\pi GM$. Furthermore, the latter path in the *complex*-ρ plane is precisely the one which has access to the interior void region $0 < \mathcal{R}e(\rho) < 2GM$.

Continuing upwards along the positive imaginary radial axis from $\mathbf{r} = i2\pi GM$ to $\mathbf{r} = i4\pi GM$ it leads to a path which is mapped to a path in the *complex*-ρ plane starting at $\rho = i\pi GM$ and ending at $i\infty$. The maps of the path along the negative imaginary radial axis from $\mathbf{r} = -i0^+$ to $\mathbf{r} = -i4\pi GM$ lead to the complex conjugates (in the *complex*-ρ plane) of the previous paths.

Consequently, one learns that the map of the $finite$ interval in the imaginary radial axis ranging from from $\mathbf{r} = i\beta_{r,min} = -i4\pi GM$ to $\mathbf{r} = i\beta_{r,max} = i4\pi GM$ yields a path in the *complex*-ρ plane which *covers all* of the imaginary ρ-axis, so that the span of the values in β_ρ is $\pm\infty$.

Concluding, we then have found that the magnitude of the finite interval $[-i4\pi GM, +i4\pi GM]$ in the imaginary radial axis is $8\pi GM$, and which is precisely equal to the inverse Hawking temperaure $\beta_H = \frac{1}{T_H}$ in $\hbar = c = k_B = 1$

units. Furthemore, among those paths in the *complex*-ρ plane is the path which has *access* to the *interior* of the black hole : the interior void region $0 < \mathcal{R}e(\rho) < 2GM$.

There seems to be a caveat because the inverse Hawking temperature β_H is the length of the circle S^1_β obtained from a compactification of the Euclidean time in Thermal Field Theory and resulting after a Wick rotation, $t_M = it_E$, from Minkowski time to Euclidean time. Thus $\beta_r \neq \beta_t$. However, one must not forget that upon crossing the horizon into the black hole interior the roles of r, t are *exchanged* due to a signature flip. Therefore, one can affirm that the finite interval in the imaginary radial axis is indeed related to the inverse Hawking temperature $\beta_H = \beta_{r,max} - \beta_{r,min} = 8\pi GM$.

The black hole entropy also admits a simple geometrical interpretation in terms of the *area* of a rectangular strip in the *complex*-ρ plane. From eqs-(6,7) we can infer that the points $\pm i\pi GM$ can be chosen to be two of the vertices (lying in the imaginary ρ-axis) of the rectangular strip, while the other two remaining vertices are located at $2GM \pm i\pi GM$. We explained earlier how the paths that explore the interior region of the black hole are those taken along the imaginary radial axis from $\mathbf{r} = \pm i0^+$ to $\mathbf{r} = \pm i2\pi GM$, and which are then mapped to the paths in the *complex*-ρ plane starting at $\rho = 2GM + i0$ (lying in the real ρ axis) and ending at $\rho = 0 \pm i\pi GM$ (in the imaginary ρ axis), respectively.

The infinitesimal region (the infinitesimal circle which shrinks to zero) around the origin $\mathbf{r} = 0 + i0 = \mathbf{0}$ is mapped to a *bifurcation* point in the real ρ axis at $\rho = 2GM + i0$, when the radius shrinks to zero. One circumnavigates *around* the pole at $\mathbf{r} = 0 + i0$, as usual, by means of going around the pole (clockwise or counter-clockwise) along the infinitesimal circle. The counter-clockwise (clockwise) rotation of $\pm\frac{\pi}{2}$ will position us in the positive (negative) imaginary axis. A rotation of $\pm\pi$ will bring us into the $r < 0$ region. The bifurcation point at $\rho = 2GM + i0$ is also consistent with a bifurcate horizon of the Penrose diagram of the extended Schwarzschild solution involving the black and white hole regions connected via a wormhole throat.

The area of the rectangular strip whose 4 vertices are located at $\pm i\pi GM$ and $2GM \pm i\pi GM$ is given by

$$2 \times \pi GM \times 2GM \ = \ 4\pi (GM)^2 \ = \ \frac{1}{4} \ 4\pi \ (2GM)^2 = \frac{A}{4} \tag{8}$$

and which is the black hole entropy in Planck units $L_P^2 = 1$ associated with the area of a spherical horizon of radius $2GM$. We hope that all these findings in this work are more than just mere numerical coincidences.

It is our belief that one of the goals of attaining a theory of Quantum Gravity is to implement a space-time-matter unification. Einstein argued that a spacetime point is devoid of any physical meaning unless a point-mass is attached to it, like it happens with the point-mass located at the origin in the Hilbert-Schwarzschild solution. In our interpretation, after taking advantage of the active diffs symmetry of General Relativity which allowed us to shift the radial location of the horizon all the way towards the singulariry, is that one can plunge into the "interior" of the point mass via the introduction of complex coordinates. In otherwords, as we plunge into this interior the $unfolding$, the *emergence* of the *thermal* dimensions (via the introduction of complex coordinates) takes place.

The source of the black hole entropy *is* its *mass*. In [6] we showed that the Euclideanized Einstein-Hilbert action associated to a scalar curvature $\mathcal{R} = \frac{4GM\delta(r)}{r^2}$ (the delta function singularity is due to the point-mass source) when the Euclidean thermal interval is chosen to be equal to $\beta_H = 8\pi GM$, yields the black hole entropy. So there is an Euclidean action/Black Hole entropy correspondence in this case. The Schwarzschild metric leads to a vanishing Ricci tensor and scalar curvature $\mathcal{R} = 0$, hence in order to arrive at a key delta function singularity at the origin one has to replace r for $|r|$ in the metric (1). More precisely, one needs to replace

$$1 - \frac{2GM}{r} \to 1 - \frac{2GM\Theta(r)}{r} \tag{9}$$

in the metric of eq-(1), where $\Theta(r)$ is the Heaviside step function, $\Theta(r) = 1$, for $r > 0$; $\Theta(r) = -1$, for $r < 0$; and $\Theta(r = 0) = 0$, the arithmetic mean of $1, -1$. Despite that $\Theta(0) = 0$, the straight use of L'Hopital's rule in eq-(9) at $r = 0$ gives $\delta(r = 0) = \infty$, and as expected, the metric is singular at $r = 0$, since $\frac{d\Theta(r)}{dr} = \delta(r)$. It is the derivatives of the step function appearing in eq-(9) which will generate the $\delta(r)$ terms in the curvature. If one wishes to be fully mathematically rigorous, one needs to recur to the Colombeau's theory of distributions instead of the Dirac delta distributions.

The introduction of complex radial coordinates leads to complex metrics of the form

$$g_{\mu\nu}(r + i\beta_r) \;=\; \gamma_{\mu\nu}(r, \beta_r) \;+\; i\beta_{\mu\nu}(r, \beta_r) \tag{10}$$

Most recently, Witten [9] has argued that for various reasons, it seems necessary to include complex saddle points in the "Euclidean" path integral of General Relativity and which was motivated by recent work of Kontsevich and Segal on *complex* metrics in Quantum Field Theory, and earlier work of Louko and Sorkin on topology change from a real time point of view.

Another way of incorporating complex coordinates and complex metrics is by exploring the geometry of the cotangent bundle of spacetime (phase space) within the context of Finsler geometry. The complex coordinates $x^\mu + i\beta^\mu$ have a correspondence with $x^\mu + ip^\mu$ where p^μ are the momentum coordinates. One simply has to impose the following correspondence between the inverse-temperature and the momentum

$$\beta^\mu \;\leftrightarrow\; \frac{T^\mu}{T_\nu T^\nu} \;\leftrightarrow\; \frac{p^\mu}{p_\nu p^\nu} \;=\; \frac{p^\mu}{p^2} \tag{11a}$$

such that

$$d\beta_\mu d\beta^\mu \;\leftrightarrow\; \frac{dp_\mu dp^\mu}{p^4}, \quad p^2 = p_\nu p^\nu \tag{11b}$$

The line element in the $8D$ cotangent bundle is

$$(ds)^2 \;=\; g_{\mu\nu}(x,p)\, dx^\mu dx^\nu \;+\; h^{ab}(x,p)\, (dp_a + N_{a\mu}(x,p)dx^\mu)\, (dp_b + N_{b\nu}(x,p)dx^\nu) \tag{12}$$

where $g_{\mu\nu}(x,p), h^{ab}(x,p)$ are the base spacetime and internal space metrics, respectively, with $a,b = 0,1,2,3$, $\mu,\nu = 0,1,2,3$, and $N_{a\mu}(x,p)$ is the nonlinear connection. The number of total components of $g_{\mu\nu}, h^{ab}, N_{a\mu}$ is $10 + 10 + 16 = 36 = (8 \times 9)/2$ which is more than sufficient to accommodate the $10 + 10$ components of the complex metric $\gamma_{\mu\nu}, \beta_{\mu\nu}$. The idea is to write down the vacuum field equations associated with the $8D$ cotangent bundle metric (12), to find the spherically symmetric static solutions, and to investigate how an analytical complex extension of the Schwarzschild metric might fit into the former cotangent bundle metric solutions. The mere presence of a mass is already indicating that a phase space picture (the cotangent bundle) should be more appropriate to embrace than the mere base spacetime picture in the full quantization process.

To sum up, we have seen how the crucial **active diffs** symmetry of General Relativity allowed us to shift the radial location $r = 2GM$ of the horizon in the metric (1) to the $r = 0^+$ location of the diffeomorphic metric (2). Note that $\rho(r = 0^+) = 2GM$ is the horizon for the metric (2). In doing so, one ended up with a spacetime void $0 < \rho < 2GM$ surrounding the singularity at $\rho(r = 0) = 0$. In order to explore the "interior" region of this void one is required to introduced complex radial coordinates, whose imaginary components had a direct link to the inverse Hawking temperature, and which furnished a path in the complex ρ plane that provided access to the sought-after interior region $0 < \mathcal{R}e(\rho) < 2GM$.

In addition, it allowed us to show the the black hole entropy $\frac{A}{4}$ (in Planck units) is equal to the *area* of a rectangular strip in the complex ρ plane associated to this above path. The gist of the physical interpretation behind this construction is that there is an emergence of thermal dimensions which unfolds as one plunges into the interior void region via the use of complex coordinates. And whose imaginary components capture the span of the thermal dimensions. The filling of the void leads to an *emergent* internal/thermal dimension via the imaginary part β_r of the complex radial variable $\mathbf{r} = r + i\beta_r$.

In this fashion, we hope to attain a merger of microscopic spacetime with thermodynamics. Quantum Mechanics involves complex numbers. The wavefunction Ψ is complex. Thus, a quantization of gravity may require the introduction of complex coordinates (like in Twistors) and complex metrics. This was not necessary in the quantization of Yang-Mills and Electrodynamics because gravity has a very different symmetry group : the infinite dimensional diffeomorphisms that led to a spacetime void surrounding the singularity and justified the introduction of complex coordinates and metrics. This is the key difference.

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