Inexplicability of the Beth's experiment within the framework of Maxwell's electrodynamics

R I Khrapko[1]
Physics Department, Moscow Aviation Institute, Moscow 125993, Russia

It is shown that the electromagnetic field in the well-known Beth's experiment contains no linear momentum. This means that the angular momentum of the field in the Beth's experiment is zero, since the angular momentum, by definition, is the moment of the linear momentum. Nevertheless, the half-wave plate in the Beth's experiment receives an angular momentum from the field, which, by definition, does not have this angular momentum. This means that the definition of the angular momentum of an electromagnetic field should be changed to explain the Beth's experiment. The angular momentum of an electromagnetic field contains a spin term that does not depend on linear momentum.
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1. Introduction

The well-known Beth's experiment [1] proves that circularly polarized light contains an angular momentum, as predicted by Sadowsky [2] and Poynting [3]. According to Beth's idea, a beam of circularly polarized light passes through a half-wave plate, which changes the chirality of the light and, accordingly, changes the direction of rotation of the electromagnetic vectors and the direction of the angular momentum of light to opposite directions. As a result, by virtue of the angular momentum conservation law, the half-wave plate receives twice the amount of angular momentum contained in the beam. However, as it was just proved at the conference [4], in the Beth's experiment, in reality, there is no rotation of electromagnetic mass-energy. Moreover, there is no mass-energy flow at all. The Poynting vector $\mathbf{E}\times\mathbf{H}$ and the linear momentum density $\varepsilon_0\mathbf{E}\times\mathbf{B}$ are equal to zero in the Beth's apparatus. This has also been proven earlier [5-8]. As a consequence, the electromagnetic field in the Beth's apparatus has no angular momentum, according to the existing definition of the angular momentum of an electromagnetic field [9-13]

$$\mathbf{J} = \varepsilon_0\int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV\ . \qquad (1)$$

According to definition (1), the angular momentum of an electromagnetic field is the moment of the linear momentum of the field, and it is equal to zero in the Beth's apparatus. So the experimentally recorded transfer of the angular momentum of light to the plate occurs in the absence of any angular momentum in the light, according to (1). Therefore, the receipt of the angular momentum by the half-wave plate from the electromagnetic field in the Beta experiment is inexplicable within the framework of definition (1).

The point is that in the Beth's experiment, light that has passed through a half-wave plate passes through it a second time after being reflected from a mirror covered with a quarter-wave plate (Fig. 4). And such a mirror does not change the chirality of light when reflected. Therefore, the light that has passed through the half-wave plate returns to it after reflection with the same chirality. But circularly polarized beams of the same chirality, having the opposite direction, create the opposite rotation of electromagnetic vectors. Therefore, when such beams interference in the Beth's apparatus, a pulsation of the field vectors arises without rotation at any point in space around the half-wave plate. This is shown in Section 2 by a simple calculation. Therefore, to explain the Beth's experiment, a revision of the definition (1) of the angular momentum of the electromagnetic field is required. An explanation of the Beta experiment is given in Section 3.

[1] Email: khrapko_ri@hotmail.com, khrapko_ri@mai.ru , http://khrapkori.wmsite.ru

2. Poynting vector in the Beth's experiment.

A simple model of a right-handed circularly polarized light beam directed along z-axis with the plane phase front was proposed by Jackson [10]:

$$\mathbf{E}_1 = \exp(iz - it)[\mathbf{x} + i\mathbf{y} + \mathbf{z}(i\partial_x - \partial_y)]E_0(r), \quad r^2 = x^2 + y^2, \tag{2}$$

$$\mathbf{H}_1 = \exp(iz - it)[-i\mathbf{x} + \mathbf{y} + \mathbf{z}(\partial_x + i\partial_y)]E_0(r), \tag{3}$$

Here $\mathbf{E}_1$ and $\mathbf{H}_1$ are complex vectors of the electromagnetic field, $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are unit coordinate vectors. ∂_x, ∂_y mean partial derivatives with respect to *x* and *y*. For simplicity, $\omega = k = c = \varepsilon_0 = \mu_0 = 1$. Index 1 in (2), (3) means that the formulas describe the primary beam after passing through the half-wave plate. The beam amplitude is indicated by $E_0(r)$. The function $E_0(r)$ is considered constant throughout the entire beam area, that is, under the condition $r < R$, where R is the radius of the beam. However, on the surface of the beam, where $r \approx R$, the function $E_0(r)$ quickly decreases to zero.

We mark the reflected beam incident on the plate with index 2. The beam has the same helicity as the primary beam passing through the plate (that is, it has the same mutual direction of momentum and spin). Therefore, formulas for it are obtained from formulas (2), (3) by changing the signs of z and y:

$$\mathbf{E}_2 = \exp(-iz - it)[\mathbf{x} - i\mathbf{y} - \mathbf{z}(i\partial_x + \partial_y)]E_0(r), \tag{4}$$

$$\mathbf{H}_2 = \exp(-iz - it)[-i\mathbf{x} - \mathbf{y} - \mathbf{z}(\partial_x - i\partial_y)]E_0(r) \tag{5}$$

Adding the primary and reflected beams and writing out explicitly the real parts of the complex expressions, we obtain the components of the resulting electromagnetic field

$$E_x = \Re[\exp(iz - it) + \exp(-iz - it)]E_0 = 2E_0 \cos z \cos t, \tag{6}$$

$$E_y = \Re[i\exp(iz - it) - i\exp(-iz - it)]E_0 = -2E_0 \sin z \cos t, \tag{7}$$

$$\begin{aligned} E_z &= \Re[\exp(iz - it)(i\partial_x - \partial_y) + \exp(-iz - it)(-i\partial_x - \partial_y)]E_0 \\ &= -2(\sin z\partial_x + \cos z\partial_y)E_0 \cos t \end{aligned} \tag{8}$$

$$H_x = \Re[-i\exp(iz - it) - i\exp(-iz - it)]E_0 = -2E_0 \cos z \sin t, \tag{9}$$

$$H_y = \Re[\exp(iz - it) - \exp(-iz - it)]E_0 = 2E_0 \sin z \sin t, \tag{10}$$

$$\begin{aligned} H_z &= \Re[\exp(iz - it)(\partial_x + i\partial_y) + \exp(-iz - it)(-\partial_x + i\partial_y)]E_0 \\ &= 2(\sin z\partial_x + \cos z\partial_y)E_0 \sin t \end{aligned}, \tag{11}$$

and the resulting electromagnetic field in the vector form is

$$\mathbf{E} = 2[\mathbf{x}\cos z - \mathbf{y}\sin z) - \mathbf{z}(\sin z\partial_x + \cos z\partial_y)]E_0 \cos t, \tag{12}$$

$$\mathbf{H} = -2[\mathbf{x}\cos z - \mathbf{y}\sin z) - \mathbf{z}(\sin z\partial_x + \cos z\partial_y)]E_0 \sin t. \tag{13}$$

It can be seen that the electric and magnetic fields are parallel to each other everywhere. Therefore, the Poynting vector is zero everywhere. There is no movement. There is no momentum.

3. Classical spin

Concerning electromagnetic radiation of circular polarization, **Poynting** expressed the idea that there is an angular momentum of the electromagnetic field, independent of the linear momentum: “If we put *E* for the energy in unit volume and *G* for the torque per unit area, we have $G = E\lambda / 2\pi$” [3, p. 565]. This statement proclaims the existence of a *density* of angular momentum that is not related to linear momentum.

With such an angular momentum in mind, independent of linear momentum, **Weyssenhoff** introduced the concept of a spin liquid [14]: “By spin-fluid we mean a fluid each element of which possesses besides energy and linear momentum also a certain amount of angular momentum, proportional – just as energy and the linear momentum – to the volume of the element”. Thus, according to Poynting, circularly polarized electromagnetic radiation is a spin liquid.

Hehl writes that spin is not associated with a motion of matter [15]. "The current density in Dirac's theory can be split into a convective part and a polarization part. The polarization part is determined by the spin distribution of the electron field. It should lead to *no* energy flux in the rest system of the electron because the genuine spin 'motion' take place only within a region of the order of the Compton wavelength of the electron". This is especially important for us because there is no movement in the Beth's experiment.

The Variation Principle automatically takes into account that the total angular momentum of an electromagnetic field consists of an orbital part and a spin part [16-18]. The total angular momentum is

$$J^{\lambda\mu} = \int (r^{\lambda}T^{\mu\nu} - r^{\mu}T^{\lambda\nu})dV_{\nu} + \int \Upsilon^{\lambda\mu\nu}dV_{\nu}\,, \tag{14}$$

where $T^{\mu\nu}$ is the energy-momentum tensor and

$$\Upsilon^{\lambda\mu\nu} = -A^{\lambda}F^{\mu\nu} + A^{\mu}F^{\lambda\nu} \tag{15}$$

is the spin tensor. The orbital part (1) in the Beta experiment is zero.

The sense of the spin tensor (15) is as follows. The component Υ^{ijt} is a volume density of spin. This means that $dS^{ij} = \Upsilon^{ijt}dV$ is the spin of electromagnetic field inside the spatial element dV. The component Υ^{ijk} is the flux density of spin flowing in the direction of the x^k axis. For example, $dS_z / dt = dS^{xy} / dt = d\tau^{xy} = \Upsilon^{xyz}da_z$ is the *z*-component of spin passing through the surface element da_z per unit time, i.e. the torque acting on the surface element. I.e. Υ^{xyz} is the Poynting's *G*, and spin density is proportional to energy density. The spin tensor is now successfully used to calculate the spin angular momentum of light [19-26]. However, there is no spin tensor in Maxwell's electrodynamics. The role of the Belinfante-Rosenfeld procedure [27,28] in the annihilation of the spin tensor (15) is analyzed in detail [29,30].

Let us now calculate the spin flux in the resulting electromagnetic field (12), (13) adjacent to the Beth's half-wave plate on one side. We calculate sequentially: first we calculate the vector potential $\mathbf{A}$, and then, using formula (15), the component Υ^{xyz} of the spin tensor $\Upsilon^{\lambda\mu\nu}$,

$$\mathbf{A} = -\int \mathbf{E}dt = -2(\mathbf{x}\cos z - \mathbf{y}\sin z)\, E_0 \sin t\,, \tag{16}$$

$$\Upsilon^{xyz} = -A^x F^{yz} + A^y F^{xz} = A^x H_x + A^y H_y = 4E_0^2 \sin^2 t\,,\ \ <\Upsilon^{xyz}> = 2E_0^2\,. \tag{17}$$

A similar calculation for the other side of the plate gives the same result. Thus, as a result of the existence of the spin flows to two sides, the plate receives the resultant torque

$$\tau_{\text{tot}} = 4\pi R^2 E_0^2 = 4P\,, \tag{18}$$

where P represents the power of the beam. This is consistent with the outcome of the Beth's experiment.

4. Illustrations

Now we illustrate the content of this article. See please. Figures 1, 2, 3 present the interference of an incident beam and the beam reflected by an ordinary mirror. Figures 5, 6, 7 present the interference in the Beth's apparatus.

In Figure 1, the left helix of the right-hand circular polarization wave moves translationally upward along the z-axis with the speed of light V. Electric field E is represented by red arrows, magnetic field H is represented by blue arrows. The right half of the figure shows a side view of the wave. The crosses in circles represent the tails of the arrows. The dots within the circles represent the noses of the arrows. The left half of the figure shows cross sections of the wave by xy-planes at three different locations. The direction of rotation of the E-H pair of vectors observed at these locations is shown. The spins of the photons are directed along the z-axis; we say that they have +z-spin. The direction of the spins coincides with the direction of the wave velocity. Therefore, the spin flux is positive. At the same time, this means the existence of a downward –z-spin flux. The spin flux situation is similar to the momentum flux situation, i.e. to pressure situation. Positive pressure in a vertical cylinder means that the +z-directed momentum passes through the upper end of the cylinder and, at the same time, the –z-momentum passes down through the lower end of the cylinder. But we do not know how to depict a flow in the picture.

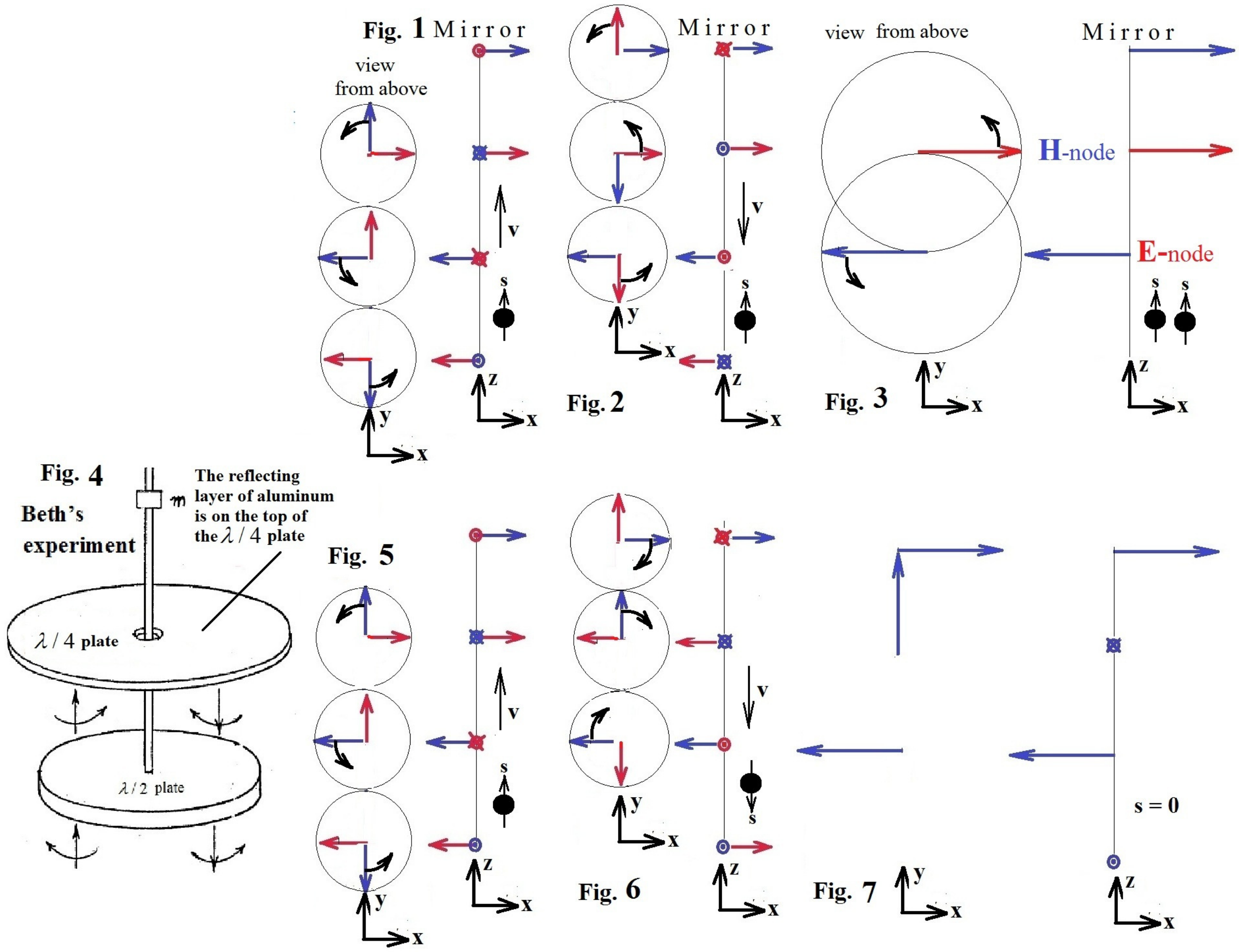

Figure 2 shows the same wave after reflection from an ordinary mirror. It moves in the opposite +z-direction. But the pair of vectors E-H rotates in the same way as in Figure 1. Accordingly, the direction of the photon spins remains + z. However, the speed changes direction to the opposite direction. Therefore, the spin flux is negative. This is analogous to negative pressure. This wave has left-hand circular polarization.

Figure 3 shows the resulting standing wave of circular polarization. The total vectors E and H rotate in the same way as in Figures 1 and 2. However, now the E and H-fields have nodes. In some places there is no electric field, and the magnetic field is doubled, in other places there is no magnetic field, and the electric field is doubled. The volume density of the spin is doubled and still has the +z direction. But the spin flux is zero. Spin is without spin flux! This is natural, because the spin flux onto the mirror is zero. The average speed of movement of the electromagnetic mass-energy is zero.

Figure 4 shows a portion of the Beth's apparatus. We are considering the space between the half-wave plate and the quarter-wave plate with mirror sputtering.

Figure 5 shows the same wave of right-hand circular polarization as in Fig. 1. The direction of the photon spins coincides with the direction of the velocity and with the direction of the +z-spin flux. So, we have a -z-spin flux down again. But now this wave is used in the Beth's experiment. It emerges from the half-wave plate and is directed at the mirror covered with a quarter-wave plate.

Figure 6 shows the wave of Figure 5 after reflection from the mirror covered with the quarter-wave plate. When reflected from such a mirror, the wave passes through the quarter-wave plate twice. Therefore, the quarter wave plate plays the role of a half wave plate. But a half-wave plate changes the chirality of the transmitted wave to the opposite one. Therefore, in contrast to the ordinary reflection as in Figure 2, in the Beth's experiment, the reflected wave retains the right-hand circular polarization. Its speed is downward. Spin of photons is directed downward as well. It is the –z-spin. We have a -z-spin flux down. Thus, the total flux of the –z-spin down to the half-wave plate is doubled. The plate experiences a torque corresponding to this doubled flux from the considering space. This torque is directed against the z-axis.

The standing electromagnetic wave arising in the Beth's experiment (Figure 7) differs significantly from the usual standing wave shown in Figure 3. There are no nodes in such a wave. For example, at the depicted time moment, there is a doubled magnetic field in all space. Over time, this magnetic field, without changing its direction, is replaced by an electric field, because the vectors of both fields are obtained by adding the vectors of the primary and reflected waves, which have opposite rotation. In this case, the vectors E and H of the fields always and everywhere coincide in direction. This means that the Poynting vector is identically zero. There is no rotation, and even no movement of the electromagnetic mass-energy. All fluxes are equal to zero, except for the spin flux. In this case, the volume density of the spin is equal to zero due to the fact that the spins of the primary and reflected waves have the opposite direction.

5. Conclusion

The concept of spin of electromagnetic radiation, which goes back to Sadowsky & Poynting [2,3], according to which the angular momentum is proportional to the electromagnetic energy, requires the spin term to be present in the definition of the angular momentum of electromagnetic radiation (14). Definition (1) is not correct. The Beth's experiment proves this.

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