Relativity in Function Spaces and the need for Fractional Exterior Calculus

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Abstract

We look at Lorentz transformations from the perspective of functional analysis and show that the theory of functional analysis so far has neglected a critical point by not taking into consideration inputs of functions when measuring distances in function spaces.

Keywords — relativity in function spaces, metric space, function space, half-forms, fractional forms, functional analysis, fractional calculus

Contents

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1 Motivation

Building on clear physical foundations, in [1] we arrived at a transformation of the form

$$\phi' = \frac{c^2}{\sqrt{1+\frac{\phi}{2c^2}}},$$

for scalar function $\phi : \mathbb{R}^4 \to \mathbb{R}$. But something must be missing here, as a scalar function remains invariant under any coordinate transformation. As we mentioned in [1] there are two possible solutions for this problem:

- ϕ is not really a scalar. It is but *a component* of some vector.
- ϕ is a scalar function, but its functional nature points us to consider it in a function space in which it is a 'vector', hence it is no longer required *there* to remain invariant.

The first solution does not bring any new mathematics and solves the problem in complete analogy to special relativity: it turns out (see[1]) that if we look at ϕ as a function on $\mathbb{R}$ and not one on $\mathbb{R}^3$, we can solve the problem using familiar vector analysis. But what if we insist to see ϕ as a function on $\mathbb{R}^4$? By comparing the second solution with special relativity and using a 'historical thought experiment' we can see that the second solution is yet to be applied to special relativity itself! Imagine you are somewhere in a history in which you only have one spatial dimension, you do not consider time to be a proper dimension, and have found a transformation

$$x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}},$$

you are amazed 'why on earth a scalar is being transformed?' Apart from special relativity you can well take the other route: as you are quite used to $x(t)$, 'well, maybe I should consider the *function space* of $x(t)$s in which $x(t)$ is *not* a scalar'. You will be guided by the Pythagorean theorem to introduce the following distance on the function space of $x(t)$s

$$ds = \sqrt{\big(x'(t) - x(t)\big)^2}\; dt = |x'(t) - x(t)|dt \Rightarrow s = \int |x'(t) - x(t)|dt,$$

which is quite expected as this is one of the natural distances for a function space[2]. Yet, this metric is using an absolute time

$$t' = t,$$

thus to consider special relativity we must consider

$$s(x'(t'), x(t));$$

in other words we must *pay attention also to the 'inputs' of functions*, something which the current theory of functional analysis completely misses![1] **The current theory of functional analysis is as premature as Euclidean geometry for it has so far neglected inputs of functions.**

[1] To the best of my knowledge, the closest that mathematical analysis has come to this insight, is via the *Skorokhod metric*[3], in relation to *càdlàg* functions. I maintain that Skorokhod metric is quite inelegant for a firm physical theory.

2 Heuristic path

Therefore let us begin by analogy to the Minkowski metric and consider

$$d_M(f(x), g(y))(x, y) := \sqrt{c^2(x-y)^2 - \big(f(x) - g(y)\big)^2}, \tag{1}$$

where c is the speed of light. But this d_M is not a proper metric on a function space *solely*; it also takes into consideration the 'inputs' of functions, something which is not considered at all in the current theory of functional analysis, to the best of my knowledge. It is easy to see this point: let

$$f(x) = g(y), \quad x \neq y$$

If this was a proper metric on a function space, we would expect to have

$$d_M(f(x), f(y)) \stackrel{?}{=} 0,$$

yet in this case (1) yields

$$d_M = c|x - y|$$

signifying its *hybrid* nature.

To turn (1) into a proper metric for the function space solely, it is natural to try to 'get rid' of the 'inputs' by integrating over them. Therefore we have to define the *differential* of d_M. The first guess would be

$$d^2 d_M(f(x), g(y)) = \sqrt{\big(f(x) - g(y)\big)^2 - (x-y)^2}\ dxdy,$$

but this faces two problems: first, the sign is different from (1). To solve this problem one might be tempted to simply take

$$\sqrt{\big(f(x) - g(y)\big)^2 - c^2(x-y)^2}$$

but this ruins an elegant property of (1): demanding the expression under square root in (1) to be positive, results in

$$\forall (x \neq y) \quad |f(x) - g(y)| \leq c|x - y|, \tag{2}$$

meaning that our functions are **speed of light-Lipschitz** continuous! More on this will be said later. Therefore we must seek another solution; the most straightforward one is to introduce

$$A(x, y) := \begin{cases} 1, & x = y \\ -1, & x \neq y \end{cases}$$

and write

$$c^2(x-y)^2 + A(x, y)\big(f(x) - g(y)\big)^2;$$

which does the job.

The second problem is that we expect (1) to pass us over to its functional aspect by letting $x = y$, which results in

$$d^2 d_M(f(x), g(y)) = |f(x) - g(x)|dxdx,$$

as we are going to have to finally integrate this differential, –quite roughly-speaking– we are going to need one of the dxes on the right-hand-side, so

$$\frac{d^2}{dx}d_M = |f(x) - g(x)|dx,$$

the only way to make sense of the 'fraction' on the left-hand-side which would allow integration seems to be

$$d(\frac{dd_M}{dx}) = |f(x) - g(x)|dx,$$

upon integration it gives

$$\frac{dd_M}{dx} = \int |f(x) - g(x)|dx,$$

which is absurd as the right-hand-side is *not a function of* x; rendering it not well-defined. *Even* **if** we ignore this problem –rather silly but just to make sure–, changing the integration variable to t to avoid confusion, and letting the domain of integration to be $[a, b]$, we have

$$\int_a^b |f(t) - g(t)|dt = A = constant,$$

so

$$\frac{dd_M}{dx} = A \Rightarrow d_M = Ax + C,$$

which is an absurdity as this d_M is not even a metric. So it seems that we are having a 'surplus' of differentials: on the left-hand-side of

$$\frac{d^2}{dx}d_M = |f(x) - g(x)|dx,$$

there is a 'surplus' dx and one 'surplus' d [again quite roughly-speaking]. Everything would be perfect had we had

$$dd_M = |f(x) - g(x)|dx,$$

yielding the L^1-norm distance $d = \int |f(x) - g(x)|dx$.
So let us define

$$dd_M(f(x), g(y)) = \sqrt{c^2(x-y)^2 + A(x,y)\big(f(x) - g(y)\big)^2}\sqrt{d}x\sqrt{d}y \tag{3}$$

and give it a chance! Let

$$x = y; \tag{4}$$

Before continuing, recall our expectation: $x = y$ should 'wash away' the 'input-dependent' aspect of (1) and yield a proper metric for function space. Since from Fractional calculus[4] we have

$$\sqrt{d}x\sqrt{d}x = dx \tag{5}$$

then

$$d\ d_M(f(x), g(x)) = \sqrt{\big(f(x) - g(x)\big)^2}dx = |f(x) - g(x)|\ dx$$

$$\Rightarrow \boxed{d(f(x), g(x)) = \int |f(x) - g(x)| \; dx} \tag{6}$$

Note that it might seem that (1) is the product metric of space of functions and $\mathbb{R}$ but notice that

$$d = \sqrt{d_1^2(f, g) + d_2^2(x, y)}$$

'glues' two metrics together while respecting their 'independence': f, g and x, y in the above (product) metric are 'dummies'. You do not need to know anything about d_2 to find d_1. This is not at all the case for (1) as it is in fact 'intertwining' functions and their inputs.

3 Lipschitz cones and Light cones

If we recall that for a Lipschitz continuous function, there exists a double cone whose origin can be moved along the graph so that the whole graph always stays outside the cone

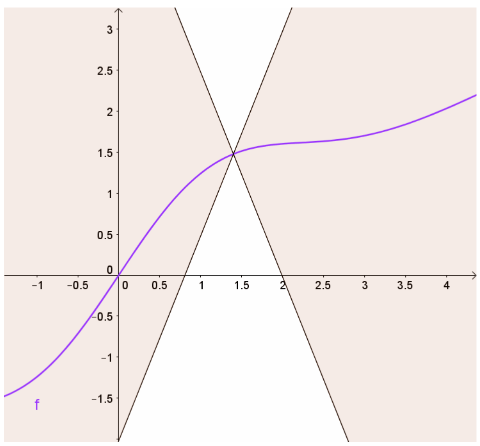

Figure 1: Geometry of Lipschitz continuity [Picture from the *Wikipedia* page on *Lipschitz continuity*]

and from the appearance of speed of light in (1) we can see a profound connection between geometry of Lipschitz functions space and that of Minkowski, i.e. *light cones of special relativity and 'Lipschitz cones' are deeply connected.*

4 Outline of a program

4.1 (n,m)-integrals and symbols

The most strikingg feature of (1) that is inevitable is its use of

$$\sqrt{d}x\sqrt{d}y,$$

whose understanding requires us to first consider

$$\sqrt{d}x,$$

i.e. **half-forms**; more generally *p-forms that* $p \in \mathbb{Q}$, *at least.* The integral

$$\int f(x,y)\sqrt{d}x\sqrt{d}y$$

is a hybrid between

$$\int f(x)dx,$$

and

$$\int f(x,y)dxdy,$$

but it is none exactly: it is *almost a double integral* in that it has two variables, but it is not! To be precise, we better define

Definition 1. ***Multipleness* of an integral**

$$m(d_1^{\alpha_1}x, d_2^{\alpha_2}y) := \sum_i \alpha_i. \tag{7}$$

We call a n-variable integral with multipleness m, a **(n,m)-ingegral**. As a special case for our purpose, which is a (2,1)-ingegral, we first define

Symbol. (1,1/2)-integral *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a smooth function;*

$$\textstyle\int_I f(x)\sqrt{d}x, \tag{8}$$

and then

Symbol. (2,1)-integral *Let* $z : I \times J \subset \mathbb{R}^2 \to \mathbb{R}$ *be a smooth function;*

$$\textstyle\int_J\int_I z(x,y)\sqrt{d}x\sqrt{d}y. \tag{9}$$

Using this notation, and

Definition 2.

$$A(x,y) := \begin{cases} 1, & x = y \\ -1, & x \neq y \end{cases} \tag{10}$$

we can write our metric (1) concisely

Definition 3. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *and* $g : J \subset \mathbb{R} \to \mathbb{R}$ *be speed-of-light–Lipschitz continuous functions. We define the 'distance' of the two functions by*

$$\boxed{s(f(x),g(y)) = \textstyle\int_J\int_I \sqrt{c^2(x-y)^2 + A(x,y)\big(f(x)-g(y)\big)^2}\,\sqrt{d}x\sqrt{d}y} \tag{11}$$

4.2 Requirements on a future theory of (n,m)-integrals

No coherent theory of these forms and integrals exists as far as I can see and it would be a naïve expectation to demand it from me now in this very paper! To talk quite formally $\int f(x,y)\sqrt{dxdy}$ requires us to make the n in

$$\int f\ d^n\mathbf{x} := \int f(x_1, ..., x_n)\ dx_1 \cdots dx_n$$

more than a mere notation; $n \in \mathbb{Q}$, maybe even $n \in \mathbb{R}$.
History of mathematics shows us that the way to make progress is to first *identify our expectations from a theory of rational-forms* and then construct a theory such that our expectations are satisfied. These expectations will then be seen to be theorems in the axiomatic system of the theory that we are after. Accordingly I name all the statements as 'theorems' but since they are not possible to prove now, until proved, all of them logically remain hypotheses or conjectures. In order to arrive at (1) consistently we seem to have used two theorems:

Theorem 1. *(11) is a metric.*

But for *what* space is the vital question that I prefer to leave open. (11) cannot be a simple metric for a function space for the reasons explained earlier, but all the requirements of a metric are easily seen to be satisfied except for the triangle inequality, and triangle inequality shows again that (11) is not a simple well-studied metric in the scope of current theory functional analysis: in that case we must have

$$d(f,h) \leq d(f,g) + d(g,h),$$

but again this triangle inequality is incomplete in that it does not pay attention to inputs of functions.

Theorem 2.

$$\int_I \int_I \sqrt{d}x\sqrt{d}x \equiv \int_I dx \tag{12}$$

Corollary (L^1 distance)**.** *In (11) let*

$$x = y,$$

then

$$\int_I \int_I |f(x) - g(x)|\sqrt{d}x\sqrt{d}x,$$

using theorem 2,

$$s(f,g) = \int_I |f(x) - g(x)|\ dx.$$

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