Elliptic Equations of Heat Transfer and Diffusion in Solids

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Abstract

We propose an alternative system of equations for the heat transfer and diffusion in solids, which leads to the second-order elliptical equations describing evolution of temperature and concentration fields with finite rate of propagation. The comparison of heat and mass transfer within the frames of parabolic and elliptic equations are discussed.

1 Introduction

The process of heat transfer in solid is described by a parabolic equation based on two assumptions. The first is the continuity of heat propagation

$$\frac{\partial q}{\partial t} + (\nabla \cdot \mathbf{q}) = 0, \tag{1}$$

where q is the volume density of heat, $\mathbf{q}$ is the volume density of heat flux. The second assumption is Fourier's law, which establishes the relationship between heat flux and temperature gradient

$$\mathbf{q} = -\kappa \nabla \theta, \tag{2}$$

where κ is the coefficient of thermal conductivity, θ is a temperature. On the other hand, for the systems that do not perform mechanical work we have

$$dq = c\rho d\theta, \tag{3}$$

where c is specific heat capacity of material, ρ is the mass density. Substituting (2) and (3) into equation (1), we obtain the classical heat equation [1] in the following form:

$$\frac{\partial \theta}{\partial t} - \alpha \Delta \theta = 0, \tag{4}$$

where $\alpha = \dfrac{\kappa}{c\rho}$ is the coefficient of thermal diffusivity.

However the disadvantage of the Fourier law of thermal conductivity (2) is that it leads us to an equation of parabolic type (4), which describes the instantaneous propagation of heat in space [2,3]. However, this contradicts the physical nature of the heat transfer process and the theory of relativity, which requires a finite transfer rate of physical interactions. To overcome this drawback, the following modification of Fourier law was proposed [4-7]:

$$\tau \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -\kappa \nabla \theta, \tag{5}$$

where τ is relaxation time. This modification takes into account the inertia of the heat transfer process and leads us to the Cattaneo-Vernotte wave equation of hyperbolic type [4-7]:

$$\tau \frac{\partial^2 \theta}{\partial t^2} + \frac{\partial \theta}{\partial t} - \alpha \Delta \theta = 0, \tag{6}$$

which is widely discussed in a literature [8-24]. However, eliminating the paradox of instantaneous heat propagation [2,8,9], this equation leads to other paradoxical results associated with the wave nature of processes such as interference of temperature waves, their reflection from the boundaries of the body and the formation of shock heat waves [10-24].

A similar situation occurs in the diffusion of impurities in solids. The continuity condition

$$\frac{\partial n}{\partial t} + (\nabla \cdot \mathbf{n}) = 0, \tag{7}$$

(here n is the impurity concentration, $\mathbf{n}$ is the diffusive flux) combined with Fick's law

$$\mathbf{n} = -D \nabla n, \tag{8}$$

(here D is diffusion coefficient) leads us to the parabolic equation for the diffusion flow

$$\frac{\partial n}{\partial t} - D \Delta n = 0. \tag{9}$$

However, despite the fact that the diffusion and heat transfer equations are the same the hyperbolic diffusion equation and diffusion waves are not discussed in a literature.

In the present paper, we propose an alternative approach to the description of heat and diffusive mass transfer, which leads to an elliptic second order equations describing a different dynamics of heat propagation and diffusion in solids.

2 Modified equation of heat conduction

In hydrodynamics the partner to continuity condition is the equation for the mass flux acceleration (Euler equation) [25]. Based on this analogy we can suppose a similar equation for the heat flux. But unlike hydrodynamics, the proposed equation should describe the deceleration of heat flux and can be written as

$$\frac{1}{s^2}\frac{\partial \mathbf{q}}{\partial t} - \nabla q = 0, \tag{10}$$

where parameter s is a rate of heat propagation. Further, we assume that $s = const$, which is determined by the properties of the material. In addition, we take into account that the circulation of the heat flux in a closed loop should be equal to zero. Then the complete system of equations describing the heat transfer process is written in the following form:

$$\frac{\partial q}{\partial t} + (\nabla \cdot \mathbf{q}) = 0, \tag{11}$$

$$\frac{1}{s^2}\frac{\partial \mathbf{q}}{\partial t} - \nabla q = 0, \tag{12}$$

$$[\nabla \times \mathbf{q}] = 0. \tag{13}$$

The equation (11) of this system is, as before, the equation of continuity. The equation (12) shows that heat flux acceleration is directed along the heat gradient (deceleration) and describes the process of relaxation of the heat flux. Equation (13) shows that the heat flow is the vortex free.

The system (11) - (13) can be transformed to the following elliptic equations

$$\frac{1}{s^2}\frac{\partial^2 q}{\partial t^2} + \Delta q = 0, \tag{14}$$

$$\frac{1}{s^2}\frac{\partial^2 \mathbf{q}}{\partial t^2} + \Delta \mathbf{q} = 0, \tag{15}$$

which resemble wave equations in form, but describe non-propagating damped waves. Assuming (3), from equation (14) we have the following elliptical equation for the temperature field $\theta(\mathbf{r}, t)$:

$$\frac{1}{s^2}\frac{\partial^2 \theta}{\partial t^2} + \Delta \theta = 0, \tag{16}$$

which is an alternative to the parabolic equation (4).

3 Comparison of parabolic and elliptic equations of heat transfer

Let us compare the parabolic and elliptic equations in detail. As one can see directly the stationary states described by equations (4) and (16) coincide. The differences are only in the dynamics of the arrival of the physical system to these states. Parabolic equation (4) admits a solution in the form of plane waves

$$\theta = A \exp\left[i\omega t + i(\mathbf{k} \cdot \mathbf{r})\right] \tag{17}$$

with dispersion law

$$i\omega \;= -\,\alpha k^2. \tag{18}$$

Here ω is the frequency, $\mathbf{k}$ is wave vector ($k = |\mathbf{k}|$). Relation (18) shows that solution (17) is damping function.

On the other hand, elliptic equation (16) also admits solutions in the form of plane waves (17), but with different dispersion dependence

$$i\omega = \pm\, sk. \tag{19}$$

Note that only solutions with $i\omega = \; -\, sk$ are physically meaningful.

In particular, let us consider the cooling a plate with thickness l uniformly heated to a temperature θ^* and with zero temperature at the boundaries. The solution to this problem in the frame of parabolic equation is expressed by the following Fourier series [1]:

$$\theta_p = \frac{4\theta^*}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n{+}1)}\sin\frac{(2n{+}1)\,\pi x}{l}\exp\left[-d_{np}t\right] \tag{20}$$

with decrement of temperature damping

$$d_{np} = \frac{\alpha\,(2n+1)^2\,\pi^2}{l^2}. \tag{21}$$

Hence, it can be seen that harmonics with high numbers n decay rapidly that contributes to the rapid equalization of sharp temperature gradients, and the cooling process is mainly determined by the zero harmonic:

$$\theta_{p0} = \frac{4\theta^*}{\pi}\sin\frac{\pi x}{l}\exp\left[-\frac{\alpha\pi^2}{l^2}t\right]. \tag{22}$$

On the other hand, the solution to the problem of cooling the plate in the case of elliptical equation (16) is expressed by the following series:

$$\theta_e = \frac{4\theta^*}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n{+}1)}\sin\frac{(2n{+}1)\,\pi x}{l}\exp\left[-d_{ne}t\right] \tag{23}$$

with decrement of temperature damping

$$d_{ne} = \frac{s\,(2n+1)\,\pi}{l}. \tag{24}$$

The zero harmonic is expressed as follows:

$$\theta_{e0} = \frac{4\theta^*}{\pi}\sin\frac{\pi x}{l}\exp\left[-\frac{s\pi}{l}t\right]. \tag{25}$$

Thus, comparing damping parameters (21) and (24) one can see that in case of elliptical equation the higher harmonics decay more slowly than in case of parabolic equation.

4 Elliptic equations of diffusion flow in a solid

By analogy, an equation describing the decceleration of the diffusion flux can be written in the following form:

$$\frac{1}{a^2}\frac{\partial \mathbf{n}}{\partial t} - \nabla n = 0, \tag{26}$$

where parameter a is a speed of diffusion. Also we take into account that the circulation of the diffusion flux in a closed loop should be equal to zero. Then the complete system of equations describing the diffusion process is written in the following form:

$$\frac{\partial n}{\partial t} + (\nabla \cdot \mathbf{n}) = 0, \tag{27}$$

$$\frac{1}{a^2}\frac{\partial \mathbf{n}}{\partial t} - \nabla n = 0, \tag{28}$$

$$[\nabla \times \mathbf{n}] = 0. \tag{29}$$

The system (27) - (29) can be transformed to the following elliptic equations

$$\frac{1}{a^2}\frac{\partial^2 n}{\partial t^2} + \Delta n = 0, \tag{30}$$

$$\frac{1}{a^2}\frac{\partial^2 \mathbf{n}}{\partial t^2} + \Delta \mathbf{n} = 0, \tag{31}$$

which describe the diffusion mass transfer with finite rate.

5 Conclusion

Thus, equations (10) and (26) are the alternative laws of variation of the heat and diffusive fluxes, which lead to a second-order differential equation of elliptic type, describing the evolution of the temperature and concentration fields. Solutions of elliptic equation have the same spatial distributions of temperature and concentration fields as in the case of parabolic equation, but describe a different dynamics of field propagation. In contrast to the hyperbolic equation, which describes the propagation of heat in the form of harmonic waves, the elliptic equation predicts the diffusion propagation of heat with a finite transfer rate. We believe that from a physical point of view, the elliptic equation is more suitable for describing heat transfer than parabolic and hyperbolic equations, as well as for the description of impurity diffusion.

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