A Rigorous Examination on Cantor's Diagonal Argument

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Abstract

It was proved in this paper that diagonal argument actually only proves that the real numbers cannot be listed as complete, which has nothing to do with if the real numbers are countable or not. The confusion of the two different concepts - Countable and "complete listing" - is the reason why the diagonal argument fails. It was proved that real numbers are countable.

Keywords: mathematical foundation; diagonal argument; real numbers; uncountable; countable

1 Introduction

Diagonal argument has a history of more than 100 years. Although there have been controversies for a long time[1-4], it still enjoys a high reputation in mathematics up till now. All doubts about it have been ignored by the defenders of the argument[5].

In recent years, the author has discussed the diagonal argument with many authoritative figures, including famous academicians.

As some of the discussions between me and some well-known personalities are in-depth, extensive and complicated[6], it is difficult to reach a broad and in-depth consensus at the moment.

However, the author's latest work has found that there is actually no need for very complicated discussions.It is enough to overthrow Cantor's theory and silence all those who oppose me as long as use a similar to Cantor's , simple but clever and more rigorous, reasoning (see Theorems 1, 2 in this article). The logic that Cantor thought was as solid as a rock in fact is very weak. There was no way out other than to collapse in a single blow.

2 Cantor's diagonal argument

Cantor's diagonal argument is very simple (by contradiction):

Assuming that the real numbers are countable, according to the definition of countability, the real numbers in the interval [0,1) can be listed one by one:

$$a_1,a_2,a_3......\qquad (1)$$

Now write the above real numbers as

$$a_1=0.a_{11}a_{12}a_{13}...$$
$$a_2=0.a_{21}a_{22}a_{23}...$$
$$a_3=0.a_{31}a_{32}a_{33}...$$
$$\ldots\ldots$$

Among them, a_{ij} represents the j^{th} decimal place of the real number a_i.

Let

$$b=0.b_1b_2b_3\ldots\qquad (2)$$

and

$b_i \neq a_{ii}$ （i=1,2,3,…） （3）

Since eq.(3) guarantees that for any real number a_i, b has one decimal place, b_i, which is different from the i-th decimal place a_{ii} of the real number a_i, which ensures that for any real number a_i,

$a_i \neq b$ （i=1,2,3,…） （4）

It seems true that b is a real number not in eq.(1), which contradicts "According to the definition of countability, the real numbers in the interval [0, 1) can be listed one by one." Therefore, Cantor believed that he had proved that the real number is uncountable.

3 My recent work

In order to prevent logical cycles, when we question set theory, we should try to avoid using set theory viewpoints. For example, when discussing the diagonal argument, except for the countable definition, any other concepts of set theory are forbidden.

Cantor believed that finding a b outside eq.(1) proves that the real number is uncountable.

However, is this correct?

Clearly, under the countable assumption, the fact that the existence of b outside eq.(1) simplely means that eq.(1) does not contain all real numbers in the interval [0,1) completely.

On the other hand, under the countable assumption, since eq.(1) does not contain all real numbers, it is very normal to find a b outside eq.(1)! Therefore, the existence of b outside eq.(1) can not prove that the real numbers are uncountable!

Above very simple and clear analysis can be expressed more strictly as follows:

Theorem 1 Under the countable assumption, only a part of real numbers in the interval [0,1) can be listed.

Proof Under the countable assumption, if all the real numbers in [0,1) can be listed, then there must be a last real number in (1), so the diagonal can be extended to this last real number, and then the diagonal method can be used to construct a real number b that is not listed, which results in a contradictory. **Proof finished**

From the above proof, it can be seen that if there is no last real number, the diagonal method may not be true: In this case, In the process of continuous listing of real numbers, the possibility that the subscript on the left of eq.(3) is always greater than that of right can not be excluded absolutely, that is, some real numbers may not satisfy eq.(3) and eq.(4), and the diagonal argument therefore fails.

Theorem 2 Under the countable assumption, finding a b outside eq.(1) in the interval [0,1) does not prove that the real numbers are uncountable.

Proof According to Theorem 1, under the countable assumption, eq.(1) only lists a part of real numbers in the interval [0,1), so there are other real numbers besides eq.(1) in the interval [0,1), and the possibility that b is one of these real numbers cannot be ruled out. That is, the existence of b in the interval [0,1) does not constitute a contradiction to the countable hypothesis. **Proof finished**

Cantor thought that as long as the real numbers in the interval [0,1) are countable, they could be listed completely. But this is wrong. In fact, not only it is impossible to list all real numbers in the interval [0,1) under the countable assumption according to Theorem 1, but also it is impossible to list all natural numbers which are countable:

Theorem 3 Only a part of natural numbers can be listed.

Proof if all the natural numbers can be listed whose values from small to large, then there must be a last natural number whose value is the largest. But this natural number plus 1 is still a natural number and its value is bigger, which results in a contradictory. **proof finished.**

Therefore, it is wrong to confuse the two different concepts： uncountable and "incomplete listing ". The latter means that we can not list the numbers completely.

Cantor confuses the difference between the two concepts. What he actually proved is that the listing of real numbers is incomplete, but he believed that he proved that real numbers are uncountable.

The other two proofs to prove that real numbers are uncountable, namely the nested interval method and Cantor's theorem, also have the same logical error.

This error results in an international jokes that continues to this day. The mainstream mathematics circle, which obviously lacks the ability to correct errors, should reflect on it.

No wonder the famous mathematician Poincaré said: Our descendants will find that the mathematics community is sick. In my opinion, it is not only sick, but also very badly sick: such an obvious logical error, having appeared for more than 100 years, no one in the mainstream mathematics circle has figured it out up till now?!

In fact, the countability of real numbers can be proved without complicated reasoning at all.

If we randomly cut to the real number axis with a thin enough knifea, there must be a distance from the zero point where we cut, and this distance is a random real number we get.

Theorem 4 Real numbers are countable.

Proof: Take a real number randomly from the set $R=\{x|x \text{ is real number}\}$, denoted by x_1, and then take another real number randomly from the set $R-\{x_1\}$, denoted by x_2,.... The process continues indefinitely, then the numbers taken out are already one-to-one correspondence with the natural number according to their subscripts. Since every time a number must be fetched, so any number may or may not be fetched, thus, there is no number that can never be fetched. **Proof Finished**

Of course, this process will not end. This is normal: the elements of any infinite set can never be taken out completely. For example, even if a natural number is taken randomly from R or from $\{x|x \text{ is natural number}\}$, it cannot be taken out completely.

However, if you always take the number from one infinite proper subset of R, for example, always take the natural number in R, the process will also not stop, which will lead to the result that other elements cannot be obtained.

Since there can be many proper subsets in R, for example,

0.10000...

0.11000...

.....

or

0.20000...

0.22000...

.....

or

0.10000...

0.12000...

0.121....

....

and so on, from the perspective of probability theory, it is not difficult for anyone with common sense of probability theory to prove (the proof is omitted) that the probability of always taking elements from one proper subset in multiple or even infinitely many proper subsets tends to 0 when the numbers taken out infinitely increase; while the probability of not always taking elements from one proper subset tends to 1. Let's take a step back. even in the unlikely event that the numbers were alwans taken from one proper subset, it can be re-selected again. Therefore, no matter what the situation is, the countability of real numbers can always be proved in the end.

4 Conclusions and discussion

1)To confuse the two concepts: countable and "complete listing " is the reason why the diagonal argument fails.

2)Real numbers are countable.

Since Peano (1858-1932), Dedekind (1831-1916), Cantor (1845-1918), Hilbert (1862-1943) Russell (1872-1970) and others introduced some logical cycles and self-contradictions in the foundation of mathematics, people seem to have gradually adapted to these non-strict things and lost their sensitivity to self-contradictions and logical cycles. As a result, the failure to find the logical error in the diagonal argument for a long time is just a tip of the iceberg.

If this situation cannot be changed significantly, the development of mathematics will inevitably decline, and the status of mathematics in the scientific community will gradually become marginalized. For example, the mathematics department of an existing university has been abolished, and this phenomenon may gradually become common in the future. The impact factor of mathematics journals, which is not very high now, may become lower and lower in the future.

For this reason, removing a large number of errors in the foundation of mathematics and strengthening the cultivation of strict thinking ability is an urgent task in the foundation of mathematics.

The author will do further work in this area.

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