Invalidating Cantor's Continuum Hypothesis and Solving Hilbert's #1 Problem II

Stephane H. Maes[1]

v2, October 23, 2022 (July 15, 2021)

Abstract:

In this short paper, we provide a mathematical proof that in set theory, developed in a mathematical universe following the ZFC axioms, Cantor's continuum hypothesis does not hold: the cardinality of the continuous set of all reals is $\mathfrak{c}$, and not $\aleph_1$, i.e., there are infinity $\aleph_1$ (and maybe more than one) between $\mathfrak{c}$, the cardinality of the continuum, and the cardinality of the infinite set of naturals, $\aleph_0$.

The proof is derived from combinatorics, relying on ZFC solely for the model of Cantor and Gödel defining $\aleph_0$. It provides input to the still unresolved first of Hilbert famous 23 math problems of interest.

This paper, resolves the first of the 23 Hilbert problems with invalidation of the continuum hypothesis.

1. Introduction

The context of the of this discussion can be found in [1], that describes the continuum hypothesis (term typically used instead of conjecture) of Cantor's and Gödel analysis [2, 5,6]. It is also the still unresolved first of Hilbert famous 23 math problems of interest [4]. It is formulated as:

> *The continuum hypothesis is that there is no set whose cardinality is strictly between that of the integers and that of the real numbers.* *(1)*

Some argued that the work of Gödel [2] then Cohen [7] would have resolved it. But this merely hinted that the conjecture cannot be proven, or disproven, within ZFC (as well as without the axiom of choice) [3], and assuming that ZFC is consistent. So the continuum hypothesis is independent of ZFC [8], that's all we know so far. Our proof does not rely on ZFC, other than for the definition of the cardinality of $\mathbb{N}$, and elementary set theory.

[1] is motivated by recent progresses in complementing ZFC with additional axioms to resolve the dilemma. Two were proposed: Martin's principle [9] and (*) [10-12]. It initially looked like different mathematical universes would exist depending on what additional axioms are added to ZFC to validate or invalidate the continuum hypothesis [1,8-12]. A recent result [13], shows that this may not be the case and seems to favor the invalidity of the continuum hypothesis [1]. Yet none of these work settle the continuum hypothesis.

[1] shmaes.physics@gmail.com

Cite as: Stephane H Maes, (2022), "Invalidating Cantor's Continuum Hypothesis and Solving Hilbert's #1 Problem II", viXra:2107.0143v2, https://vixra.org/pdf/2107.0143v2.pdf, https://shmaes.wordpress.com/2021/07/18/invalidating-cantors-continuum-hypothesis-and-solving-hilberts-1-problem/, October 23, 2022 (v1 was July 15, 2021).

We provide a proof that the cardinality of $\mathbb{R}$, noted as $\mathfrak{c}$, is not $\aleph_1$ as proposed by Cantor with his continuum conjecture / hypothesis [1,2].

This version corrects our usage, associated with the notations of $\aleph_n$ (for n > 0), and $\mathfrak{c}$, that we had incorrectly redefined, versus the commonly used definitions as in [15,16]. As a result, our proof also appeared incorrect, as it did not expand on the consequences of (4), but rather symbolically played with product of $\mathbb{N}$, an internal intermediate step of the proof without the whole setup, which lead to another unfortunate notation that could lead to several apparent incorrect statements, when taken at face value. The notation confusing $\mathfrak{c}$, $\aleph_1$ did not help knowledgeable readers. When the comment was made about the interpretation of our incorrect short end statement, we accepted the inappropriate notation and simplifications, and re-cast the proof more extensively, and used the accepted definitions for $\aleph_n$. The principle of the proof however is still exactly the same, just more detailed, extensive and rigorous in its notations. We are grateful to the reader who pointed out the confusions and resulting false statements.

2. Proof: Computing the cardinality of hard to describe set of Reals

In this paper we only rely on ZFC to allow the definition of the cardinality of the infinite set of naturals, $\aleph_0$, as set cardinality and the cardinality of all countable ordinal numbers , $\aleph_1$. Beyond that, we do not use ZFC.

The approach is not affected by the methodology to count the cardinality of $\mathbb{R}$ (e.g. à la [7]), or by the current new axioms and their compatibility or incompatibilities [8-12].

The sketch of the proof is:

- $\aleph_0$ is the cardinality of the naturals space, i.e. $\mathbb{N}$. $\aleph_0$ is the power of denumerably infinite sets [2].
- The cardinality of $\mathbb{R}$ is obtained as: $\aleph_0$ (for naturals before the decimal point) x $\aleph_0$ (for the position of first non-zero digit after the decimal point) x ($\aleph_0$ (for value of the following digits without 0, for the natural number that it consists of, i.e. cardinality of $\mathbb{N}$) + a looped repeat of previous steps after decimal point, now applies after the first non-zero) (2)
- => Cardinality of $\mathbb{R}$ is $\mathfrak{c} = \aleph_0 \times \aleph_0 \times \aleph_0 + (1 + \aleph_0 (1 + \dots (\dots))) / \aleph_0 \approx 2^{\aleph_0}$ (3), remembering that $\aleph_0 \times \aleph_0 = \aleph_0$: we have 2 multiplies $\aleph_0$ times.
- (3) validates the cardinality of power set of $\mathbb{N}$, as in [16-18], and validates the algorithm (2).

On the other hand:

- Consider a set of the points on $\mathbb{R}$ defined with algorithm of (2), but where at each level after the first iteration, a finite set of digits are set to zeros, and they are determined by a random function ℓ that also tends to set to zero more and more digits beyond a certain position associated to them in the real that is being built): S_ℓ (4)
  - Such a set S_ℓ is not countable (as it requires constructing it exactly as for the algorithm of (2)). There is no bijection between $\mathbb{N}$, and the set constructed by (4). (5)
  - Yet the cardinality of S_ℓ is strictly smaller than $\mathfrak{c}$, something seen when applying the construction algorithm. (6)

(6) proves that the continuum hypothesis is false.

(5) confirmed that hunches of others that if there was a way to invalidate the hypothesis, it had to be via complicated to describe real sets [14], as is S_ℓ. (5) holds because we can always a priori find such a ℓ and an associated set S_ℓ , where no continuum segment no matter what. Some sets S_ℓ may by random have continuous segments. A ℓ that would produce for example a set S_ℓ à la Cantor set would be unacceptable [19]. These would not be usable in the proof. So one should not just pick one set S_ℓ, but a suitable set S_ℓ that is not spurious from the point of view of the proof. If the picked S_ℓ has continuous segments, re-run the algorithm, of course, checking that ℓ is appropriate to implement (4). The non-countability of a suitable S_ℓ relies on the fact that we use the same algorithms as (2), and so inherits the same countability approach. This is the easiest way to argue and prove the countability of it. A ℓ that would produce for example a set à la Cantor set would be unacceptable [19].

QED.

3. Conclusions

We think that, while rather obvious, this reasoning is a huge step forward as it was not apparently understood so far if [1] is to be believed.

Indeed, we note indeed the essential independence from ZFC as expected, the absence of the need to add axioms, and the fact that the result appear true in mathematics, instead of possibly sometimes true and sometimes false depending on the axioms behind a model.

In fact, the work of Gödel and Cohen clearly identified that the continuum hypothesis is independent of the ZFC (or ZF per [8]). The mathematics community decided to therefore try to find additional axioms that can help decide. Our approach is different 1) somehow we dropped the need of axioms (other than as sustaining Mathematics and Logics with set theory) 2) showed in a framework that does not rely on them that in fact there is no degree of freedom: the continuum hypothesis is wrong when makes sense (i.e. defined); which of course maintains a link to set theory and ZFC/ZF axioms.

Of course, it would be of interest to see what additional axioms are actually equivalent to our proof. It is for future work or collaboration. On the basis of bypassing axioms, some may consider that it is an Physicist's or Engineer's proof. It is correct, that is what we have provided. We challenge others to help or produce the framework that they would desire, beyond this, to be satisfied.

On the basis of this paper, we argue that the first of the 23 Hilbert problems is now potentiall resolved with invalidation of the continuum hypothesis.

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