Tutorial: Electric and Magnetic Dipoles—Their Fields and Angular-Orientation Dynamics

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Abstract The idealized electric dipole is visualized as two charges of equal strength and opposite sign which are kept a fixed nonzero distance apart that is arbitrarily short. This configuration's electric dipole-moment vector is the charge strength multiplied by the fixed-length vector from the negative to the positive charge. We work out the electric fields of electric dipoles, and then obtain their angular-orientation equations of motion in a constant electric field by attributing a moment of inertia to them, which results in electric-dipole dynamics analogous to that of a pendulum. The idealized magnetic dipole is visualized as a closed current loop of arbitrarily small nonzero spatial extent. When its closed loop lies in a plane its magnetic dipole-moment vector is the loop current divided by the speed of light times the area the loop encloses times the unit vector normal to that plane. The vector-function form of the magnetic field of a magnetic dipole is identical to that of the electric field of an electric dipole. We obtain the angular-orientation equation of motion of a magnetic dipole in a constant magnetic field by attributing a proportional mass current to its circulating charge current, which results in magnetic-dipole dynamics analogous to gyroscope precession. The closely related quantum spin dynamics of spin-1/2 electrons in a constant magnetic field is then also discussed.

1. The electric fields produced by electric dipoles

Given a static point charge q at location $\mathbf{r} = \mathbf{d}/2$ together with the opposite static point charge $-q$ at location $\mathbf{r} = -\mathbf{d}/2$, the resulting static electric potential is,

$$V(\mathbf{r}; -q, -\mathbf{d}/2; q, \mathbf{d}/2) = q/|\mathbf{r} - \mathbf{d}/2| - q/|\mathbf{r} + \mathbf{d}/2| =$$
$$(q/|\mathbf{r}|)\big[\big(1\big/\sqrt{1 - (\mathbf{d}\cdot\mathbf{r}/|\mathbf{r}|^2) + (|\mathbf{d}|/(2|\mathbf{r}|))^2}\,\big) - \big(1\big/\sqrt{1 + (\mathbf{d}\cdot\mathbf{r}/|\mathbf{r}|^2) + (|\mathbf{d}|/(2|\mathbf{r}|))^2}\,\big)\big]. \tag{1.1a}$$

For $|\mathbf{d}| \ll |\mathbf{r}|$ the electrostatic potential $V(\mathbf{r}; -q, -\mathbf{d}/2; q, \mathbf{d}/2)$ of Eq. (1.1a) is adequately represented by its approximation through first order in $(|\mathbf{d}|/|\mathbf{r}|)$, which is,

$$V(\mathbf{r}; \mathbf{m}) = (\mathbf{m}\cdot\mathbf{r})\big/|\mathbf{r}|^3, \text{ where the electric dipole-moment vector } \mathbf{m} \equiv (q\mathbf{d}). \tag{1.1b}$$

Implicit in the passage from the electrostatic potential $V(\mathbf{r}; -q, -\mathbf{d}/2; q, \mathbf{d}/2)$ of Eq. (1.1a) to the electrostatic electric dipole-moment potential $V(\mathbf{r}; \mathbf{m})$ of Eq. (1.1b) is the mathematical idealization of taking joint limits $|\mathbf{d}| \to 0$ and $|q| \to \infty$ in such a way that the electric dipole-moment vector $\mathbf{m} \equiv (q\mathbf{d})$ remains fixed.

With the electrostatic electric dipole-moment potential $V(\mathbf{r}; \mathbf{m})$ of Eq. (1.1b) in hand, the corresponding electrostatic electric dipole-moment field $\mathbf{E}(\mathbf{r}; \mathbf{m})$ is straightforwardly obtained by taking the negative of the gradient of $V(\mathbf{r}; \mathbf{m})$,

$$\mathbf{E}(\mathbf{r}; \mathbf{m}) = -\nabla_{\mathbf{r}} V(\mathbf{r}; \mathbf{m}) = -\nabla_{\mathbf{r}}\big((\mathbf{m}\cdot\mathbf{r})\big/|\mathbf{r}|^3\big) = \big(3(\mathbf{m}\cdot\mathbf{r})\mathbf{r} - |\mathbf{r}|^2\mathbf{m}\big)\big/|\mathbf{r}|^5. \tag{1.2}$$

2. The net force and torque a constant electric field exerts on an electric dipole

A constant electric field $\mathbf{E}$ exerts equal and opposite forces on the two equal-strength, opposite-sign charges q and $-q$ of an electric dipole. We take those two charges as being rigidly joined in our electric dipole model, so the constant electric field $\mathbf{E}$ exerts zero net force on that dipole.

Since the forces exerted by the constant electric field $\mathbf{E}$ on the two charges of the electric dipole are equal and opposite, and we regard those two charges as rigidly joined with fixed distance $|\mathbf{d}|$ between them, that constant electric field exerts a net torque on the electric dipole about its center when the electric dipole's electric dipole-moment vector $q\mathbf{d} = \mathbf{m}$ isn't parallel or anti-parallel to that constant electric field $\mathbf{E}$. This net torque about the rigid electric dipole's center clearly has its maximum magnitude when the dipole-moment vector $q\mathbf{d} = \mathbf{m}$ is perpendicular to the electric field $\mathbf{E}$, in which case that torque magnitude is obviously equal to $2|q\mathbf{E}||\mathbf{d}/2| = |\mathbf{m}||\mathbf{E}|$. More generally, when the angle between the dipole-moment vector $q\mathbf{d} = \mathbf{m}$ and the constant electric field $\mathbf{E}$ is θ, the two lever arms of the net torque about the electric dipole's center each have length $|\mathbf{d}/2||\sin\theta|$, so that torque magnitude is $2|q\mathbf{E}||\mathbf{d}/2||\sin\theta| = |\mathbf{m}||\mathbf{E}||\sin\theta|$.

With the magnitude $|\mathbf{m}||\mathbf{E}||\sin\theta|$ of the net torque which the constant electric field $\mathbf{E}$ exerts on the electric dipole of vector moment $q\mathbf{d} = \mathbf{m}$ about its center now in hand, we next need to determine whether the full correct expression for that net torque is $+|\mathbf{m}||\mathbf{E}|\sin\theta$ or $-|\mathbf{m}||\mathbf{E}|\sin\theta$. In the first case that torque

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would correspond to the potential energy function $V_+(\theta) \equiv +|\mathbf{m}||\mathbf{E}|\cos\theta$, while in the second case that torque would correspond to the potential energy function $V_-(\theta) \equiv -|\mathbf{m}||\mathbf{E}|\cos\theta$ because,

$$-dV_\pm(\theta)/d\theta = \mp|\mathbf{m}||\mathbf{E}|d\cos\theta/d\theta = \pm|\mathbf{m}||\mathbf{E}|\sin\theta.$$

We note that the potential energy function $V_+(\theta) = +|\mathbf{m}||\mathbf{E}|\cos\theta$ has a maximum at $\theta = 0$, which corresponds to an unstable equilibrium, and a minimum at $\theta = \pi$, which corresponds to a stable equilibrium, whereas the potential energy function $V_-(\theta) = -|\mathbf{m}||\mathbf{E}|\cos\theta$ has a minimum at $\theta = 0$, which corresponds to a stable equilibrium, and a maximum at $\theta = \pi$, which corresponds to an unstable equilibrium.

In the case of the electric dipole which has the electric dipole-moment vector $q\mathbf{d} = \mathbf{m}$, its charge q experiences the force $q\mathbf{E}$ *in* the direction of the constant electric field $\mathbf{E}$, while its charge $-q$ experiences the force $-q\mathbf{E}$ *counter* to the direction of the constant electric field $\mathbf{E}$, which produces a net torque on that dipole whose direction is such as to *decrease* the existing angle θ which separates the direction of the dipole-moment vector $q\mathbf{d} = \mathbf{m}$ from the direction of the constant electric field $\mathbf{E}$, a fact which is particularly obvious when that existing angle of separation θ is equal to $\pi/2$. Therefore the angle $\theta = 0$ between the direction of the dipole-moment vector $q\mathbf{d} = \mathbf{m}$ and the direction of the constant electric field $\mathbf{E}$ corresponds to *a stable equilibrium,* whereas the angle $\theta = \pi$ between those two directions corresponds to *an unstable equilibrium,* which implies that *the correct potential energy function is* $V_-(\theta) = -|\mathbf{m}||\mathbf{E}|\cos\theta$ *and the corresponding correct net torque is* $-|\mathbf{m}||\mathbf{E}|\sin\theta$.

Our next task is to combine this potential energy $V_-(\theta) = -|\mathbf{m}||\mathbf{E}|\cos\theta$ for the electric dipole of moment $\mathbf{m}$ in the constant electric field $\mathbf{E}$ with that electric dipole's kinetic energy of angular motion to obtain its Lagrangian and equations of angular orientation dynamics.

3. The angular orientation dynamics of an electric dipole in a constant electric field

The directional orientation of an electric dipole's electric dipole-moment vector $\mathbf{m}$ in three dimensions is described by the unit vector,

$$\hat{\mathbf{n}}_\mathbf{m} \equiv (\mathbf{m}/|\mathbf{m}|) = (\cos\phi\sin\theta,\ \sin\phi\sin\theta,\ \cos\theta), \tag{3.1a}$$

where the z-axis of our coordinate system points in the direction of the constant electric field $\mathbf{E}$, i.e.,

$$\mathbf{E} = (0, 0, |\mathbf{E}|). \tag{3.1b}$$

If we attribute *the moment of inertia* $\mathcal{I}$ to the electric dipole with electric dipole-moment vector $\mathbf{m}$ in the constant electric field $\mathbf{E}$, then that electric dipole's *kinetic energy of angular motion* T is,

$$T = (1/2)\mathcal{I}|d\hat{\mathbf{n}}_\mathbf{m}/dt|^2 =$$
$$(1/2)\mathcal{I}|(-\dot\phi\sin\phi\sin\theta + \dot\theta\cos\phi\cos\theta,\ \dot\phi\cos\phi\sin\theta + \dot\theta\sin\phi\cos\theta,\ -\dot\theta\sin\theta)|^2 =$$
$$(1/2)\mathcal{I}\big((\dot\phi)^2\sin^2\theta + (\dot\theta)^2\big), \tag{3.2a}$$

which, *together* with this electric dipole's *potential energy* $V_-(\theta) = -|\mathbf{m}||\mathbf{E}|\cos\theta$ *in the constant electric field* $\mathbf{E}$ produces its *Lagrangian* L,

$$L = T - V_-(\theta) = (1/2)\mathcal{I}\big((\dot\phi)^2\sin^2\theta + (\dot\theta)^2\big) + |\mathbf{m}||\mathbf{E}|\cos\theta. \tag{3.2b}$$

The two canonical (angular) momenta inherent in this Lagrangian are the polar angular momentum L_θ and the azimuthal angular momentum L_ϕ, namely,

$$L_\theta \equiv \partial L/\partial\dot\theta = \mathcal{I}\dot\theta \quad \text{and} \quad L_\phi \equiv \partial L/\partial\dot\phi = \mathcal{I}\dot\phi\sin^2\theta. \tag{3.2c}$$

The corresponding two generalized-force equations inherent in the Eq. (3.2b) Lagrangian L are,

$$dL_\theta/dt = \partial L/\partial\theta \ \Rightarrow\ \mathcal{I}\ddot\theta = -|\mathbf{m}||\mathbf{E}|\sin\theta + \mathcal{I}(\dot\phi)^2\sin\theta\cos\theta \quad \text{and}$$
$$dL_\phi/dt = \partial L/\partial\phi \ \Rightarrow\ \mathcal{I}d\big(\dot\phi\sin^2\theta\big)/dt = 0, \tag{3.2d}$$

so azimuthal angular momentum is conserved. Eqs. (3.2d) are analogous to the equations of motion of a massless rigid-arm pendulum of length l which supports a mass m at its bottom while hanging from a frictionless universal pivot at its top in the practically constant gravitational field at the earth's surface that is characterized by acceleration g; to switch Eqs. (3.2d) to the pendulum equations of motion only requires the substitution of ml^2 for $\mathcal{I}$ and the substitution of mgl for $|\mathbf{m}||\mathbf{E}|$. It is well known that these equations of motion can be reduced to quadrature, as we shall explicitly show further on, but the consequent elliptic integrals aren't elementary functions. We next point out specialized exact solutions of Eqs. (3.2d) which have time-independent θ and $\dot{\phi}$ (and therefore ϕ which are linear in time), and then develop sinusoidal approximations which are adequate to describe sufficiently small-amplitude time-dependent excursions around these time-independent exact solutions.

4. Aspects of the angular behavior of an electric dipole in a constant electric field

It is convenient to write the Eq. (3.2d) equations of angular motion for an electric dipole of electric dipole moment $\mathbf{m}$ and moment of inertia $\mathcal{I}$ in a constant electric field $\mathbf{E}$ in terms of a single constant ω which has the dimensions of frequency,

$$\ddot{\theta} = -\omega^2 \sin\theta + (\dot{\phi})^2 \sin\theta\cos\theta \text{ and } d\big(\dot{\phi}\sin^2\theta\big)/dt = 0, \text{ where } \omega^2 \equiv (|\mathbf{m}||\mathbf{E}|/\mathcal{I}). \tag{4.1}$$

The simplest solutions of Eq. (4.1) are those of stable and unstable *static equilibrium,*

$$\theta(t) = 0 \text{ and } \theta(t) = \pi. \tag{4.2}$$

Another specialized class of simple exact solutions of the two Eq. (4.1) equations of motion are those of *time-independent polar angle* $\theta(t) = \theta(t_0)$, where $0 < \theta(t_0) < \pi/2$, in conjunction with time-independent time rate of change of the azimuthal angle $\phi(t)$ such that $(\dot{\phi}(t))^2 = (\dot{\phi}(t_0))^2 = (\omega^2/\cos(\theta(t_0)))$, which implies that $\phi(t)$ is linear in time,

$$\phi(t) = \phi(t_0) \pm \omega(t - t_0)\sqrt{1/\cos\theta(t_0)}. \tag{4.3}$$

This particular class of specialized exact solutions however clearly excludes the possibility that $\dot{\phi}(t) = 0$, in which case Eqs. (4.1) describe electric dipole angular motion *restricted to a single plane,* i.e., $\phi(t) = \phi(t_0)$, for which the equation of motion for the polar angle $\theta(t)$ that is given by Eq. (4.1) becomes,

$$\ddot{\theta} = -\omega^2 \sin\theta. \tag{4.4a}$$

Eq. (4.4a) has, inter alia, the specialized time-independent solution $\theta(t) = 0$, i.e., stable static equilibrium. Eq. (4.4a), however, *in addition* describes time-dependent single-plane excursions of $\theta(t)$ around its specialized time-independent solution $\theta(t) = 0$. When these time-dependent excursions $\theta(t)$ around the specialized time-independent solution $\theta(t) = 0$ are of sufficiently small amplitude, specifically when $|\theta(t)| \ll 1$, then these time-dependent excursions $\theta(t)$ are adequately approximated by the solutions of,

$$\ddot{\theta} = -\omega^2 \theta, \tag{4.4b}$$

whose general solution $\theta(t)$ is the sinusoidal form,

$$\theta(t) = \theta(t_0)\cos(\omega(t - t_0)) + (\dot{\theta}(t_0)/\omega)\sin(\omega(t - t_0)), \tag{4.4c}$$

which, when the initial values $\theta(t_0)$ and $\dot{\theta}(t_0)$ satisfy $|\theta(t_0)| \ll 1$ and $|\dot{\theta}(t_0)/\omega| \ll 1$, clearly satisfies the required small-amplitude excursion condition $|\theta(t)| \ll 1$.

We next similarly exhibit the particular form of the equation of motion for the polar angle $\theta(t)$ given by Eq. (4.1) which has, inter alia, the specialized time-independent exact solution $\theta(t) = \theta(t_0)$, where $0 < \theta(t_0) < \pi/2$. That equation *in addition* describes the time-dependent excursions of $\theta(t)$ around $\theta(t_0)$, for which we shall work out a sinusoidal approximation that is adequate when the amplitude of those excursions is sufficiently small.

In order to carry out this program, we must first fully decouple the Eq. (4.1) equation $\ddot{\theta} = -\omega^2 \sin\theta + (\dot{\phi})^2 \sin\theta\cos\theta$ for $\theta(t)$ from the Eq. (4.1) equation $d(\dot{\phi}\sin^2\theta)/dt = 0$ for $\dot{\phi}(t)$ by formally solving the latter for $\dot{\phi}(t)$ as follows,

$$\dot{\phi}(t) = \big(\dot{\phi}(t_0)\sin^2(\theta(t_0))/\sin^2(\theta(t))\big), \tag{4.5a}$$

which, incidentally, implies that,

$$\phi(t) = \phi(t_0) + \textstyle\int_{t_0}^{t} \big(\dot{\phi}(t_0)\sin^2(\theta(t_0))/\sin^2(\theta(t'))\big)\, dt'. \tag{4.5b}$$

We then insert the Eq. (4.5a) result for $\dot{\phi}(t)$ into the Eq. (4.1) equation $\ddot{\theta} = -\omega^2 \sin\theta + (\dot{\phi})^2 \sin\theta\cos\theta$ for $\theta(t)$ to obtain,

$$\ddot{\theta}(t) = -\omega^2 \sin(\theta(t)) + \big((\dot{\phi}(t_0))^2 \sin^4(\theta(t_0)) \cos(\theta(t))/\sin^3(\theta(t))\big). \tag{4.5c}$$

In order for this decoupled Eq. (4.5c) equation for $\theta(t)$ to have the specialized time-independent solution $\theta(t) = \theta(t_0)$, we must choose the following particular value for $\dot{\phi}(t_0)$,

$$\dot{\phi}(t_0) = \pm\omega\sqrt{1/\cos(\theta(t_0))}, \tag{4.5d}$$

which is consistent with the specialized Eq. (4.3), whereupon the decoupled Eq. (4.5c) becomes,

$$\ddot{\theta}(t) = -\omega^2 \sin(\theta(t)) + \omega^2\big(\sin^4(\theta(t_0)) \cos(\theta(t))/\big(\cos(\theta(t_0)) \sin^3(\theta(t))\big)\big), \tag{4.5e}$$

which indeed has the specialized time-independent solution $\theta(t) = \theta(t_0)$. The particular Eq. (4.5d) choice $\dot{\phi}(t_0) = \pm\omega\sqrt{1/\cos(\theta(t_0))}$ causes Eq. (4.5b) to become,

$$\phi(t) = \phi(t_0) \pm \omega\sqrt{1/\cos(\theta(t_0))} \int_{t_0}^{t} \big(\sin^2(\theta(t_0))/\sin^2(\theta(t'))\big)\, dt', \tag{4.5f}$$

which reduces to Eq. (4.3) in the case of the specialized time-independent solution $\theta(t) = \theta(t_0)$ of Eq. (4.5e).

We next wish solve Eq. (4.5e) for small-amplitude time-dependent excursions $\theta(t)$ around its specialized time-independent solution $\theta(t) = \theta(t_0)$ (where $0 < \theta(t_0) < \pi/2$), just as we solved Eq. (4.4a) for small-amplitude time-dependent excursions $\theta(t)$ (which are described by Eq. (4.4c)) around its specialized time-independent solution $\theta(t) = 0$.

Small-amplitude time-dependent excursions $\theta(t)$ around the specialized time-independent solution $\theta(t) = \theta(t_0)$ (where $0 < \theta(t_0) < \pi/2$) of Eq. (4.5e) have the representation,

$$\theta(t) = \theta(t_0)[1 + \epsilon(t; \theta(t_0), \dot{\theta}(t_0))], \text{ where } \epsilon(t = t_0; \theta(t_0), \dot{\theta}(t_0)) = 0 \text{ and } |\epsilon(t; \theta(t_0), \dot{\theta}(t_0))| \ll 1. \tag{4.6a}$$

We insert the Eq. (4.6a) representation of $\theta(t)$ into Eq. (4.5e), and keep terms only through first order in $\epsilon(t; \theta(t_0), \dot{\theta}(t_0))$. This produces a homogeneously linear, second-order differential equation for $\epsilon(t; \theta(t_0), \dot{\theta}(t_0))$ that has sinusoidal solutions. We now illustrate the laborious expansions through first order in $\epsilon(t; \theta(t_0), \dot{\theta}(t_0))$ needed to carry this out, starting with,

$$\sin(\theta(t)) = \sin\big(\theta(t_0) + \theta(t_0)\epsilon(t; \theta(t_0), \dot{\theta}(t_0))\big) \approx \sin(\theta(t_0)) + \cos(\theta(t_0))\theta(t_0)\epsilon(t; \theta(t_0), \dot{\theta}(t_0)) =$$
$$\sin(\theta(t_0))\big[1 + \theta(t_0)\big(\cos(\theta(t_0))/\sin(\theta(t_0))\big)\epsilon(t; \theta(t_0), \dot{\theta}(t_0))\big]. \tag{4.6b}$$

Similarly,

$$\cos(\theta(t)) \approx \cos(\theta(t_0))\big[1 - \theta(t_0)\big(\sin(\theta(t_0))/\cos(\theta(t_0))\big)\epsilon(t; \theta(t_0), \dot{\theta}(t_0))\big], \tag{4.6c}$$

and therefore,

$$\omega^2\big(\sin^4(\theta(t_0)) \cos(\theta(t))/\big(\cos(\theta(t_0)) \sin^3(\theta(t))\big)\big) \approx$$
$$\omega^2 \sin(\theta(t_0))\Big[1 - \theta(t_0)\Big(\big(\sin(\theta(t_0))/\cos(\theta(t_0))\big) + 3\big(\cos(\theta(t_0))/\sin(\theta(t_0))\big)\Big)\epsilon(t; \theta(t_0), \dot{\theta}(t_0))\Big]. \tag{4.6d}$$

Eq. (4.6d) together with,

$$-\omega^2 \sin(\theta(t)) \approx \omega^2 \sin(\theta(t_0))\big[-1 - \theta(t_0)\big(\cos(\theta(t_0))/\sin(\theta(t_0))\big)\epsilon(t; \theta(t_0), \dot{\theta}(t_0))\big], \tag{4.6e}$$

which follows from Eq. (4.6b) and,

$$\ddot{\theta}(t) = \theta(t_0)\, d^2\epsilon(t; \theta(t_0), \dot{\theta}(t_0))/dt^2, \tag{4.6f}$$

which follows from Eq. (4.6a), shows that the result of inserting the Eq. (4.6a) representation of $\theta(t)$ into Eq. (4.5e) and keeping terms only through first order in $\epsilon(t; \theta(t_0), \dot{\theta}(t_0))$ produces the following homogeneously linear, second-order differential equation for $\epsilon(t; \theta(t_0), \dot{\theta}(t_0))$,

$$d^2\epsilon(t; \theta(t_0), \dot{\theta}(t_0))/dt^2 =$$
$$-\omega^2 \sin(\theta(t_0))\Big[\big(\sin(\theta(t_0))/\cos(\theta(t_0))\big) + 4\big(\cos(\theta(t_0))/\sin(\theta(t_0))\big)\Big]\epsilon(t; \theta(t_0), \dot{\theta}(t_0)) =$$
$$-\omega^2\big[\big(1 + 3\cos^2(\theta(t_0))\big)/\cos(\theta(t_0))\big]\epsilon(t; \theta(t_0), \dot{\theta}(t_0)), \tag{4.6g}$$

whose solution that satisfies $\epsilon(t = t_0; \theta(t_0), \dot{\theta}(t_0)) = 0$ and $|\epsilon(t; \theta(t_0), \dot{\theta}(t_0))| \ll 1$ (see Eq. (4.6a)) is,

$$\epsilon(t;\theta(t_0),\dot\theta(t_0)) = \epsilon(\theta(t_0),\dot\theta(t_0))\sin\big(\omega(\theta(t_0))(t-t_0)\big), \text{ where } |\epsilon(\theta(t_0),\dot\theta(t_0))| \ll 1$$
$$\text{and } \ \omega(\theta(t_0)) \equiv \omega\,\sqrt{(1+3\cos^2(\theta(t_0)))/\cos(\theta(t_0))}\,. \tag{4.6h}$$

So from Eqs. (4.6h) and (4.6a),

$$\theta(t) = \theta(t_0)\big[1+\epsilon(\theta(t_0),\dot\theta(t_0))\sin\big(\omega(\theta(t_0))(t-t_0)\big)\big], \text{ where } |\epsilon(\theta(t_0),\dot\theta(t_0))| \ll 1, \tag{4.6i}$$

represents small-amplitude time-dependent excursions around the specialized time-independent solution $\theta(t) = \theta(t_0)$ of Eq. (4.5e) for any $\theta(t_0)$ which satisfies $0 < \theta(t_0) < \pi/2$. Eq. (4.6i) implies that,

$$\dot\theta(t_0) = \theta(t_0)\,\epsilon(\theta(t_0),\dot\theta(t_0))\,\omega(\theta(t_0)), \tag{4.6j}$$

so for $0 < \theta(t_0) < \pi/2$, $\epsilon(\theta(t_0),\dot\theta(t_0))$ is explicitly given in terms of $\theta(t_0)$ and $\dot\theta(t_0)$ by,

$$\epsilon(\theta(t_0),\dot\theta(t_0)) = \big(\dot\theta(t_0)/(\theta(t_0)\,\omega(\theta(t_0)))\big) = \big(\dot\theta(t_0)/(\theta(t_0)\,\omega)\big)\sqrt{\cos(\theta(t_0))/(1+3\cos^2(\theta(t_0)))}\,, \tag{4.6k}$$

which for $0 < \theta(t_0) < \pi/2$ allows the $\theta(t)$ of Eq. (4.6i) to be written,

$$\theta(t) = \theta(t_0) + \big(\dot\theta(t_0)/\omega(\theta(t_0))\big)\sin\big(\omega(\theta(t_0))(t-t_0)\big), \tag{4.6l}$$

provided that $|\dot\theta(t_0)| \ll (\theta(t_0)\,\omega(\theta(t_0))) = \big(\theta(t_0)\,\omega\,\sqrt{(1+3\cos^2(\theta(t_0)))/\cos(\theta(t_0))}\,\big)$.

We next insert the Eq. (4.6i) polar-angle result $\theta(t)$ for small-amplitude time-dependent excursions around the time-independent solution $\theta(t) = \theta(t_0)$, where $0 < \theta(t_0) < \pi/2$, into Eq. (4.5f) to obtain the small-amplitude excursions around the corresponding azimuthal-angle result $\phi(t) = \phi(t_0) \pm \omega\sqrt{1/\cos(\theta(t_0))}(t-t_0)$ which is given by Eq. (4.3). Through first order in $\epsilon(\theta(t_0),\dot\theta(t_0))$, insertion of the $\theta(t)$ of Eq. (4.6i) into the Eq. (4.5f) expression for $\phi(t)$, namely,

$$\phi(t) = \phi(t_0) \pm \omega\sqrt{1/\cos(\theta(t_0))}\int_{t_0}^{t}\big(\sin^2(\theta(t_0))/\sin^2(\theta(t'))\big)\,dt',$$

produces,

$$\phi(t) = \phi(t_0) \pm$$
$$\omega\sqrt{1/\cos(\theta(t_0))}\int_{t_0}^{t}\big[1 - 2\big(\theta(t_0)\cos(\theta(t_0))/\sin(\theta(t_0))\big)\epsilon(\theta(t_0),\dot\theta(t_0))\sin(\omega(\theta(t_0))(t'-t_0))\big]\,dt' =$$
$$\phi(t_0) \pm \omega\sqrt{1/\cos(\theta(t_0))}(t-t_0)\mp$$
$$2\sqrt{1/(1+3\cos^2(\theta(t_0)))}\big(\theta(t_0)\cos(\theta(t_0))/\sin(\theta(t_0))\big)\epsilon(\theta(t_0),\dot\theta(t_0))\big(1-\cos\big(\omega(\theta(t_0))(t-t_0)\big)\big), \tag{4.7}$$

where $0 < \theta(t_0) < \pi/2$, $|\epsilon(\theta(t_0),\dot\theta(t_0))| \ll 1$, and we know from Eq. (4.6h) that,

$$\omega(\theta(t_0)) = \omega\,\sqrt{(1+3\cos^2(\theta(t_0)))/\cos(\theta(t_0))}\,.$$

The entity $\epsilon(\theta(t_0),\dot\theta(t_0))$ in the Eq. (4.7) result can be replaced by $\big(\dot\theta(t_0)/(\theta(t_0)\,\omega(\theta(t_0)))\big)$ (see Eq. (4.6k)) *provided that the condition* $|\epsilon(\theta(t_0),\dot\theta(t_0))| \ll 1$ *is replaced by the condition* $|\dot\theta(t_0)| \ll (\theta(t_0)\,\omega(\theta(t_0)))$.

We have so far focused on the specialized time-independent exact solutions of the Eq. (4.5c) decoupled polar-angle $\theta(t)$ equation of motion, and on the development of sinusoidal approximations to small-amplitude time-dependent excursions around those time-independent exact solutions. The decoupled polar-angle Eq. (4.5c) can, however, be reduced to quadrature in terms of elliptic integrals, so we conclude this discussion of the angular orientation dynamics of electric dipoles in constant electric fields by carrying out that reduction. It is useful to first abbreviate the initial-value combination $\dot\phi(t_0)\sin^2(\theta(t_0))$, which is the conserved azimuthal angular momentum L_ϕ divided by the moment of inertia $\mathcal{I}$, as $\omega\alpha$, whereupon the Eq. (4.5b) expression for the azimuthal angle $\phi(t)$ in terms of the polar angle $\theta(t)$ simplifies to,

$$\phi(t) = \phi(t_0) + \omega\int_{t_0}^{t}\big(\alpha/\sin^2(\theta(t'))\big)\,dt', \tag{4.8a}$$

and the Eq. (4.5c) decoupled equation of motion for the polar angle $\theta(t)$ simplifies to,

$$\ddot\theta(t) = \omega^2\big(-\sin(\theta(t)) + \big(\alpha^2\cos(\theta(t))/\sin^3(\theta(t))\big)\big). \tag{4.8b}$$

Multiplying Eq. (4.8b) by $2\dot\theta(t)$ and then integrating with respect to t yields,

$$(\dot\theta(t))^2 = \omega^2\big(2\cos(\theta(t)) - \big(\alpha^2/\sin^2(\theta(t))\big) + \beta\big), \tag{4.8c}$$

where the dimensionless constant of integration β is given in terms of initial values by,

$$\beta = (\dot{\phi}(t_0)\sin(\theta(t_0))/\omega)^2 + (\dot{\theta}(t_0)/\omega)^2 - 2\cos(\theta(t_0)), \tag{4.8d}$$

so β is the system's *conserved energy* $(T + V_-)$ *divided by* $(\mathcal{I}\omega^2/2) = (|\mathbf{m}||\mathbf{E}|/2)$.

Multiplying Eq. (4.8c) by $\sin^2(\theta(t)) = (1 - \cos^2(\theta(t)))$, allows its right side to be expressed entirely in terms of powers of $\cos(\theta(t))$, while its left side becomes $(\sin(\theta(t))\dot{\theta}(t))^2 = (d\cos(\theta(t))/dt)^2$. Taking the square root of both sides of that result produces,

$$d\cos(\theta(t))/dt = \omega\sqrt{(2\cos(\theta(t))(1 - \cos^2(\theta(t)))) - \alpha^2 + (\beta(1 - \cos^2(\theta(t))))}, \tag{4.8e}$$

which immediately reduces to quadrature as,

$$\omega(t - t_0) = \int_{\cos(\theta(t_0))}^{\cos(\theta(t))} \sqrt{1/[(2x(1 - x)(1 + x)) - \alpha^2 + (\beta(1 - x)(1 + x))]}\, dx. \tag{4.8f}$$

Because its integrand is the square root of the inverse of a *cubic* form in the integration variable x, the integral in Eq. (4.8f) cannot be expressed in terms of elementary functions; it is termed an elliptic integral.

We next work out the magnetic fields produced by magnetic dipoles, which can be visualized as electric-current closed-loop circuits of arbitrarily small nonzero spatial extent.

5. The magnetic fields produced by magnetic dipoles

A magnetic dipole's closed-loop circuit of constant electric current I is described by a closed curve $\mathbf{r}(s)$, where we conveniently choose the parameter s to be the curve's running arc length, which implies that,

$$0 \le s \le l, \text{ where l is the curve's length, and } |\mathbf{r}'(s)| = 1, \tag{5.1a}$$

so $\mathbf{r}'(s)$ is *a unit vector* which describes the constant current's *local direction at the loop location* $\mathbf{r}(s)$, $0 \le s \le l$. In addition, the fact that the loop is *closed* of course implies that,

$$\mathbf{r}(l) = \mathbf{r}(0). \tag{5.1b}$$

When the closed loop *lies in a plane,* the *area* it encloses *times the unit vector normal to that plane* is,

$$\mathbf{area}([\mathbf{r}(s)]) = (1/2)\int_0^l (\mathbf{r}(s) \times \mathbf{r}'(s))\, ds, \tag{5.1c}$$

and the magnitude $|\mathbf{area}([\mathbf{r}(s)])|$ of that area is regarded as arbitrarily small, but nonzero.

We denote the total charge which is circulating in the closed loop as Q, and assume that it is uniformly distributed throughout the closed loop's length l, with every part of it circulating through the loop at the constant speed v that is given by,

$$v = I/(Q/l) = (lI)/Q. \tag{5.2a}$$

Therefore the circulating charge's local velocity $\mathbf{v}(s)$ at the loop location $\mathbf{r}(s)$ is,

$$\mathbf{v}(s) = v\,\mathbf{r}'(s) = ((lI)/Q)\,\mathbf{r}'(s). \tag{5.2b}$$

The charge of an infinitesimal length δs of the loop is $(Q/l)\delta s$, so the current density of an infinitesimal length δs of the loop at the loop location $\mathbf{r}(s)$ is,

$$\mathbf{j}(\mathbf{r}; \delta s, \mathbf{r}(s)) = ((Q/l)\delta s)\,\mathbf{v}(s)\,\delta^{(3)}(\mathbf{r} - \mathbf{r}(s)) = \\ (Q/l)\,((lI)/Q)\,\mathbf{r}'(s)\,\delta^{(3)}(\mathbf{r} - \mathbf{r}(s))\,\delta s = I\,\mathbf{r}'(s)\,\delta^{(3)}(\mathbf{r} - \mathbf{r}(s))\,\delta s. \tag{5.2c}$$

Therefore the current density of the *entire* closed loop is,

$$\mathbf{j}(\mathbf{r}) = I\int_0^l \mathbf{r}'(s)\,\delta^{(3)}(\mathbf{r} - \mathbf{r}(s))\, ds. \tag{5.2d}$$

The Eq. (5.2d) current density $\mathbf{j}(\mathbf{r})$ for the *entire* closed loop satisfies the *magnetostatic* physical consistency requirement $\nabla_{\mathbf{r}} \cdot \mathbf{j}(\mathbf{r}) = 0$ because,

$$\nabla_{\mathbf{r}} \cdot \mathbf{j}(\mathbf{r}) = I\int_0^l (\mathbf{r}'(s) \cdot (\nabla_{\mathbf{r}}\,\delta^{(3)}(\mathbf{r} - \mathbf{r}(s))))\, ds = -I\int_0^l (d(\delta^{(3)}(\mathbf{r} - \mathbf{r}(s)))/ds)\, ds = \\ -I\big(\delta^{(3)}(\mathbf{r} - \mathbf{r}(l)) - \delta^{(3)}(\mathbf{r} - \mathbf{r}(0))\big) = 0, \text{ since } \mathbf{r}(l) = \mathbf{r}(0). \tag{5.2e}$$

To obtain the magnetic field $\mathbf{B}(\mathbf{r})$ produced by a closed current loop magnetic dipole, we first work out its vector potential $\mathbf{A}(\mathbf{r})$ using the divergence-free closed-loop current density $\mathbf{j}(\mathbf{r})$ of Eq. (5.2d),

$$\mathbf{A}(\mathbf{r}) = (1/c)\int \mathbf{j}(\mathbf{r}_1)/|\mathbf{r}-\mathbf{r}_1|\,d^3\mathbf{r}_1 = (I/c)\int_0^l\!\left(\int \mathbf{r}'(s)\delta^{(3)}(\mathbf{r}_1-\mathbf{r}(s))/|\mathbf{r}-\mathbf{r}_1|\,d^3\mathbf{r}_1\right)ds =$$
$$(I/c)\int_0^l(\mathbf{r}'(s)/|\mathbf{r}-\mathbf{r}(s)|)\,ds = (I/(c|\mathbf{r}|))\int_0^l\left[\mathbf{r}'(s)\Big/\sqrt{1-2((\mathbf{r}(s)\cdot\mathbf{r})/|\mathbf{r}|^2)+(|\mathbf{r}(s)|/|\mathbf{r}|)^2}\,\right]ds. \quad (5.3a)$$

For $\max_{\{s|\,0\le s\le l\,\}}|\mathbf{r}(s)| \ll |\mathbf{r}|$ the vector potential $\mathbf{A}(\mathbf{r})$ of Eq. (5.3a) is adequately represented by its approximation through first order in $(|\mathbf{r}(s)|/|\mathbf{r}|)$, which is,

$$\mathbf{A}(\mathbf{r}) = (I/(c|\mathbf{r}|))\int_0^l(\mathbf{r}'(s)(1+((\mathbf{r}(s)\cdot\mathbf{r})/|\mathbf{r}|^2)))\,ds =$$
$$(I(\mathbf{r}(l)-\mathbf{r}(0))/(c|\mathbf{r}|)) + (I/(c|\mathbf{r}|^3))\int_0^l\mathbf{r}'(s)(\mathbf{r}(s)\cdot\mathbf{r})\,ds = (I/(c|\mathbf{r}|^3))\int_0^l\mathbf{r}'(s)(\mathbf{r}(s)\cdot\mathbf{r})\,ds, \quad (5.3b)$$

because $\mathbf{r}(l) = \mathbf{r}(0)$. We next carry out integration by parts on the last integral in Eq. (5.3b),

$$\int_0^l\mathbf{r}'(s)(\mathbf{r}(s)\cdot\mathbf{r})\,ds = (\mathbf{r}(l)(\mathbf{r}(l)\cdot\mathbf{r})-\mathbf{r}(0)(\mathbf{r}(0)\cdot\mathbf{r})) - \int_0^l\mathbf{r}(s)(\mathbf{r}'(s)\cdot\mathbf{r})\,ds = -\int_0^l\mathbf{r}(s)(\mathbf{r}'(s)\cdot\mathbf{r})\,ds, \quad (5.3c)$$

because $\mathbf{r}(l) = \mathbf{r}(0)$. The result of Eq. (5.3c) implies that,

$$\int_0^l\mathbf{r}'(s)(\mathbf{r}(s)\cdot\mathbf{r})\,ds = (1/2)\int_0^l(\mathbf{r}'(s)(\mathbf{r}(s)\cdot\mathbf{r})-\mathbf{r}(s)(\mathbf{r}'(s)\cdot\mathbf{r}))\,ds = (1/2)\int_0^l((\mathbf{r}(s)\times\mathbf{r}'(s))\times\mathbf{r})\,ds, \quad (5.3d)$$

which together with the result for $\mathbf{A}(\mathbf{r})$ in Eq. (5.3b) yields that,

$$\mathbf{A}(\mathbf{r}) = ((\mathbf{m}\times\mathbf{r})/|\mathbf{r}|^3),\text{ where }\mathbf{m}\equiv(I/(2c))\int_0^l(\mathbf{r}(s)\times\mathbf{r}'(s))\,ds. \quad (5.3e)$$

The Eq. (5.3e) entity $\mathbf{m}\equiv(I/(2c))\int_0^l(\mathbf{r}(s)\times\mathbf{r}'(s))\,ds$, which pertains to a magnetic dipole's closed loop $\mathbf{r}(s)$ and constant electric current I, is that magnetic dipole's *magnetic dipole-moment vector*—when the magnetic dipole's closed loop lies in a plane, $\mathbf{m} = (I/c)\,\mathbf{area}([\mathbf{r}(s)])$ (see Eq. (5.1c)). Magnetic dipole-moment vectors have the dimensions of charge times length, which matches those of electric dipole-moment vectors. The magnetic dipole is mathematically idealized by taking the joint limits $\max_{\{s|\,0\le s\le l\,\}}|\mathbf{r}(s)|\to 0$ and $I\to\infty$ in such a way that the magnetic dipole-moment vector $\mathbf{m} = (I/(2c))\int_0^l(\mathbf{r}(s)\times\mathbf{r}'(s))\,ds$ remains fixed.

The Eq. (5.3e) vector potential $\mathbf{A}(\mathbf{r};\mathbf{m})\equiv((\mathbf{m}\times\mathbf{r})/|\mathbf{r}|^3)$ for a magnetic dipole corresponds to the magnetic field $\mathbf{B}(\mathbf{r};\mathbf{m}) = \nabla_{\mathbf{r}}\times\mathbf{A}(\mathbf{r};\mathbf{m}) = \nabla_{\mathbf{r}}\times((\mathbf{m}\times\mathbf{r})/|\mathbf{r}|^3)$, which is (tediously) worked out to be,

$$\mathbf{B}(\mathbf{r};\mathbf{m}) = \nabla_{\mathbf{r}}\times((\mathbf{m}\times\mathbf{r})/|\mathbf{r}|^3) = \mathbf{m}(\nabla_{\mathbf{r}}\cdot(\mathbf{r}/|\mathbf{r}|^3)) - (\mathbf{m}\cdot\nabla_{\mathbf{r}})(\mathbf{r}/|\mathbf{r}|^3) =$$
$$\mathbf{m}(3+|\mathbf{r}|(\partial/\partial|\mathbf{r}|))(1/|\mathbf{r}|^3) - (\mathbf{m}+((\mathbf{m}\cdot\mathbf{r})(\mathbf{r}/|\mathbf{r}|)(\partial/\partial|\mathbf{r}|)))(1/|\mathbf{r}|^3) = \big(3(\mathbf{m}\cdot\mathbf{r})\mathbf{r}-|\mathbf{r}|^2\mathbf{m}\big)\big/|\mathbf{r}|^5, \quad (5.4)$$

a vector-function form identical to that of the electric field of an electric dipole given by Eq. (1.2).

If for the charge Q which is uniformly distributed along the length l of the closed current loop $\mathbf{r}(s)$, $0\le s\le l$, and which circulates around that loop at constant speed $v = (lI)/Q$, there is *a corresponding mass M with precisely those same properties,* then the angular momentum vector $\mathbf{L}$ of that circulating mass M comes out *to be proportional to the current loop's magnetic dipole-moment vector* $\mathbf{m}$.

The angular momentum $\delta\mathbf{L}(\delta s,\mathbf{r}(s))$ of the motion of the circulating mass in the infinitesimal length δs of the part of the loop located at $\mathbf{r}(s)$ is,

$$\delta\mathbf{L}(\delta s,\mathbf{r}(s)) = \mathbf{r}(s)\times(((M/l)\delta s)\mathbf{v}(s)) = (\mathbf{r}(s)\times((M/l)v\,\mathbf{r}'(s)))\delta s =$$
$$(\mathbf{r}(s)\times((M/l)((lI)/Q)\mathbf{r}'(s)))\delta s = (M/Q)I(\mathbf{r}(s)\times\mathbf{r}'(s))\delta s. \quad (5.5a)$$

Therefore the *net angular momentum* $\mathbf{L}$ of the motion of the circulating mass M in the *entire* loop is,

$$\mathbf{L} = (M/Q)I\int_0^l(\mathbf{r}(s)\times\mathbf{r}'(s))\,ds = (2Mc/Q)\mathbf{m}, \quad (5.5b)$$

where $\mathbf{m} = (I/(2c))\int_0^l(\mathbf{r}(s)\times\mathbf{r}'(s))\,ds$ is the loop's magnetic dipole-moment vector given by Eq. (5.3e).

In the next section we shall see that the net force which a constant magnetic field exerts on a magnetic dipole is zero, and that the net torque which such a field exerts on a magnetic dipole is perpendicular to its dipole-moment vector $\mathbf{m}$, and therefore, from Eq. (5.5b), perpendicular to its angular momentum $\mathbf{L}$. Thus a magnetic dipole's response to a constant magnetic field is similar to a gyroscope's response to torque that is perpendicular to its angular momentum, i.e., it *precesses.*

6. The precession dynamics of magnetic dipoles in constant magnetic fields

Just as we assumed that the two equal and opposite charges q and $-q$ of an electric dipole $\mathbf{m} = (q\mathbf{d})$ *are rigidly attached to each other by an idealized strut of length* $|\mathbf{d}|$ *which has no charge or mass,* we shall assume that *the idealized structure* of the constant-current I magnetic-dipole closed loop of length l and shape $\mathbf{r}(s)$, $0 \le s \le l$, *which confines the charge* Q *and mass* M *that circulates through it at constant speed* $v = (lI)/Q$ *is completely rigid and has no charge or mass.*

We begin by calculating *the net force* $\mathbf{F}$ which a constant magnetic field $\mathbf{B}$ exerts on such a constant-current magnetic-dipole closed loop. The force $\delta\mathbf{F}(\delta s, \mathbf{r}(s))$ which the constant magnetic field $\mathbf{B}$ exerts on the infinitesimal length δs of the part of the closed loop located at $\mathbf{r}(s)$ is,

$$\delta\mathbf{F}(\delta s, \mathbf{r}(s)) = (((Q/l)\delta s)(\mathbf{v}(s)/c) \times \mathbf{B}) = ((Q/l)(v\,\mathbf{r}'(s)/c) \times \mathbf{B})\delta s =$$
$$((Q/l)(((lI)/Q)\mathbf{r}'(s)/c) \times \mathbf{B})\delta s = (I/c)(\mathbf{r}'(s) \times \mathbf{B})\delta s. \tag{6.1a}$$

Therefore *the net force* $\mathbf{F}$ which a constant magnetic field $\mathbf{B}$ exerts on the *entire* loop is,

$$\mathbf{F} = (I/c)\int_0^l (\mathbf{r}'(s) \times \mathbf{B})\, ds = (I/c)\big[(\mathbf{r}(l) \times \mathbf{B}) - (\mathbf{r}(0) \times \mathbf{B})\big] = \mathbf{0}, \tag{6.1b}$$

since $\mathbf{r}(l) = \mathbf{r}(0)$ because the loop is closed.

We next calculate *the net torque* $\mathbf{trq}$ which a constant magnetic field $\mathbf{B}$ exerts on such a constant-current magnetic-dipole closed loop. The torque $\delta\mathbf{trq}(\delta s, \mathbf{r}(s))$ which that field exerts on the infinitesimal length δs of the part of the closed loop located at $\mathbf{r}(s)$ is obtained from the Eq. (6.1a) result for $\delta\mathbf{F}(\delta s, \mathbf{r}(s))$,

$$\delta\mathbf{trq}(\delta s, \mathbf{r}(s)) = \mathbf{r}(s) \times \delta\mathbf{F}(\delta s, \mathbf{r}(s)) = \mathbf{r}(s) \times \big[(I/c)(\mathbf{r}'(s) \times \mathbf{B})\delta s\big] =$$
$$(I/c)\big[\mathbf{r}(s) \times (\mathbf{r}'(s) \times \mathbf{B})\big]\delta s = (I/c)\big[\mathbf{r}'(s)(\mathbf{r}(s)\cdot\mathbf{B}) - \mathbf{B}(\mathbf{r}(s)\cdot\mathbf{r}'(s))\big]\delta s. \tag{6.1c}$$

Therefore *the net torque* $\mathbf{trq}$ which a constant magnetic field $\mathbf{B}$ exerts on the *entire* loop is,

$$\mathbf{trq} = (I/c)\int_0^l \big[\mathbf{r}'(s)(\mathbf{r}(s)\cdot\mathbf{B}) - \mathbf{B}(\mathbf{r}(s)\cdot\mathbf{r}'(s))\big]\, ds =$$
$$(I/c)\int_0^l \mathbf{r}'(s)(\mathbf{r}(s)\cdot\mathbf{B})\, ds - (I/(2c))\mathbf{B}\big(|\mathbf{r}(l)|^2 - |\mathbf{r}(0)|^2\big) = (I/c)\int_0^l \mathbf{r}'(s)(\mathbf{r}(s)\cdot\mathbf{B})\, ds, \tag{6.1d}$$

where we have used the fact that $\mathbf{r}(l) = \mathbf{r}(0)$. We next refer to Eq. (5.3d), where it is pointed out that,

$$\int_0^l \mathbf{r}'(s)(\mathbf{r}(s)\cdot\mathbf{r})\, ds = (1/2)\int_0^l ((\mathbf{r}(s) \times \mathbf{r}'(s)) \times \mathbf{r})\, ds = \big((1/2)\int_0^l (\mathbf{r}(s) \times \mathbf{r}'(s))\, ds\big) \times \mathbf{r},$$

in which we can replace $\mathbf{r}$ by $\mathbf{B}$. Combining doing so with the expression for $\mathbf{trq}$ given by Eq. (6.1d), and also noting the Eq. (5.3e) definition, $\mathbf{m} \equiv (I/(2c))\int_0^l (\mathbf{r}(s) \times \mathbf{r}'(s))\, ds$, yields the result,

$$\mathbf{trq} = (I/c)\int_0^l \mathbf{r}'(s)(\mathbf{r}(s)\cdot\mathbf{B})\, ds = \big((I/(2c))\int_0^l (\mathbf{r}(s) \times \mathbf{r}'(s))\, ds\big) \times \mathbf{B} = \mathbf{m} \times \mathbf{B}. \tag{6.1e}$$

Since the net torque $\mathbf{trq}$ exerted by the constant magnetic field $\mathbf{B}$ on a closed-loop magnetic dipole is equal to the time rate of change of that dipole's angular momentum $\mathbf{L}$, Eq. (6.1e) implies that,

$$d\mathbf{L}/dt = \mathbf{m} \times \mathbf{B} = (Q/(2Mc))(\mathbf{L} \times \mathbf{B}), \tag{6.1f}$$

where we have applied the relation $\mathbf{L} = (2Mc/Q)\mathbf{m}$ between $\mathbf{L}$ and $\mathbf{m}$ that is given by Eq. (5.5b). Note that $d\mathbf{m}/dt = (Q/(2Mc))(\mathbf{m} \times \mathbf{B})$ *as well,* since $\mathbf{L} = (2Mc/Q)\mathbf{m}$, so Eq. (6.1f) as well describes the motion of a magnetic dipole-moment vector $\mathbf{m}$ in a constant magnetic field $\mathbf{B}$. Eq. (6.1f) implies conservation *of the magnitude* $|\mathbf{L}|$ of the angular momentum $\mathbf{L}$ of a magnetic dipole in a constant magnetic field $\mathbf{B}$ because,

$$d|\mathbf{L}|^2/dt = 2(\mathbf{L}\cdot(d\mathbf{L}/dt)) = 2(Q/(2Mc))(\mathbf{L}\cdot(\mathbf{L} \times \mathbf{B})) = 0, \tag{6.2a}$$

Thus a constant magnetic field $\mathbf{B}$ changes *only the direction* of the angular momentum $\mathbf{L}$ and magnetic moment $\mathbf{m}$ of a magnetic dipole. In *addition,* a constant magnetic field $\mathbf{B}$ changes *only those components of the magnetic dipole's angular momentum* $\mathbf{L}$ *which aren't parallel to it* because Eq. (6.1f) implies that,

$$d(\mathbf{L}\cdot\mathbf{B})/dt = (Q/(2Mc))((\mathbf{L} \times \mathbf{B})\cdot\mathbf{B}) = 0. \tag{6.2b}$$

The expression of *the solution* of the Eq. (6.1f) equation for the angular momentum $\mathbf{L}$ of a magnetic dipole in a constant magnetic field $\mathbf{B}$ *becomes more compact* when that equation *is rewritten in the form,*

$$d\mathbf{L}/dt = \omega_{|\mathbf{B}|}(\mathbf{L} \times \hat{\mathbf{u}}_\mathbf{B}), \text{ where, } \omega_{|\mathbf{B}|} \equiv ((Q|\mathbf{B}|)/(2Mc)) \text{ and } \hat{\mathbf{u}}_\mathbf{B} \equiv (\mathbf{B}/|\mathbf{B}|), \text{ so } |\hat{\mathbf{u}}_\mathbf{B}|^2 = 1. \tag{6.2c}$$

Probably the easiest way to solve Eq. (6.2c) is via its second-order in time version (whose general solution needs to be specialized to be that of the first-order in time Eq. (6.2c)), namely,

$$d^2\mathbf{L}/dt^2 = \omega_{|\mathbf{B}|}((d\mathbf{L}/dt) \times \hat{\mathbf{u}}_\mathbf{B}) = \omega^2_{|\mathbf{B}|}((\mathbf{L} \times \hat{\mathbf{u}}_\mathbf{B}) \times \hat{\mathbf{u}}_\mathbf{B}) = \omega^2_{|\mathbf{B}|}(\hat{\mathbf{u}}_\mathbf{B} \times (\hat{\mathbf{u}}_\mathbf{B} \times \mathbf{L})) =$$
$$\omega^2_{|\mathbf{B}|}(\hat{\mathbf{u}}_\mathbf{B}(\hat{\mathbf{u}}_\mathbf{B} \cdot \mathbf{L}) - \mathbf{L}) = -\omega^2_{|\mathbf{B}|}(\mathbf{L} - \hat{\mathbf{u}}_\mathbf{B}(\hat{\mathbf{u}}_\mathbf{B} \cdot \mathbf{L})). \tag{6.2d}$$

The vector entity $\hat{\mathbf{u}}_\mathbf{B}(\hat{\mathbf{u}}_\mathbf{B} \cdot \mathbf{L})$ in the last equality of Eq. (6.2d) is precisely the part of the vector $\mathbf{L}$ that is *parallel* to the constant magnetic field $\mathbf{B}$, which we know from Eq. (6.2b) *to be independent of time* t, so we can express it as its value *at the initial time* t_0, namely, $\hat{\mathbf{u}}_\mathbf{B}(\hat{\mathbf{u}}_\mathbf{B} \cdot \mathbf{L}) = \hat{\mathbf{u}}_\mathbf{B}(\hat{\mathbf{u}}_\mathbf{B} \cdot \mathbf{L}(t_0))$, whereupon we can rewrite Eq. (6.2d) in the form,

$$d^2(\mathbf{L} - \hat{\mathbf{u}}_\mathbf{B}(\hat{\mathbf{u}}_\mathbf{B} \cdot \mathbf{L}(t_0)))/dt^2 = -\omega^2_{|\mathbf{B}|}(\mathbf{L} - \hat{\mathbf{u}}_\mathbf{B}(\hat{\mathbf{u}}_\mathbf{B} \cdot \mathbf{L}(t_0))), \tag{6.2e}$$

which is merely a vector form of the simple harmonic oscillator equation, whose full solution in terms in terms of initial values at the initial time t_0 is,

$$(\mathbf{L}(t) - \hat{\mathbf{u}}_\mathbf{B}(\hat{\mathbf{u}}_\mathbf{B} \cdot \mathbf{L}(t_0))) = (\mathbf{L}(t_0) - \hat{\mathbf{u}}_\mathbf{B}(\hat{\mathbf{u}}_\mathbf{B} \cdot \mathbf{L}(t_0)))\cos(\omega_{|\mathbf{B}|}(t - t_0)) +$$
$$((d\mathbf{L}/dt)_{t=t_0}/\omega_{|\mathbf{B}|})\sin(\omega_{|\mathbf{B}|}(t - t_0)), \tag{6.2f}$$

from which we see that the full solution of Eq. (6.2d) for $\mathbf{L}(t)$ is,

$$\mathbf{L}(t) = \hat{\mathbf{u}}_\mathbf{B}(\hat{\mathbf{u}}_\mathbf{B} \cdot \mathbf{L}(t_0)) + (\mathbf{L}(t_0) - \hat{\mathbf{u}}_\mathbf{B}(\hat{\mathbf{u}}_\mathbf{B} \cdot \mathbf{L}(t_0)))\cos(\omega_{|\mathbf{B}|}(t - t_0)) +$$
$$((d\mathbf{L}/dt)_{t=t_0}/\omega_{|\mathbf{B}|})\sin(\omega_{|\mathbf{B}|}(t - t_0)). \tag{6.2g}$$

We in fact wish to solve *the first-order in time* equation given by Eq. (6.2c), from which we can see that,

$$(d\mathbf{L}/dt)_{t=t_0} = \omega_{|\mathbf{B}|}(\mathbf{L}(t_0) \times \hat{\mathbf{u}}_\mathbf{B}). \tag{6.2h}$$

We substitute the right side of Eq. (6.2h) into Eq. (6.2g) to obtain the solution of Eq. (6.2c),

$$\mathbf{L}(t) = \hat{\mathbf{u}}_\mathbf{B}(\hat{\mathbf{u}}_\mathbf{B} \cdot \mathbf{L}(t_0)) + (\mathbf{L}(t_0) - \hat{\mathbf{u}}_\mathbf{B}(\hat{\mathbf{u}}_\mathbf{B} \cdot \mathbf{L}(t_0)))\cos(\omega_{|\mathbf{B}|}(t - t_0)) + (\mathbf{L}(t_0) \times \hat{\mathbf{u}}_\mathbf{B})\sin(\omega_{|\mathbf{B}|}(t - t_0)). \tag{6.2i}$$

By applying the facts that $(\hat{\mathbf{u}}_\mathbf{B} \times \hat{\mathbf{u}}_\mathbf{B}) = \mathbf{0}$ and that,

$$((\mathbf{L}(t_0) \times \hat{\mathbf{u}}_\mathbf{B}) \times \hat{\mathbf{u}}_\mathbf{B}) = -(\mathbf{L}(t_0) - \hat{\mathbf{u}}_\mathbf{B}(\hat{\mathbf{u}}_\mathbf{B} \cdot \mathbf{L}(t_0))),$$

it is readily verified that the $\mathbf{L}(t)$ given by Eq. (6.2i) *satisfies* Eq. (6.2c). In addition, by applying the facts that $|\hat{\mathbf{u}}_\mathbf{B}|^2 = 1$ and $(\hat{\mathbf{u}}_\mathbf{B} \cdot (\mathbf{L}(t_0) \times \hat{\mathbf{u}}_\mathbf{B})) = 0$ it is easily verified that the $\mathbf{L}(t)$ given by Eq. (6.2i) satisfies,

$$(\hat{\mathbf{u}}_\mathbf{B} \cdot \mathbf{L}(t)) = (\hat{\mathbf{u}}_\mathbf{B} \cdot \mathbf{L}(t_0)). \tag{6.2j}$$

By also applying the facts that $(\mathbf{L}(t_0) \cdot (\mathbf{L}(t_0) \times \hat{\mathbf{u}}_\mathbf{B})) = 0$ and that,

$$|\mathbf{L}(t_0) - \hat{\mathbf{u}}_\mathbf{B}(\hat{\mathbf{u}}_\mathbf{B} \cdot \mathbf{L}(t_0))|^2 = |\mathbf{L}(t_0) \times \hat{\mathbf{u}}_\mathbf{B}|^2 = |\mathbf{L}(t_0)|^2 - (\hat{\mathbf{u}}_\mathbf{B} \cdot \mathbf{L}(t_0))^2,$$

it can furthermore be verified that the $\mathbf{L}(t)$ given by Eq. (6.2i) satisfies,

$$|\mathbf{L}(t)|^2 = |\mathbf{L}(t_0)|^2. \tag{6.2k}$$

Eq. (6.2i) describes *precession* of the angular momentum $\mathbf{L}(t)$, specifically *rotation* with angular frequency $\omega_{|\mathbf{B}|} = ((Q|\mathbf{B}|)/(2Mc))$ *of that part of* $\mathbf{L}(t)$ which is *perpendicular* to the constant magnetic field $\mathbf{B}$.

The spin $\mathbf{s}$ of a spin-1/2 electron *precesses similarly to* $\mathbf{L}$ in a constant magnetic field. That *quantum system* is described by a Hamiltonian operator, so it is worthwhile to obtain *the classical Hamiltonian* H that yields the precession equation $d\mathbf{L}/dt = (Q/(2Mc))(\mathbf{L} \times \mathbf{B})$ of Eq. (6.1f). Since $d\mathbf{L}/dt = \{\mathbf{L}, H\}$, where $\{\mathbf{L}, H\}$ *is the classical Poisson bracket of* $\mathbf{L}$ *with* H, $\{\mathbf{L}, H\}$ must be equal to $(Q/(2Mc))(\mathbf{L} \times \mathbf{B})$. We can *deduce* this H from the classical Poisson brackets *of the components of* $\mathbf{L}$. In detail those components are,

$$\mathbf{L} = (L_x, L_y, L_z) = (\mathbf{r} \times \mathbf{p}) = ((yp_z - zp_y),\ (zp_x - xp_z),\ (xp_y - yp_x)).$$

The particular classical Poisson bracket $\{L_x, L_y\}$ is the *antisymmetric* combination of the dot products of the gradients with respect to $\mathbf{r}$ and $\mathbf{p}$ of L_x and L_y, namely,

$$\{L_x, L_y\} \equiv \big[(\nabla_\mathbf{r} L_x(\mathbf{r},\mathbf{p}) \cdot \nabla_\mathbf{p} L_y(\mathbf{r},\mathbf{p})) - (\nabla_\mathbf{r} L_y(\mathbf{r},\mathbf{p}) \cdot \nabla_\mathbf{p} L_x(\mathbf{r},\mathbf{p}))\big] =$$
$$\big[(\nabla_\mathbf{r}(yp_z - zp_y) \cdot \nabla_\mathbf{p}(zp_x - xp_z)) - (\nabla_\mathbf{r}(zp_x - xp_z) \cdot \nabla_\mathbf{p}(yp_z - zp_y))\big] =$$
$$\big[((0, p_z, -p_y) \cdot (z, 0, -x)) - ((-p_z, 0, p_x) \cdot (0, -z, y))\big] = (xp_y - yp_x) = L_z.$$

The *antisymmetry* of Poisson brackets furthermore implies that $\{L_y, L_x\} = -L_z$, and it also implies that $\{L_x, L_x\} = \{L_y, L_y\} = \{L_z, L_z\} = 0$. The *remaining* Poisson brackets of L_x, L_y and L_z *are similar to* $\{L_x, L_y\}$, and produce the well-known simple cyclic closed Lie algebra of the rotation group,

$$\{L_x, L_y\} = -\{L_y, L_x\} = L_z, \quad \{L_y, L_z\} = -\{L_z, L_y\} = L_x, \quad \{L_z, L_x\} = -\{L_x, L_z\} = L_y. \tag{6.3a}$$

The nine Poisson brackets of L_x, L_y and L_z given above imply that,

$$\{L_x, (\mathbf{L}\cdot\mathbf{B})\} = \{L_x, (L_xB_x + L_yB_y + L_zB_z)\} = L_zB_y - L_yB_z = -(L_yB_z - L_zB_y) = -(\mathbf{L}\times\mathbf{B})_x, \tag{6.3b}$$

and they likewise imply that $\{L_y, (\mathbf{L}\cdot\mathbf{B})\} = -(\mathbf{L}\times\mathbf{B})_y$ and $\{L_z, (\mathbf{L}\cdot\mathbf{B})\} = -(\mathbf{L}\times\mathbf{B})_z$, so,

$$\{\mathbf{L}, (\mathbf{L}\cdot\mathbf{B})\} = -(\mathbf{L}\times\mathbf{B}). \tag{6.3c}$$

Eq. (6.3c) implies that the particular classical Hamiltonian,

$$H \equiv -(Q/(2Mc))(\mathbf{L}\cdot\mathbf{B}), \tag{6.3d}$$

yields the precession equation $d\mathbf{L}/dt = \{\mathbf{L}, H\} = (Q/(2Mc))(\mathbf{L}\times\mathbf{B})$ of Eq. (6.1f).

We next briefly discuss the spin-1/2 electron in a constant magnetic field $\mathbf{B}$, whose quantum Hamiltonian operator is largely *modeled* on the classical Hamiltonian $H = -(Q/(2Mc))(\mathbf{L}\cdot\mathbf{B})$ of Eq. (6.3d).

7. The spin-1/2 electron in a constant magnetic field

The Hamiltonian operator for a spin-1/2 electron in a constant magnetic field $\mathbf{B}$ is required to have only *two* energy eigenstates to accord with spectroscopic observation. If that Hamiltonian operator's structure is akin to that of the classical Hamiltonian $H = -(Q/(2Mc))(\mathbf{L}\cdot\mathbf{B})$ of Eq. (6.3d) for the angular momentum $\mathbf{L}$ of a magnetic dipole in a constant magnetic field $\mathbf{B}$, then the three *components* s_x, s_y and s_z of the spin-1/2 electron's *spin vector* $\mathbf{s}$ must, like that Hamiltonian operator itself, *have only two eigenstates,* and therefore must be 2×2 Hermitian *matrices.* In *addition,* s_x, s_y and s_z must have *quantum* Poisson-bracket relations *that are appropriate to angular momentum,* namely *that are formally the same* as the simple cyclic *classical* Poisson bracket relations for L_x, L_y and L_z which are given in Eq. (6.3a) and accord with the closed Lie algebra for the rotation group. Since the *quantum* Poisson bracket for two quantum operators (or matrices) is $(-i/\hbar)$ *times their commutator,* the three fundamental angular-momentum *commutation relations* which the three 2×2 Hermitian spin-1/2 *spin matrices* s_x, s_y and s_z *must adhere to* are,

$$s_xs_y - s_ys_x = i\hbar s_z, \qquad s_ys_z - s_zs_y = i\hbar s_x, \qquad s_zs_x - s_xs_z = i\hbar s_y. \tag{7.1a}$$

To satisfy the three Eq. (7.1a) angular-momentum commutation relations *with* 2×2 *Hermitian matrices,* Wolfgang Pauli first *explicitly exhibited* three 2×2 Hermitian matrices σ_x, σ_y, and σ_z which mutually anticommute and satisfy the following cyclic closed *matrix multiplication relations,*

$$\sigma_x\sigma_y = -\sigma_y\sigma_x = i\sigma_z, \qquad \sigma_y\sigma_z = -\sigma_z\sigma_y = i\sigma_x, \qquad \sigma_z\sigma_x = -\sigma_x\sigma_z = i\sigma_y, \qquad \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1. \tag{7.1b}$$

Each one of these three 2×2 Hermitian matrices obviously has the two eigenvalues ± 1. Furthermore,

$$s_x \equiv (\hbar/2)\sigma_x, \qquad s_y \equiv (\hbar/2)\sigma_y, \qquad s_z \equiv (\hbar/2)\sigma_z, \tag{7.1c}$$

are 2×2 Hermitian matrices *which satisfy* Eq. (7.1a), and each of them has the two eigenvalues $\pm(\hbar/2)$. Pauli's *explicit* 2×2 *Hermitian forms of* σ_x*,* σ_y*, and* σ_z are as follows: σ_z has $(1, -1)$ along its main diagonal and $(0, 0)$ along its secondary diagonal, σ_x has $(0, 0)$ along its main diagonal and $(1, 1)$ along its secondary diagonal, and σ_y has $(0, 0)$ along its main diagonal and $(-i, i)$ along its secondary diagonal. These very convenient forms of σ_x, σ_y, and σ_z *aren't unique,* however, since *any unitary* 2×2-matrix transformation of them remains Hermitian and satisfies the crucial matrix multiplication relations of Eq. (7.1b).

The *minimal adaption* of the classical Hamiltonian $H = -(Q/(2Mc))(\mathbf{L}\cdot\mathbf{B})$ of Eq. (6.3d) for a magnetic-dipole *one-dimensional loop* of charge Q and mass M in a constant magnetic field $\mathbf{B}$ to the *quantum* spin-1/2 electron in that magnetic field *replaces* Q by the electron charge $-e$, M by the electron mass m_e and the classical $\mathbf{L}$ by the quantum $\mathbf{s} = (s_x, s_y, s_z)$ of Eq. (7.1c). The difference between the two energy eigenvalues $\pm((e|\mathbf{B}|\hbar)/(4m_ec))$ of this quantum Hamiltonian corresponds to a transition-photon angular frequency of $((e|\mathbf{B}|)/(2m_ec))$, which exactly matches the classical Hamiltonian's precession angular frequency $\omega_{|\mathbf{B}|} = ((Q|\mathbf{B}|)/(2Mc))$ of Eqs. (6.2i) and (6.2c) when $Q = e$ and $M = m_e$. But the spectroscopic observation *is double this angular frequency,* so the above *adapted* Hamiltonian is *adjusted* to be $H = (e/(m_ec))(\mathbf{s}\cdot\mathbf{B})$.

That the electron's spinning charge and mass *aren't* adequately modeled by a *one-dimensional loop* of that charge and mass isn't too surprising; the spectroscopic observation might indicate, as one example, that the electron's mean circulating-charge radius is larger than its mean circulating-mass radius. The electron's structure can't be definitively determined by experiments which scatter projectiles off of electrons because their *resolution* is *limited* to approximately the Compton wavelength of the electron by the electron-positron pair production which accompanies such scattering at sufficiently high energies and confuses the electron's location—quantum scattering cross sections are invariant under the interchange of identical particles. One would in fact presume that the electron's structure reflects the Compton wavelengths of the vector bosons of the weak interactions, which are vastly smaller than the electron's Compton wavelength. That the electron's structure is determined by the weak interactions is *suggested* by the *existence* of the muon and tau particles, which have electron-like characteristics but much higher masses than that of the electron, and which decay *only weakly;* the electron would appear to be the ground state of a weak-interaction spectroscopic family.