Unification Archimedes constant π, golden ratio φ, Euler's number e and imaginary number i

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10/10/2021

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Abstract

In the first chapter we will present the definitions,history,relations and arithmetics approximations of Archimedes constant π,golden ratio φ,Euler's number e and imaginary number i. A total of 220 expressions.
In the second chapter we will present exact expressions between Archimedes constant π,golden ratio φ,Euler's number e and imaginary number i. A total of 127 expressions.
In the third chapter we will present approximations expressions between Archimedes' constant π,the golden ratio φ,the number Euler e and the imaginary number i. A total of 112 expressions.

1. Basic mathematical constants

1.1 Archimedes constant π

Introduction

Archimedes constant π is a mathematical constant that expresses the ratio of the circumference C to the diameter δ of a circle. It is symbolized by the Greek letter π from the middle of the 18th century. It is referred to as Archimedes' constant because the method of determining π is attributed to Greek Mathematician Archimedes from the perimeter P_n of a regular polygon with n sides that describe a circle with diameter d. The number π is an implicit and transcendental number.
It appears in many types in all fields of mathematics and physics. It is found in many types of trigonometry and geometry,especially in terms of circles, ellipses or spheres. It is also found in various types from other disciplines, such as Cosmology,numbers,Statistics,fractals, thermodynamics,engineering,and electromagnetism. Number is celebrated on the "day of π" and the record of counting the digits of π is often mentioned in news headlines. Several people tried to memorize the value of π as accurately as possible,leading to a record of over 67.000 digits.

History of Archimedes constant π

The constant π has been known for almost 4,000 years. The ancient Babylonians calculated the area of a circle by taking 3 times the square of its radius, which gave a value of π=3. A Babylonian disk in the period 1.900–1.680 BC. indicates a value π=3,125. A papyrus in the period 1.650 BC. gives us a picture of the mathematics of ancient Egypt. The Egyptians calculated the area of a circle with a formula that gave the approximate value π=3,1605. The first calculation of π was made by Archimedes of Syracuse (287-212 BC.X.),one of the greatest mathematicians of the ancient world. Archimedes approached the area of a circle using the Pythagorean Theorem to find the areas of two regular polygons. Archimedes knew that he had not found the value of π but only an approximation within them. In this way,Archimedes showed that $3+10/71<\pi<3+1/7$. A similar approach was used by Zu Chongzhi (429–501),a brilliant Chinese mathematician and astronomer. Zu Chongzhi calculated the value of the ratio of the circumference of a circle to its diameter to be 355/113.

Definition

The constant is defined as:

$$\pi=C/\delta$$

C is the circumference C of a circle,
δ is the diameter of a circle.
This definition of π is valid only in flat Euclidean Geometry,while if extended to convex Non-Euclidean Geometries

the ratio does not remain constant. There are other definitions of π based on infinite calculus or trigonometry that are not based on a circle.

Numerical value of Archimedes constant π

To 50 decimal places the value of Archimedes constant π is:

π=3,14159265358979323846264338327950288419716939937510.........

In January 2.020,Timothy Mullican announced the calculation of 50 trillion digits in 303 days.

Exact types of Archimedes constant π

An exact formula for the constant π from the inverse tangents of the unit fractions is the Machin formula:

$\pi=16\cdot\tan^{-1}(1/5)-4\cdot\tan^{-1}(1/239)$

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Generalized Machin Types:

$\pi=20\cdot\tan^{-1}(1/7)+8\cdot\tan^{-1}(3/79)$

$\pi=4\cdot\tan^{-1}(1/\sqrt{2})+2\cdot\tan^{-1}(1/\sqrt{8})$

An insignificant monovalent Machin type formula is given by the identity:

$\pi=4\cdot\tan^{-1}1$

Also the type of Euler Machin:

$\pi=4\cdot\tan^{-1}(1/2)+4\cdot\tan^{-1}(1/3)$

Also Hermann's formula:

$\pi=8\cdot\tan^{-1}(1/2)-4\cdot\tan^{-1}(1/7)$

Still the Hutton guy:

$\pi=8\cdot\tan^{-1}(1/3)+4\cdot\tan^{-1}(1/7)$

Three Machin type terms include Gauss's Machin type:

$\pi=48\cdot\tan^{-1}18+32\cdot\tan^{-1}57-20\cdot\tan^{-1}239$

Also Strassnitzky's formula:

$\pi=4\cdot\tan^{-1}2-4\cdot\tan^{-1}5+4\cdot\tan^{-1}8$

Other types of Machin are:

$\pi=24\cdot\tan^{-1}8+8\cdot\tan^{-1}57+4\cdot\tan^{-1}239$

$\pi=16\cdot\tan^{-1}5-4\cdot\tan^{-1}70+4\cdot\tan^{-1}99$

$\pi=32\cdot\tan^{-1}10-4\cdot\tan^{-1}239-16\cdot\tan^{-1}515$

$\pi=20\cdot\tan^{-1}7+16\cdot\tan^{-1}53+8\cdot\tan^{-1}4.443$

$\pi=12\cdot\tan^{-1}4+28\cdot\tan^{-1}239+4\cdot\tan^{-1}1.985$

Finally other types of Machin by Borwein and Bailey are:

$\pi=48\cdot\tan^{-1}49+128\cdot\tan^{-1}57-20\cdot\tan^{-1}239+48\cdot\tan^{-1}110.443$

$\pi=176\cdot\tan^{-1}57+28\cdot\tan^{-1}239-48\cdot\tan^{-1}682+96\cdot\tan^{-1}1.296$

Approximations Archimedes constant π

The 68 arithmetics approximations of Archimedes' constant π are:

1) $\pi\approx 22/7=3{,}14$...........

2) $\pi\approx\sqrt{2}+\sqrt{3}\approx 3{,}14$...........

3) $\pi\approx 31^{1/3}=3{,}141$...........

4) $\pi\approx 19\cdot\sqrt{2}/16=3{,}141$...........

5) $\pi\approx(4/5)^{-4}+7/10=3{,}141$...........

6) $\pi\approx 9/5+9/5^{1/2}=3{,}141$...........

7) $\pi\approx(7/3)\cdot(1+\sqrt{3}/5)=3{,}141$.........

8) $\pi\approx 333/106=3{,}1415$.........

9) $\pi\approx[(40/3)-\sqrt{12})])^{1/2}=3{,}1415$.........

10) $\pi\approx (3/14)^{4}\cdot(193/5)^{2}=3{,}1415$...........

11) $\pi\approx(553/312)^{2}=3{,}1415$...........

12) $\pi\approx 7^{7}/4^{9}=3{,}1415$...........

13) $\pi\approx 43^{7/23}=3{,}1415$...........

14) $\pi\approx 3+[1/5\cdot[(2/5)^{1/10}+1/2]=3{,}14159$...........

15) $\pi\approx\ln 2.198/\sqrt{6}=3{,}14159$...........

16) $\pi\approx(296/167)^{2}=3{,}14159$...........

17) $\pi\approx 99^{2}/2.206\cdot\sqrt{2}=3{,}141592$...........

18) $\pi\approx 4-(106/166)^{1/3}=3{,}141592$...........

19) $\pi\approx(689/396)/\ln(689/396)=3{,}141592$...........

20) $\pi\approx(99/80)\cdot[7/(7-3\cdot\sqrt{2})]=3{,}141592$...........

21) $\pi \approx (66^3+86^2)/55^3$=3,141592..........

22) $\pi \approx (47^3+20^3)/30^3-1$=3,141592..........

23) $\pi \approx 689/396 \cdot \ln(689/396)$=3,141592.........

24) $\pi \approx [(150^{3/2}-1)/6]^{1/5}$=3,141592..........

25) $\pi \approx 355/113$=3,141592..........

With error accuracy $8 \cdot 10^{-8}$. This approximation is the most important because it is expressed in simple fractions and has relatively good accuracy.

26) $\pi \approx (66 \cdot \sqrt{2})/(33 \cdot \sqrt{29}-148)$=3,1415926..........

27) $\pi \approx 2+[1+(413/750)]^{1/2}$=3,1415926..........

28) $\pi \approx (13/4)^{1.181/1.216}$=3,1415926..........

29) $\pi \approx (2^{300}/6 \cdot 7^{103})^{1/5}$=3,1415926..........

30) $\pi \approx (2.143/22)^{1/4}$=3,14159265..........

31) $\pi \approx (77.729/254)^{1/6}$=3,14159265..........

32) $\pi \approx (9/67)^{1/2} \cdot \ln(5.280)$=3,14159265..........

33) $\pi \approx 103.283/32.876$=3,14159265..........

34) $\pi \approx 3 \cdot \ln(5.280)/\sqrt{67}$=3,14159265..........

35) $\pi \approx [9^2+(19^2/22)]^{1/4}$ =3,14159265..........

36) $\pi \approx [97+(9/22)]^{1/4}$=3,14159265..........

37) $\pi \approx \ln(\pi+20-9 \cdot 10^{-40})$=3,14159265..........

38) $\pi \approx 97^{272/1.087}$=3,14159265..........

39) $\pi \approx (3/\sqrt{67}) \cdot \ln(5.280)$=3,14159265..........

40) $\pi \approx 103.993/33.102$=3,141592653..........

41) $\pi \approx 22/17+37/47+88/83$=3,141592653..........

42) $\pi \approx (63/25)+(17+\sqrt{5})/(7+\sqrt{5})$=3,141592653..........

43) $\pi \approx (63/25) \cdot [(17+15 \cdot \sqrt{5})/(7+15 \cdot \sqrt{5})$=3,141592653..........

44) $\pi \approx [31+(62^2+14)/28^2]^{1/3}$=3,141592653..........

45) $\pi \approx 312.689/99.532$=3,141592653..........

46) $\pi \simeq 3+(1/7)-1/[790+(5/6)]=3{,}141592653$...........

47) $\pi \simeq [31+(25/3.983)]^{1/3}=3{,}141592653$...........

48) $\pi \simeq \ln[23+(1/22)+(2/21)]=3{,}141592653$...........

49) $\pi \simeq 427/596+405/167=3{,}141592653$...........

50) $\pi \simeq (48/23)\cdot\ln(60.318/13.387)=3{,}1415926535$...........

51) $\pi \simeq (125/123)\cdot\ln(28.102/1.277)=3{,}1415926535$.........

52) $\pi \simeq \{(19/60)+[1/(3\cdot 123.499)^{1/2}]\}^{-1}=3{,}1415926535$.........

53) $\pi \simeq [(19/60)+(3\cdot 123.449)^{-1/2}]^{-1}=3{,}1415926535$...........

54) $\pi \simeq 2+2^{2/41}\cdot(75.757/1.329)^{1/41}=3{,}14159265358$...........

55) $\pi \simeq (355/113)-[1/(44^{4}+533)]=3{,}1415926535897$...........

56) $\pi \simeq (3\cdot\sqrt{3}/10)\cdot\{5-[10/13-6\cdot(9)^{1/3}]^{1/2}\}^{-1}=3{,}1415926535897$...........

57) $\pi \simeq \{[7+\ln(7/12)]^{1/3}+1\}^{4}-64=3{,}1415926535897$...........

58) $\pi \simeq (355/113)\cdot[1-(3\cdot 10^{-4}/3.533)]=3{,}14159265358979$...........

59) $\pi \simeq (24/\sqrt{142})\cdot\ln[(10+11\cdot\sqrt{2})^{1/2}+(10+7\cdot\sqrt{2})^{1/2}/2]=3{,}14159265358979$...........

60) $\pi \simeq (12/\sqrt{130})\cdot\ln[(3+\sqrt{13})\cdot(\sqrt{8}+\sqrt{10})/2]=3{,}14159265358979$...........

61) $\pi \simeq 357.377/(784+105^{5/2}) =3{,}14159265358979$...........

62) $\pi \simeq (3/163^{1/2})\cdot\ln(640.320)=3{,}14159265358979$...........

63) $\pi \simeq [(7+\ln 7/12)^{1/3}+1]^{4}-64=3{,}14159265358979$...........

With error accuracy 2×10^{-16}. A relatively simple and very exact expression.

64) $\pi \simeq \ln(262.537.412.640.768.744)/\sqrt{163}=3{,}141592653589793$.........

65) $\pi \simeq (12/\sqrt{190})\cdot\ln[(3+\sqrt{10})\cdot(\sqrt{8}+\sqrt{10})/2]=3{,}141592653589793$...........

66) $\pi \simeq 2+[(276.694.819.753.963/56.647)^{1/158}/2^{1/79}]=3{,}1415926535897932$...........

67) $\pi \simeq \ln(640.320^{3}+744)/163^{1/2}=3{,}14159265358979323846264338327$...........

68) $\pi \simeq \ln[(640.320^{3}+744)^{2}-2\cdot 196.884]/(2\cdot 163^{1/2})$

$\pi=3{,}14159265358979323846264338327950288419716939937$.........

Astronomical approximations

A strange astronomical approximation is:

$\pi \simeq (1/w)\cdot[(13\cdot y-6\cdot w)/13\cdot y)+3\cdot w]$

where:
w =1 week,
y=1 year.
For y=365 days:

п=3,1415926 ......

It is accurate to the first 7 decimal places. For y=365,2425 days,which is the average of the Gregorian year:

п=3,14159 ......

It is accurate to the first 5 decimal places.

Approximations inequalities

Also the mathematical constant п can be approached from the inequalities:

$3+10/71<\pi<3+1/7$

This approach was demonstrated by Archimedes using a polygon with 96 sides.

1.2 Golden radio φ

Introduction

In mathematics,two quantities have a golden ratio φ if the ratio of their sum to the largest quantity is equal to the ratio of the largest quantity to the smallest. A number that is often encountered when taking distance ratios in simple geometric shapes such as the pentagon,the pentagram,the hexagon and the dodecahedron. The golden ratio φ is a transcendental number.

History of golden radio φ

The ancient Greek mathematicians studied for the first time the golden ratio φ,due to its frequent appearance in geometry. He discovered that the golden ratio was neither an integer nor a fraction. The golden ratio is important in the geometry of regular pentagrams and pentagons. The golden word was known to the Pythagoreans. In their secret symbol,the pentacle,the golden word appears on the sides of a star as well as on the quotient of the area of the regular pentagon with vertices at the edges of the pentagon to the area of the regular pentagon. Pheidias (500 BC-432 BC),the Greek sculptor and mathematician applied the golden ratio φ to the design of sculptures for the Parthenon. Plato (Circa 428 BC-347 BC),in his views on natural science and cosmology presented in "Timaeus'' considered the golden section as the most binding of all mathematical relations and the key to physics Euclid's Elements provide the first written definition of what we now call the golden ratio. Euclid cites one for dividing the line into end and middle ratio. In the 20th century,the golden ratio is represented by the Greek letter φ,from the initial letter of the sculptor Pheidias who is said to have been one of the first to use it in his works. The golden ratio was studied regionally during the next millennium (850–930) and used in the geometric calculations of pentagons and decagons. 18th century mathematicians Abraham de Moivre,Daniel Bernoulli and Leonhard Euler used a formula based on the golden ratio that finds the value of a Fibonacci number based on its placement in the sequence.

Numerical value of the golden ratio φ

To 50 decimal places the value of φ is:

φ=1,61803398874989484820458683436563811772030917980576...........

Expressions of the golden ratio φ

In Timaeus,Plato considered the golden ratio φ to be the union of all mathematical relations and the key to the physics of the Universe. The golden ratio φ is a root of the 2nd degree polynomial:

$P(x)=x^2-x-1$

Both the golden ratio and its conjugates are roots of the 4th degree equation:

$x^4-2\cdot x^3-x^2+2\cdot x+1=0$

The basic arithmetic relation applies to the golden ratio φ:

$\varphi=(\sqrt{5}+1)/2$

Other equivalent arithmetic relations for the golden ratio φ are:

$\varphi=[(\sqrt{5}+3)/2]^{1/2}$

$\varphi=[(5+\sqrt{5})/(5-\sqrt{5})]^{1/2}$

The following also apply:

$\varphi=1+\varphi^{-1}$

$\varphi=1+1/\varphi$

$\varphi=(2+\varphi^{-1})^{1/2}$

$\varphi^{-1}=(2-\varphi^{-1})^{1/2}$

$\varphi^2=\varphi+1$

$\varphi=1+\{[1/[1+(1/\varphi)]\}$

The trigonometric relations also apply to the golden ratio φ:

$\varphi=-2\cdot\sin(666)$

$\varphi=-2\cdot\cos(144)$

$\varphi=-2\cdot\cos(6\cdot6\cdot6)$

$\varphi=-\cos(6\cdot6\cdot6)-\sin(666)$

Also the golden ratio φ satisfies the iteration relation:

$\varphi^{2n}=\varphi^{2n-1}+\varphi^{2n-2}, n \in N^+$

The following applies to the golden ratio φ:

$1=\varphi^1-\varphi^{-1}=\varphi^3-\varphi^{-3}-\varphi^2-\varphi^{-2}=\varphi^2-\varphi=\varphi^{-3}+2\cdot\varphi^{-2}=(1/2)\cdot\varphi^{-5}+(5/2)\cdot\varphi^{-2}=1$

$2=2\cdot\varphi^1-2\cdot\varphi^{-1}=\varphi^1+\varphi^{-2}=\varphi^2-\varphi^{-1}=\varphi^{-5}+5\cdot\varphi^{-2}=2$

$3=\varphi^2+\varphi^{-2}=\varphi^4+\varphi^{-4}-\varphi^3+\varphi^{-3}=5\cdot\varphi^{-1}-\varphi^{-5}=3$

$4=\varphi^3-\varphi^{-3}=\varphi^5-\varphi^{-5}-\varphi^4-\varphi^{-4}=4\cdot\varphi^1-4\cdot\varphi^{-1}=2\cdot\varphi^1+2\cdot\varphi^{-2}=2\cdot\varphi^2-2\cdot\varphi^{-1}=2\cdot\varphi^{-5}+10\cdot\varphi^{-2}=4$

$5=(2\cdot\varphi-1)^2=(\varphi^2+1)\cdot(\varphi^{-2}+1)=\varphi^2+\varphi^1+2\cdot\varphi^{-2}=5$

$6=2\cdot\varphi^2+2\cdot\varphi^{-2}=\varphi^4+\varphi^{-4}-\varphi^1+\varphi^{-1}=6$

$7=\varphi^{4}+\varphi^{-4}=\varphi^{2}+\varphi^{-2}+\varphi^{3}-\varphi^{-3}=7$

$8=\varphi^{7}-13\cdot\varphi=5\cdot\varphi-\varphi^{-5}=\varphi^{1}-\varphi^{-1}+\varphi^{4}+\varphi^{-4}=8$

$9=2\cdot\varphi^{1}-2\cdot\varphi^{-1}+\varphi^{4}+\varphi^{-4}=\varphi^{5}-\varphi^{-5}-\varphi^{1}-\varphi^{-2}=\varphi^{5}-\varphi^{-5}-\varphi^{2}+\varphi^{-1}=9$

$10=2\cdot(2\cdot\varphi-1)^{2}=2\cdot(\varphi^{2}+1)\cdot(\varphi^{-2}+1)=10$

$11=\varphi^{5}-\varphi^{-5}=\varphi^{7}-\varphi^{-7}-\varphi^{6}-\varphi^{-6}=11$

$12=\varphi^{1}-\varphi^{-1}+\varphi^{5}-\varphi^{-5}=12$

$18=\varphi^{6}+\varphi^{-6}=\varphi^{8}+\varphi^{-8}-\varphi^{7}+\varphi^{-7}=18$

$29=\varphi^{7}-\varphi^{-7}=\varphi^{9}-\varphi^{-9}-\varphi^{8}-\varphi^{-8}=29$

$47=\varphi^{8}+\varphi^{-8}=\varphi^{10}+\varphi^{-10}-\varphi^{9}+\varphi^{-9}=47$

$76=\varphi^{9}-\varphi^{-9}=\varphi^{11}-\varphi^{-11}-\varphi^{10}-\varphi^{-10}=76$

$123=\varphi^{10}+\varphi^{-10}=\varphi^{12}+\varphi^{-12}-\varphi^{11}+\varphi^{-11}=123$

$199=\varphi^{11}-\varphi^{-11}=\varphi^{13}-\varphi^{-13}-\varphi^{12}-\varphi^{-12}=199$

$322=\varphi^{12}+\varphi^{-12}=\varphi^{14}+\varphi^{-14}-\varphi^{13}+\varphi^{-13}=322$

$521=\varphi^{13}-\varphi^{-13}=\varphi^{15}-\varphi^{-15}-\varphi^{14}-\varphi^{-14}=521$

$843=\varphi^{14}+\varphi^{-14}=\varphi^{16}+\varphi^{-16}-\varphi^{15}+\varphi^{-15}=843$

$1.364=\varphi^{15}-\varphi^{-15}=\varphi^{17}-\varphi^{-17}-\varphi^{16}-\varphi^{-16}=1.364$

$2.207=\varphi^{16}+\varphi^{-16}=\varphi^{18}+\varphi^{-18}-\varphi^{17}+\varphi^{-17}=2.207$

$3.571=\varphi^{17}-\varphi^{-17}=\varphi^{19}-\varphi^{-19}-\varphi^{18}-\varphi^{-18}=3.571$

$5.778=\varphi^{18}+\varphi^{-18}=5.778$

$9.349=\varphi^{19}-\varphi^{-19}=9.349$

$271.443=\varphi^{26}+\varphi^{-26}=271.443$

$3.010.349=\varphi^{31}-\varphi^{-31}=3.010.349$

$12.752.043=\varphi^{34}+\varphi^{-34}=12.752.043$

The following also applies to the golden ratio φ:

$\varphi^{-10}=89-55\cdot\varphi=\varphi^{-11}+\varphi^{-12}=\varphi^{-8}-\varphi^{-9}$

$\varphi^{-9}=34\cdot\varphi-55=\varphi^{-10}+\varphi^{-11}=\varphi^{-7}-\varphi^{-8}$

$\varphi^{-8}=34-21\cdot\varphi=\varphi^{-9}+\varphi^{-10}=\varphi^{-6}-\varphi^{-7}$

$\varphi^{-7}=13\cdot\varphi-21=\varphi^{-8}+\varphi^{-9}=\varphi^{-5}-\varphi^{-6}$

$\varphi^{-6}=13-8\cdot\varphi=\varphi^{-7}+\varphi^{-8}=\varphi^{-4}-\varphi^{-5}$

$\varphi^{-5}=5\cdot\varphi-8=\varphi^{-6}+\varphi^{-7}=\varphi^{-3}-\varphi^{-4}=5\cdot\varphi^{2}-13=(2+\varphi)^{2}-13=(\varphi^{2}+1)^{2}-13=[(\varphi^{2}+1)^{2}-2]^{1/2}$

$\varphi^{-4}=5-3\cdot\varphi=\varphi^{-5}+\varphi^{-6}=\varphi^{-2}-\varphi^{-3}$

$\varphi^{-3}=2\cdot\varphi-3=\varphi^{-4}+\varphi^{-5}=\varphi^{-1}-\varphi^{-2}$

$\varphi^{-2}=2-\varphi=\varphi^{-3}+\varphi^{-4}=1-\varphi^{-1}$

$\varphi^{-1}=\varphi-1=\varphi^{-2}+\varphi^{-3}=\varphi^{2}-2=(2-\varphi)^{1/2}$

$\varphi^{0}=1=0+1=\varphi^{-1}+\varphi^{-2}=\varphi-\varphi^{-1}$

$\varphi^{1}=\varphi=\varphi+0=1+\varphi^{-1}$

$\varphi^{2}=\varphi+1=\varphi^{-1}+2$

$\varphi^{3}=2\cdot\varphi+1=\varphi^{2}+\varphi=2\cdot\varphi^{2}-1=\varphi^{5}-\varphi^{4}$

$\varphi^{4}=3\cdot\varphi+2=\varphi^{3}+\varphi^{2}=\varphi^{6}-\varphi^{5}$

$\varphi^{5}=5\cdot\varphi+3=\varphi^{4}+\varphi^{3}=(\varphi+2)^{2}-2$

$\varphi^{6}=8\cdot\varphi+5=4\cdot\varphi^{3}+1=\varphi^{5}+\varphi^{4}$

$\varphi^{7}=13\cdot\varphi+8=\varphi^{6}+\varphi^{5}$

$\varphi^{8}=21\cdot\varphi+13=7\cdot\varphi^{4}1=\varphi^{7}+\varphi^{6}=3\cdot\varphi^{5}+2\cdot\varphi^{4}=3\cdot\varphi^{4}+5\cdot\varphi^{3}+2\cdot\varphi^{2}=8\cdot\varphi^{3}+5\cdot\varphi^{2}$

$\varphi^{10}=55\cdot\varphi+34=\varphi^{9}+\varphi^{8}$

$\varphi^{11}=89\cdot\varphi+55=\varphi^{10}+\varphi^{9}$

$\varphi^{12}=144\cdot\varphi+89=\varphi^{11}+\varphi^{10}$

$\varphi^{13}=233\cdot\varphi+144$

$\varphi^{14}=377\cdot\varphi+233$

$\varphi^{15}=610\cdot\varphi+377$

$\varphi^{16}=987\cdot\varphi+610$

$\varphi^{17}=1.597\cdot\varphi+987=\varphi^{18}-\varphi^{16}$

$\varphi^{18}=2.584\cdot\varphi+1.597$

$\varphi^{19}=4.181 \cdot \varphi+2.584$

$\varphi^{20}=6.765 \cdot \varphi+4.181$

Approximations expressions golden ratio φ

27 approximations expressions golden ratio φ:

1) $\varphi \approx 13/8=1,6$...........

2) $\varphi \approx 144/89=1,61$...........

3) $\varphi \approx 89/55=1,618$...........

4) $\varphi \approx 377/233=1,6180$...........

5) $\varphi \approx 521/322=1,6180$...........

6) $\varphi \approx 843/521=1,6180$...........

7) $\varphi \approx 1.597/987=1,61803$...........

8) $\varphi \approx 1.364/843=1,61803$...........

9) $\varphi \approx 4.181/2.584=1,61803$...........

10) $\varphi \approx 5.778/3.571=1,61803$...........

11) $\varphi \approx 2.207/1.364=1,618033$...........

12) $\varphi \approx 3.571/2.207=1,618033$...........

13) $\varphi \approx 2^{856/1.233}=1,618033$...........

14) $\varphi \approx 9.349/5.778=1,6180339$...........

15) $\varphi \approx 10.946/6.765=1,6180339$...........

16) $\varphi \approx (5.778)^{1/18}=1,6180339$...........

17) $\varphi \approx (3.571)^{1/17}=1,61803398$...........

18) $\varphi \approx (9.349)^{1/19}=1,61803398$...........

19) $\varphi \approx 46.368/28.657=1,618033988$...........

20) $\varphi \approx 121.393/75.025=1,618033988$...........

21) $\varphi \approx 317.811/196.418=1,6180339887$...........

22) $\varphi \approx (1.860.498)^{1/30}=1,618033988749$...........

23) $\varphi \approx (2.537.720.636)^{1/45}=1,618033988749894$...........

24) $\varphi \doteq (3.461.452.808.002)^{1/60}$=1,618033988749894..........

25) $\varphi \doteq (4.721.424.167.835.364)^{1/75}$=1,618033988749894..........

26) $\varphi \doteq (6.440.026.026.380.244.498)^{1/90}$=1,618033988749894..........

27) φ≐7.778.742.049/4.807.526.976=1,6180339887498948482..........

Also the golden ratio φ can be approached from the inequalities:

$8/5<\varphi<81/50$

Golden ratio conjugate Φ

The golden ratio conjugate Φ,also called the silver ratio,is the quantity:

$\Phi=1/\varphi$

$\Phi=\varphi-1$

$\Phi=\varphi^2-2$

$\Phi=(2-\varphi)^{1/2}$

The golden ratio conjugate Φ has value:

$\Phi=(\sqrt{5}-1)/2=2/(\sqrt{5}+1)$

To 50 decimal places the value of Φ is:

Φ=0,61803398874989484820458683436563811772030917980576 ....

The following applies to the conjunction Φ of the golden ratio φ:

$\Phi^3+2\cdot\Phi^2=1$

$\Phi^5+5\cdot\Phi^2=2$

1.3 Euler's number e

Introduction

The number e is an important mathematical constant,which is the base of the natural logarithm. It is the limit of the sequence $(1+1/n)^n$ as n approaches infinity,an expression derived from the study of compound interest. The number e is sometimes called Euler's number,after the Swiss mathematician Leonhard Euler or Napier's constant.However,Euler's choice of the symbol e is said to have been retained in his honor. The constant was discovered by the Swiss mathematician Jacob Bernoulli while studying compound interest.

Like the constant π,e is irrational (that is,it cannot be represented as a ratio of integers) and transcendental (that is, it is not a root of any non-zero polynomial with rational coefficients). The number e is prominent in mathematics,along with 0,1,π and i. All five of these numbers play important and repetitive roles in mathematics and these five constants appear in a formulation of Euler's identity. Euler's number has many practical uses,especially in higher level mathematics such as calculus,differential equations,trigonometry,complex analysis,statistics,etc. From Euler's identity the following relation of the mathematical constant e can emerge:

$e=i^{-2i/\pi}$

History

The history of the mathematical constant e begins with John Napier (1.550-1.617) who defined logarithms through a process called dynamic ratio. However,this did not contain the constant itself, but simply a list of logarithms calculated from the constant. that the table was written by William Oughtred. The first known use of the constant,represented by the letter b,was the correspondence from Gottfried Leibniz to Christiaan Huygens in 1,690 and 1,691. However the discovery of the constant itself is credited to Jacob Bernoulli in 1.683 Leonhard Euler introduced the letter e as the basis for natural logarithms, writing in a letter to Christian Goldbach on November 25,1971. Leonhard Euler (1.707-1.783) named his letter after the constant e and discovered many of these remarkable properties.

Numerical value

To 50 decimal places the value of e is:

e=2,71828182845904523536028747135266249775724709369995 ....

Approximations expressions

26 approximations expressions of Euler's number e:

1) $e \approx 19/7=2,71$................

2) $e \approx 87/32=2,718$................

3) $e \approx 193/71=2,718$................

4) $e \approx 1.264/465=2,7182$............

5) $e \approx 878/323=2,7182$............

6) $e \approx (5/93^{1/\pi})^6=2,71828$............

7) $e \approx 3^{1/2}+4^{-1/100}=2,71828$............

8) $e \approx (355/113)-(69/163)=2,71828$............

9) $e \approx 2.721/1.001=2,718281$............

10) $e \approx (1/7)-[8\cdot\ln(4/5)/\ln2]=2,718281$............

11) $e \approx 3-(5/63)^{1/2}=2,718281$............

12) $e \approx 163^{32/163}=2,718281$............

13) $e \approx 23.225/8.544=2,7182818$............

14) $e \approx 2+(54^2+41^2)/80^2=2,7182818$............

15) $e \approx (1+5\cdot6^{1/5})/3=2,7182818$............

16) $e \approx (1+1/80^2)\cdot(1+1/2+1/3+1/4+1/5+1/6+1/7+1/8)=2,7182818$............

17) $e \approx (1+9^{-9})^{387.420.489}=2,71828182$............

18) $e \approx 49.171/18.089=2,718281828$............

19) $e \approx 271.801/99.990 = 2,718281828$............

20) $e \approx 517.656/190.435 = 2,7182818284$............

21) $e \approx [150-(87^3+12^5)/83^3]^{1/5} = 2,7182818284$............

22) $e \approx (37/613) \cdot [45+(35/991)] = 2,7182818284$............

23) $e \approx [8/(9.112.774^{1/4}-52)] = 2,71828182845$............

24) $e \approx 4-[(300^4-100^4-1.291^2+9^2)/91^5] = 2,718281828459$............

25) $e \approx [1.097-(55^5+311^3-11^3)/68^5]^{1/7} = 2,718281828459045$............

26) $e \approx 848.456.353/312.129.649 = 2,71828182845904523$............

Also the constant e can be approached from the inequalities:

$2+7/10<e<2+3/4$

1.4 Imaginary unit i

Introduction

The imaginary unit i is a solution to the quadratic equation $x^2+1=0$. Although there is no real number with this property,it can be used to extend the real numbers to what are called complex numbers,using addition and multiplication. Despite their misleading name,imaginary numbers are not only real but also very useful,with application in electricity,signal processing and many other applications. They are widely used in electronics,for the representation of alternating currents and in waves.

History

Although the Greek mathematician and engineer Hero of Alexandria is noted as the first to have conceived imaginary numbers,it was Rafael Bombelli who first set down the rules for multiplication of complex numbers in 1.572. The concept had appeared in print earlier,such as in work by Gerolamo Cardano. At the time,imaginary numbers and negative numbers were poorly understood and were regarded by some as fictitious or useless much as zero once was. Many other mathematicians were slow to adopt the use of imaginary numbers,including René Descartes,who wrote about them in his La Géométrie in which the term imaginary was used and meant to be derogatory. The use of imaginary numbers was not widely accepted until the work of Leonhard Euler (1.707–1.783) and Carl Friedrich Gauss (1.777–1.855). The geometric significance of complex numbers as points in a plane was first described by Caspar Wessel (1.745–1.818).

Expressions of imaginary unit i

The imaginary unit i has the ability:

$i^2=-1$

The square root of i is:

$\sqrt{i}=\pm(i+1)/2$

For the imaginary unit i:

$i^{4n}=1$

$i^{4n+1}=i$

$i^{4n+2}=-1$

$i^{4n+3}=-i$

η is any integer.Also:

$\cos i=(e+1/e)/2=e^2+1/2\cdot e$

$\sin i=(e-1/e)\cdot i/2=(e^2-1)\cdot i/2\cdot e$

1.5 Other numbers

Gelfond's constant

Gelfond's constant,in mathematics,is the number e^{π},e raised to the power π. Like e and π,this constant is a transcendental number. It was named after the Soviet mathematician Aleksandr Gelfond. Gelfond's constant were singled out in Hilbert's 7th problem as an example of numbers whose excess was an open problem. This was first established by Gelfond and may now be considered as an application of the Gelfond–Schneider theorem,noting that:

$e^{\pi}=(e^{i\pi})^{-i}=(-1)^{-i}$

The numerical value Gelfond's constant is:

$e^{\pi}\simeq 23{,}14069263277926900572$..........

17 approximations expressions Gelfond's constant:

1) $e^{\pi}\simeq 2\cdot(4\cdot\pi-1)=23{,}1$.....

2) $e^{\pi}\simeq 23+\pi^{1-e}=23{,}1$.......

3) $e^{\pi}\simeq 20+\pi=23{,}14$.....

4) $e^{\pi}\simeq 162/7=23{,}14$.....

5) $e^{\pi}\simeq \pi^2\cdot[3+(4+\sqrt{5})^{1/2}])^{1/2}=23{,}140$.........

6) $e^{\pi}\simeq 302\cdot\pi/41=23{,}140$......

7) $e^{\pi}\simeq 3.241\cdot\pi/440=23{,}1406$......

8) $e^{\pi}\simeq[29/42+(9.507)^{1/2}]=23{,}14069$......

9) $e^{\pi}\simeq 30+(1+\sqrt{2})-345/7=23{,}140692$......

10) $e^{\pi}\simeq \ln 3+(159/5)\cdot\ln 2=23{,}14069263$......

11) $e^{\pi}\simeq 7\cdot\pi+1+(-e^2+28\cdot e+189/634\cdot e)=23{,}1406926327$......

12) $e^{\pi}\simeq 6.112.351\cdot\pi/829.816=23{,}140692632779$......

13) $e^{\pi}\simeq 7\cdot\pi+(365.455/317.913)=23{,}140692632779$......

14) $e^{\pi}\simeq 17+\sqrt{3}+(172))^{1/4}+\ln(\ln 9)=23{,}1406926327792$......

15) $e^{\pi} \approx 20+\pi+1/[1.111+1/(11+1/\sqrt{2})]=23,1406926327792......$

16) $e^{\pi} \approx (421\cdot\pi-133\cdot\varphi-316)/(-176\cdot\pi+539\cdot\varphi-285)=23,14069263277926.........$

17) $e^{\pi} \approx (-381\cdot\pi+363\cdot\varphi+469\cdot e+127)/(44\cdot\pi+11\cdot\varphi+88\cdot e-361)=23,140692632779260057.........$

Number i^i

The number i^i is:

$i^i=(e^{i\pi/2})^i=e^{-\pi/2}=(e^{\pi})^{-1/2}$

The decimal expansion is given by:

$i^i \approx 0,20787957635076190854......$

Number π^e

The decimal expansion for π^e is given by:

$\pi^e \approx 22,45915771836104547342.........$

It is not known whether or not this number is transcendental.For Gelfond's constant e^{π} and number π^e:

$e^{\pi}/\pi^e+\pi^e/e^{\pi} \approx 2=2,000893.........$

Number $e^{\pi}-\pi^e$

The decimal expansion for $e^{\pi}-\pi^e$ is given by:

$e^{\pi}-\pi^e \approx 0,68153491441822353230........$

Other numbers

$\varphi^{\varphi} \approx 2,178457567938..........$

$\varphi^{\pi} \approx 4,53475716116............$

$\varphi^{e} \approx 3,69902532658..........$

$\pi^{\varphi} \approx 6,37951853648..........$

$e^{\varphi} \approx 5,04316564336...........$

$\varphi^{\varphi}\cdot\varphi^{\pi} \approx 9,8787760564777..........$

$\varphi^{e}\cdot e^{\pi} \approx 85,59800812326............$

$\varphi^{\pi}\cdot\pi^{e} \approx 101,84682653935...........$

$\Phi^{\pi}\cdot e^{\pi} \approx 104,93742163059...........$

2. Exact expressions between mathematical constants

2.1 Exact expressions between φ and π

47 exact expressions between golden ratio φ and Archimedes constant π:

1) $\varphi=2\cdot\cos(\pi/5)$

2) $\varphi=2\cdot\sin(3\cdot\pi/10)$

3) $\varphi=1/2\cdot\cos(2\pi/5)$

4) $\varphi=1/2\cdot\sin(\pi/10)$

5) $\varphi=-2\cdot\cos(4\cdot\pi/5)$

6) $\varphi=1-2\cdot\cos(3\cdot\pi/5)$

7) $\varphi=(1/2)\cdot\sec(2\cdot\pi/5)$

8) $\varphi=(1/2)\cdot\csc(\pi/10)$

9) $\varphi=\sin(3\cdot\pi/10)/\sin(\pi/6)$

10) $\varphi=\sin(\pi/6)/\sin(\pi/10)$

11) $\varphi=\sin(2\cdot\pi/5)/\sin(\pi/5)$

12) $\varphi=\sin(5\pi/6)/\sin(\pi/10)$

13) $\varphi=\sin(7\cdot\pi/10)/\sin(\pi/6)$

14) $(3-\varphi)^{1/2}=2\cdot\sin(\pi/5)$

15) $(\varphi\cdot\sqrt{5})^{1/2}=2\cdot\cos(\pi/10)$

16) $\cos(\pi/1+\varphi)+\cos(\pi/\varphi)=0$

17) $\varphi=[\pi+(5\cdot\pi^2)^{1/2}]/2\cdot\pi$

18) $(2-\varphi)^{1/2}=2\cdot\cos(2\cdot\pi/5)$

19) $[2-(1/\varphi)]^{1/2}=2\cdot\cos(3\cdot\pi/10)$

20) $[2+(1/\varphi)]^{1/2}=2\cdot\cos(\pi/5)$

21) $(2+\varphi)^{1/2}=2\cdot\cos(\pi/10)$

22) $(2-\varphi)^{1/2}=2\cdot\sin(\pi/10)$

23) $(2-\varphi^{-1})^{1/2}=2\cdot\sin(\pi/5)$

24) $(2+\varphi^{-1})^{1/2}=2\cdot\sin(3\cdot\pi/10)$

25) $(2+\varphi)^{1/2}=2\cdot\sin(2\cdot\pi/5)$

26) $[2+(2+\varphi)^{1/2}]^{1/2}=2\cdot\cos(\pi/20)$

27) $[2+(2+\varphi^{-1})^{1/2}]^{1/2}=2\cdot\cos(\pi/10)$

28) $[2+(2-\varphi^{-1})^{1/2}]^{1/2}=2\cdot\cos(3\cdot\pi/20)$

29) $[2+(2-\varphi)^{1/2}]^{1/2}=2\cdot\cos(\pi/5)$

30) $[2-(2-\varphi)^{1/2}]^{1/2}=2\cdot\cos(3\cdot\pi/10)$

31) $[2-(2-\varphi^{-1})^{1/2}]^{1/2}=2\cdot\cos(7\cdot\pi/20)$

32) $[2-(2+\varphi^{-1})^{1/2}]^{1/2}=2\cdot\cos(2\cdot\pi/5)$

33) $[2-(2+\varphi)^{1/2}]^{1/2}=2\cdot\cos(9\cdot\pi/20)$

34) $[2+(2+\varphi)^{1/2}]^{1/2}=2\cdot\sin(9\cdot\pi/20)$

35) $[2+(2+\varphi^{-1})^{1/2}]^{1/2}=2\cdot\sin(2\cdot\pi/5)$

36) $[2+(2-\varphi)^{1/2}]^{1/2}=2\cdot\sin(7\cdot\pi/20)$

37) $[2+(2-\varphi)^{1/2}]^{1/2}=2\cdot\sin(3\cdot\pi/10)$

38) $[2-(2-\varphi)^{1/2}]^{1/2}=2\cdot\sin(\pi/5)$

39) $[2-(2-\varphi^{-1})^{1/2}]^{1/2}=2\cdot\sin(3\cdot\pi/20)$

40) $[2-(2+\varphi^{-1})^{1/2}]^{1/2}=2\cdot\sin(\pi/10)$

41) $[2-(2+\varphi)^{1/2}]^{1/2}=2\cdot\sin(\pi/20)$

43) $\varphi^{-1}+(3\cdot\varphi\cdot\sqrt{5})^{1/2}=4\cdot\cos(\pi/15)$

44) $(\varphi\cdot\sqrt{5})^{1/2}=2\cdot\cos(\pi/10)$

45) $\varphi+(3\cdot\varphi^{-1}\cdot\sqrt{5})^{1/2}=4\cdot\cos(2\cdot\pi/15)$

46) $-\varphi^{-1}+(3\cdot\varphi\cdot\sqrt{5})^{1/2}=4\cdot\cos(4\cdot\pi/15)$

47) $-\varphi+(3\cdot\varphi^{-1}\cdot\sqrt{5})^{1/2}=4\cdot\cos(7\cdot\pi/15)$

2.2 Exact expressions between φ and e

11 exact expressions between golden ratio φ and Euler's number e:

1) $e=(2\cdot\varphi-1)^{2/\ln(5)}$

2) $\varphi=(1+e^{\ln5/2})/2$

3) $\varphi=e^{(1/2)\ln(\varphi/\varphi-1)}$

4) $e^{-[\ln(\varphi)+\ln(1-\varphi)]}+1=0$

5) $\varphi=e^{\ln(-1)/5}+e^{-\ln(-1)/5}$

6) $e^{\ln5/2}=2\cdot e^{(1/2)\ln(\varphi/\varphi-1)}-1$

7) $e^{\ln5/2}=2\cdot e^{\ln(-1)/5}+2\cdot e^{-\ln(-1)/5}-1$

8) $e^{(1/2)\ln(\varphi/\varphi-1)}=e^{\ln(-1)/5}+e^{-\ln(-1)/5}$

9) $\varphi=e^{[\ln(\varphi)+\ln(1-\varphi)]/5}+e^{-[\ln(\varphi)+\ln(1-\varphi)]/5}$

10) $e^{(1/2)\ln(\varphi/\varphi-1)}=e^{[\ln(\varphi)+\ln(1-\varphi)]/5}+e^{-[\ln(\varphi)+\ln(1-\varphi)]/5}$

11) $e^{[\ln(\varphi)+\ln(1-\varphi)]/1+\varphi}+e^{-[\ln(\varphi)+\ln(1-\varphi)]/1+\varphi}+e^{[\ln(\varphi)+\ln(1-\varphi)]/\varphi}+e^{-[\ln(\varphi)+\ln(1-\varphi)]/\varphi}=0$

2.3 Exact expressions between π and e

8 Exact expressions between Archimedes constant π and Euler's number e:

1) $e^{\ln5/2}=4\cdot\cos(\pi/5)-1$

2) $e^{\ln5/2}=4\cdot\sin(3\cdot\pi/10)-1$

3) $e^{\ln5/2}=1-4\cdot\cos(3\cdot\pi/5)$

4) $e^{\ln5/2}=\sec(2\cdot\pi/5)-1$

5) $e^{\ln5/2}=\csc(\pi/10)-1$

6) $2\cdot\sin(\pi/5)=[3-(1+e^{\ln5/2})/2)]^{½}$

7) $\cos[\pi/1+(1+e^{\ln5/2})/2)]+\cos[2\cdot\pi/(1+e^{\ln5/2})]=0$

8) $e^{\ln5/2}=\{[\pi+(5\cdot\pi^2)^{1/2}]/\pi\}-1$

2.4 Exact expressions between π and i

3 exact expressions between Archimedes constant π and imaginary unit i:

1) $\pi=2\cdot\ln i/i$

2) $\pi=2\cdot i\cdot\log(1-i/1+i)$

3) $\pi=-i\cdot\ln(-1)$

2.5 Exact expressions between φ and i

14 exact expressions between golden ratio φ and imaginary unit i:

1) $\varphi^{1/2}+i\cdot\varphi^{-1/2}=(1+2\cdot i)^{1/2}$

2) $\varphi^{3/2}+i\cdot\varphi^{-3/2}=2\cdot[1+(i/2)]^{1/2}$

3) $2\cdot\sin(i\cdot\ln\varphi)=i$

4) $\sin^2(i\cdot\ln\varphi)=-1/4$

5) $\varphi=[\ln(\cos\varphi+i\cdot\sin\varphi)]/i$

6) $2\cdot\sin(i\cdot\ln\varphi)=e^{-\ln\varphi}-e^{\ln\varphi}/i$

7) $2\cdot\sin(i\cdot\ln\varphi)=-i\cdot[(1/\varphi)-\varphi]$

8) $\varphi=2\cdot\cos(2\cdot\ln i/5\cdot i)$

9) $\varphi=2\cdot\sin(3\cdot\ln i/5\cdot i)$

10) $\varphi=1-2\cdot\cos(6\cdot\ln i/5\cdot i)$

11) $\varphi=(1/2)\cdot\sec(4\cdot\ln i/5\cdot i)$

12) $\varphi=(1/2)\cdot\csc(\ln i/5\cdot i)$

13) $2\cdot\sin(2\cdot\ln i/5\cdot i)=(3-\varphi)^{1/2}$

14) $\cos(2\cdot\ln i/i\cdot(1+\varphi)+\cos(2\cdot\ln i/i\cdot\varphi)=0$

2.6 Exact expressions between e and i

11 exact expressions between Euler's number e and imaginary unit i:

1) $\cos i=(e+1)/2\cdot e$

2) $\sin i=(e-1)\cdot i/2\cdot e$

3) $e^2-1=2\cdot i\cdot e\cdot\sin i$

4) $\sin i=e^{-1}-e/2\cdot i$

5) $e^{-1}-e=2\cdot i\cdot\sin i$

6) $e^{\ln5/2}=4\cdot\cos(2\cdot\ln i/5\cdot i)-1$

7) $e^{\ln5/2}=4\cdot\sin(3\cdot\ln i/5\cdot i)-1$

8) $e^{\ln5/2}=1-4\cdot\cos(6\cdot\ln i/5\cdot i)$

9) $e^{\ln5/2}=\sec(4\cdot\ln i/5\cdot i)-1$

10) $e^{\ln5/2}=\csc(\ln i/5\cdot i)-1$

11) $2\cdot\sin(2\cdot\ln i/5\cdot i)=[3-(1+e^{\ln5/2})/2]^{1/2}$

2.7 Exact expressions between π,φ and i

4 exact expressions between golden ratio φ,Archimedes constant π and imaginary unit i:

1) $\sin[(\pi/2)-i\cdot\ln\varphi]=\sqrt{5}/2$

2) $i\cdot\pi=\ln(\varphi)+\ln(1-\varphi)$

3) $\pi=-5\cdot i\cdot\log\{(1/2)\cdot[\varphi+i\cdot(4-\varphi^{2})^{1/2}]\}$

4) $\varphi=(2\cdot\varphi-1)^{2i\pi/5\ln(5)}+(2\cdot\varphi-1)^{-2i\pi/5\ln(5)}$

2.8 Exact expressions between φ,e and i

3 exact expressions between golden ratio φ,Euler's number e and imaginary unit i:

1) $e^{i\varphi}=\cos\varphi+i\cdot\sin\varphi$

2) $\varphi=e^{2\ln i/5}+e^{-2\ln i/5}$

3) $e^{i\varphi}=\cos(e^{2\ln i/5}+e^{-2\ln i/5})+i\cdot\sin(e^{2\ln i/5}+e^{-2\ln i/5})$

2.9 Exact expressions between φ,π and e

3 exact expressions between golden ratio φ,Archimedes constant π and Euler's number e:

1) $2\cdot\cos(\pi/5)=e^{(1/2)\ln(\varphi/\varphi-1)}$

2) $2\cdot\cos(\pi/5)=e^{[\ln(\varphi)+\ln(1-\varphi)]/5}+e^{-[\ln(\varphi)+\ln(1-\varphi)]/5}$

3) $\varphi=e^{\{\ln[2\cos(\pi/5)]+\ln(2\varphi-1)\}/5}+e^{-i\pi\{\ln[2\cos(\pi/5)]+\ln(\varphi-1)\}/5}$

2.10 Exact expressions between π,e and i

10 exact expressions between Archimedes constant π,Euler's number e and imaginary unit i:

1) $e^{i\pi}+1=0$

2) $e^{2i\pi}-1=0$

3) $e=i^{-2i/\pi}$

4) $i^{i}=(e^{i\pi/2})^{i}$

5) $i^{i}=e^{-\pi/2}$

6) $i^{i}=(e^{\pi})^{-1/2}$

7) $e^{i\pi}=\cos\pi+i\cdot\sin\pi$

8) $e^{-\pi}=(-1)^{i}$

9) $e^{\pi}=i^{-2i}$

10) $e^{2i\pi/5}-e^{4i\pi/5}-e^{6i\pi/5}+e^{8i\pi/5}=\sqrt{5}$

2.11 Exact expressions between π,φ,e and i

16 exact expressions that connects golden ratio φ,Archimedes constant π,Euler's number e and imaginary unit i:

1) $e^{i\pi}+\varphi=\varphi^{-1}$

2) $e^{i\varphi}=e^{i(\varphi+2\pi)}$

3) $e^{i\varphi}=e^{i\varphi+2i\pi}$

4) $e^{i\varphi}=\cos(\varphi+2\cdot\pi)+i\cdot\sin(\varphi+2\cdot\pi)$

5) $\varphi=e^{i\pi/5}+e^{-i\pi/5}$

6) $\varphi=i\cdot[(e^{i\pi/10})/(1-e^{-i\pi/5})]$

7) $e^{i\pi/10}-e^{-i\pi/10}=i/\varphi$

8) $2\cdot e^{i\pi/5}=\varphi+i\cdot(3-\varphi)^{1/2}$

9) $2\cdot e^{i\pi/5}=\varphi+i\cdot(\sqrt{5}/\varphi)^{1/2}$

10) $\varphi=2\cdot e^{i\pi/5}-2\cdot i\cdot\sin(\pi/5)$

11) $e^{i\pi/10}=(i/2\cdot\varphi)+(1-1/4\cdot\varphi^{2})^{1/2}$

12) $\varphi=(e^{2i\pi/5}-e^{4i\pi/5}-e^{6i\pi/5}+e^{8i\pi/5}+1)/2$

13) $\varphi^{2}=(e^{i\pi/5}+e^{-i\pi/5})^{2}$

14) $\varphi^{2}=(e^{2i\pi/5}+2+e^{-2i\pi/5})^{2}$

15) $\varphi^{2}=-(e^{4i\pi/5}+e^{-24i\pi/5})^{2}+1$

16) $\varphi^{2}=(e^{i\pi/5}+e^{-i\pi/5})+1$

These expressions are the most important in mathematics because they connect the four basic mathematical constants.

2.12 Exact Magic expression that connects six basic mathematical constants 0,1,π,φ,e and i

The following expression connects six basic mathematical constants,the number 0,the number 1,golden ratio φ,Archimedes constant π,Euler's number e and imaginary unit i:

$e^{i\pi/1+\varphi}+e^{-i\pi/1+\varphi}+e^{i\pi/\varphi}+e^{-i\pi/\varphi}=0$

3. Approximations expressions between mathematical constants

3.1 Approximations expressions between φ and π

25 approximations expressions that connects the golden ratio φ and Archimedes constant π:

1) $\pi \simeq \varphi^{(\pi+\varphi)/2} = 3{,}14..........$

2) $\varphi \simeq 2\cdot(\log_{\varphi}\pi)-\pi = 1{,}61.........$

3) $\pi \simeq 4/\sqrt{\varphi} = 3{,}14..........$

4) $\pi \simeq 6\cdot\varphi^{5}/(2\cdot\varphi^{5}-1) = 3{,}141.........$

5) $\varphi \simeq (\pi/2\pi-6)^{1/5} = 1{,}618...........$

6) $\pi \simeq 3\cdot(\varphi^{-3}+5)/5 = 3{,}141.........$

7) $\pi \simeq 6\cdot\varphi^{2}/5 = 3{,}141.........$

8) $\varphi \simeq (5\cdot\pi/6)^{1/2} = 1{,}6180........$

9) $(12/7)\cdot\log\pi-2\cdot\log\varphi \simeq 1 = 0{,}99997044..........$

10) $[2\cdot\varphi^{-2}-\varphi^{-3}]-[(\sqrt{2}-1)\cdot 4/\pi] \simeq 0 = 0{,}000470957..........$

11) $\pi \simeq (6/5)\cdot\log 2+(24/5)\cdot\log\varphi = 3{,}14159.........$

12) $\pi \simeq (48\cdot\varphi-37)/8\cdot\varphi = 3{,}141592.........$

13) $\varphi \simeq 37/(48-8\cdot\pi) = 1{,}6180339..........$

14) $\pi \simeq [(802\cdot\varphi-801)/(602\cdot\varphi-601)]^{4} = 3{,}1415926.........$

15) $\pi \simeq (2^{7}\cdot 5^{2}\cdot 4.463/9.181\cdot\varphi)^{1/6} = 3{,}1415926.........$

16) $\varphi \simeq 2^{7}\cdot 5^{2}\cdot 4.463/9.181\cdot\pi^{6} = 1{,}6180339..........$

17) $\pi \simeq (9.764.064\varphi+4.867.832/944.055\varphi+566.433)^{1/2} = 3{,}141592653............$

18) $\varphi \simeq 4.867.832-566.433\cdot\pi^{2}/944.055\cdot\pi^{2}-9.764.064 = 1{,}618033988........$

19) $\varphi \simeq \ln(3\cdot\pi\cdot 22^{3}/35)/\ln[(111\cdot\pi^{2}/8)+(2/21)] = 1{,}61803398874...........$

20) $\varphi \simeq (355/113\cdot\pi)^{4/565}\cdot 9349^{2/35} = 1{,}61803398874...........$

21) $\pi \simeq (35/3\cdot 22^{3})\cdot[360\cdot\varphi^{-2}-2\cdot\varphi^{-3}+(3\cdot\varphi)^{-5}]^{\varphi} = 3{,}141592653............$

22) $\pi \simeq (14.212.169/12^{6})-\varphi = 3{,}1415926535897.............$

23) $\varphi \simeq (14.212.169/12^{6})-\pi = 1{,}6180339887498...........$

24) $\varphi \simeq 151.837.964\cdot\pi/294.810.267 = 1{,}6180339887498948482...........$

25) $\varphi \simeq 11\cdot(40\cdot\pi^{2}+23\cdot\pi-70)/(502\cdot\pi^{2}-659\cdot\pi-185) = 1{,}6180339887498948482...........$

3.2 Approximations expressions between π and e

45 approximations expressions that connects Archimedes constant π and Euler's number e:

1) $\pi \approx 2\cdot e/\sqrt{3}=3,1$..........

2) $\pi \approx 128/15\cdot e=3,1$..........

3) $\pi \approx 7\cdot e^{\cos(5/2)}=3,14$..........

4) $\pi \approx (163)^{1/2}/(e)^{1/3}-6=3,14$.........

5) $\pi \approx (4\cdot e-1)^{1/2}=3,14$..........

6) $e \approx 9-2\cdot \pi=2,71$.........

7) $\pi \approx e+\tan(2/5)=3,141$.........

8) $\pi \approx 3/\sin(3\cdot e)=3,141$...........

9) $\pi \approx (2\cdot e^3+e^8)^{1/7}=3,141$.........

10) $\pi \approx [e/(\pi-e)^{3/2}]^{1/2}=3,141$.........

11) $e \approx (\pi+1)/\pi^{1/e}=2,718$...........

12) $\pi^9/e^8 \approx 10=9,999838798$..........

13) $e^\pi-\pi \approx 20=19,999099979$..........

14) $3^{(\pi+e)/4} \approx 5=4,999973083$............

15) $e^6-\pi^4-\pi^5 \approx 0=0,000017673$.........

16) $(\pi+e+163)^{1/2} \approx 13=12,994609439$...........

17) $e^\pi-\pi^{1-e} \approx 23=23,00081238$..........

18) $e^\pi-2\cdot(4\cdot\pi-1) \approx 0=0,007951404$.........

19) $(163/e)+(e/163)+(\pi/163)-60=0,000299061$.........

20) $e^\pi \approx \pi^2\cdot[3+(4+\sqrt{5})^{1/2}])^{1/2}=23,140406827$........

21) $e^\pi/\pi^e+\pi^e/e^\pi \approx 2=2,000893$...........

22) $\pi \approx 2\cdot 7/(5-e/5)=3,1415$.........

23) $\pi \approx \ln[(7-e^2)/2]=3,14159$.........

24) $e \approx (5/93^{1/\pi})^6=2,71828$.............

25) $\pi \approx [(2+e^e)^{1/2}/e]^e=3,14159$..........

26) $\pi \approx (e^2/3)+(19/28)=3,14159$..........

27) $\pi \approx (69/163)+e=3,14159$..........

28) $\pi \simeq (920/157)-e=3,14159........$

29) $\pi \simeq \ln[e \cdot \ln(4.979)]=3,141592........$

30) $\pi \simeq [\ln\ln(97+7/5)]^{e}=3,141592.........$

31) $\pi \simeq e^{75.709/66.137}=3,1415926..........$

32) $\pi \simeq 2+e^{709/5.354}=3,1415926........$

33) $e \simeq 42 \cdot \pi \cdot [(\pi/2)-\ln(3 \cdot \pi/2)]=2,71828182..........$

34) $e \simeq 2.100 \cdot (10/629)^{1/2}/\pi^{4}=2,71828182..........$

35) $\pi \simeq 10 \cdot \tanh(28 \cdot \pi/15)-(\Pi^{9}/e^{8}) \simeq 0=0,0000000006005..........$

36) $e^{\Pi}-\ln 3-(159/5) \cdot \ln 2 \simeq 0=0,00000000023..........$

37) $e \simeq (1,000924472) \cdot [2 \cdot (4/\pi)^{8}]^{(\pi/4)4}=2,718281828...........$

38) $(3.730 \cdot \pi^{5}/e)-(879 \cdot e^{3}/\pi^{4})-419.736 \simeq 0=0,0000000001622...........$

39) $e^{-\pi/9}+e^{-4\pi/9}+e^{-9\pi/9}+e^{-16\pi/9}+e^{-25\pi/9}+e^{-36\pi/9}+e^{-49\pi/9}+e^{-64\pi/9}-1 \simeq 0=0,0000000000010504.....$

40) $e^{\Pi}-7 \cdot \pi-1-(-e^{2}+28 \cdot e+189/634 \cdot e) \simeq 0,000000000000377311.......$

41) $e^{\Pi}-7 \cdot \pi-(365.455/317.913) \simeq 0=0,00000000000002805671.......$

42) $e^{\Pi}-\{20+\pi+1/[1.111+1/(11+1/\sqrt{2})]\} \simeq 0=0,000000000000063..........$

43) $e^{\Pi}-[17+\sqrt{3}+(172))^{1/4}+\ln(\ln 9)] \simeq 0=0,000000000000025...........$

44) $e \simeq [\pi-(12/\ln 889)^{2}]^{-1/4}$

With numerical value for Euler's number e=2,7182818284590........

45) $\pi \simeq (12/\ln 889)^{2}+e^{-4}$

With numerical value for Archimedes constant π=3,14159265358979...........
We observe that these relatively simple relationships with error 2×10^{-19} have amazing accuracy.

3.3 Approximations expressions between φ and e

11 approximations expressions that connects golden ratio φ and Euler's number e:

1) $\varphi \simeq (5 \cdot e+4)/4 \cdot e=1,61..........$

2) $e \simeq 4/(4 \cdot \varphi-5)=2,71...........$

3) $e \simeq \varphi^{2}+\varphi^{-5}+\varphi^{-10}+\varphi^{-13}=2,7182.........$

4) $e \simeq \varphi^{4/(\varphi+1)} \cdot 2^{1/(\varphi+1)}=2,7182.........$

5) $\varphi \approx (2^{6/e}-4)^{-1}=1,6180$............

6) $e \approx 6 \cdot 7 \cdot \varphi/5^2=2,7182$........

7) $\varphi \approx 5^2 \cdot e/6 \cdot 7=1,6180$............

8) $1/e+1/\varphi+1/71 \approx 1=0,999997937$...........

9) $1/\sqrt{e}+1/\varphi^2+1/87 \approx 1=0,999990924$...........

10) $\varphi \approx [\sqrt{63} \cdot (3-e)+1]/2=1,61803$.........

11) $\varphi \approx (711 \cdot e^2-906 \cdot e-487)/(175 \cdot e^2-56 \cdot e+283)=1,6180339887498948482$.......

3.4 Approximation expression between π and i

1 approximations expressions that connects Archimedes constant π and imaginary unit i:

1) $(\pi+20)^i \approx -1$

3.5 Approximations expressions between φ,π and e

28 approximations expressions that connects golden ratio φ,Archimedes constant π and Euler's number e:

1) $e \approx (\varphi/e^{\pi}-\pi^{e})^{\pi/e}=2,71$........

2) $\varphi \approx 357 \cdot \pi-412 \cdot e=1,61$.........

3) $e \approx 3 \cdot \varphi-\pi+1=2,71$.........

4) $(\varphi^2+e^2)/\pi^2-(3/2)^{12}/2^7 \approx 0=0,000286948$.........

5) $\varphi \approx e-(\pi)^{1/12}=1,618$.........

6) $e \approx \pi+\varphi+(1-\pi \cdot \varphi)/2=2,718$........

7) $\pi \approx [(6 \cdot e \cdot \varphi^3)/7]^{1/2}=3,141$..........

8) $\pi \approx [\varphi^2+e^2-(1/e)^2]^{1/2}=3,141$.......

9) $\pi \approx 5 \cdot \varphi \cdot e/7=3,141$.......

10) $e \approx \pi^{12/7}/\varphi^2=2,7182$........

11) $e \approx [(2 \cdot \pi^2+2 \cdot \pi-\varphi-1)/3]-\pi \cdot \varphi=2,7182$..........

12) $e \approx 7 \cdot \pi^2/6 \cdot \varphi^3=2,7182$...........

13) $\varphi \approx [7 \cdot \pi^2/6 \cdot e]^{1/3}=1,6180$........

14) $\varphi \approx 7 \cdot \pi/5 \cdot e=1,6180$.......

15) $e \approx 7 \cdot \pi/5 \cdot \varphi=2,7182$........

16) $e \approx (6 \cdot \varphi-2 \cdot \pi+2 \cdot \pi \cdot \varphi)/5=2,71828$........

17) $\varphi \approxeq [5\cdot(1+\pi)/e]-6=1,6180339........$

18) $\pi \approxeq [e\cdot(\varphi+6)/5]-1=3,1415926.........$

19) $\varphi \approxeq 14\cdot(e^{19}/10^{2}\cdot\pi^{22})^{1/5}=1,6180339.........$

20) $\pi \approxeq (2\cdot e/(e-1)-[(2\cdot\varphi-1)/(100-36/10^{-5})]=3,1415926535.........$

21) $\varphi \approxeq 14\cdot(e^{19}/10^{2}\cdot\pi^{22})^{1/5}\cdot(10\cdot e^{8}/\pi^{9})^{1/531}=1,6180339887..........$

22) $(\varphi+e)^{314.393.388.446.503}/\pi^{402.909.889.100.000}\approxeq 1,0000000000000018..........$

23) $(\varphi+e))^{314.393.388.446.503}/^{402.909.889.100.000}\approxeq \pi=3,14159265358979.........$

24) $\varphi \approxeq (1.967.981\cdot\pi-314.270\cdot e)/3.293.083=1,618033988749894...........$

25) $e \approxeq 1.400\cdot\varphi-2.410\cdot\pi+1.082/906\cdot\varphi-1.016\cdot\pi+172=2,71828182845904523..........$

26) $\varphi \approxeq -\pi^{2}+2\cdot\pi-3\cdot e-13+\sqrt{2}-\log2-\log3/-3\cdot\pi^{3}-\pi-e+7\cdot\sqrt{2}+7\cdot\sqrt{3}-3\cdot\log2$

With numerical value for golden ratio $\varphi=1,618033988749894848...........$

27) $\varphi \approxeq (1/42)\cdot[22\cdot\pi^{2}-56\cdot e^{2}+50\cdot\pi-34\cdot e-5\cdot(1+e^{2})^{1/2}+7\cdot(1+e)^{1/2}+20\cdot(1+\pi^{2})^{1/2}-24\cdot(1+\pi)^{1/2}-1]$

With numerical value for golden ratio $\varphi=1,6180339887498948482...........$

28) $e \approxeq (45.483\cdot\varphi-2.961\cdot\pi-18.765)/16.748$

This approximation has the highest accuracy with a numerical value for the Euler's number $e=2,71828182845904523536 0..........$

3.6 Approximation expression between i,π and e

1 approximation expression that connects Archimedes constant π,Euler's number e and imaginary unit i:

1) $\pi^{ie}-1\approxeq 2=1,99977658867.............$

3.7 Approximation expression between φ,i,π and e

1 approximation expression that connects golden ratio φ,Archimedes constant π,Euler's number e and imaginary unit i:

$\varphi^{2}+e^{2}+(i/e)]^{2}\approxeq \pi^{2}$

Conclusions

We presented new exact and approximations expressions that connect the golden ratio φ,Archimedes constant π,Euler's number e and imaginary unit i.

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