On the change in electrical power caused by loading an electrical circuit with an impedance

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Abstract

In this paper we apply the theory of two–ports to present and to proof two real–valued theorems and two complex–valued theorems on the difference in electrical power of an unloaded and a loaded circuit driven by a voltage source or a current source, respectively.

1 Theorems on real–valued impedances

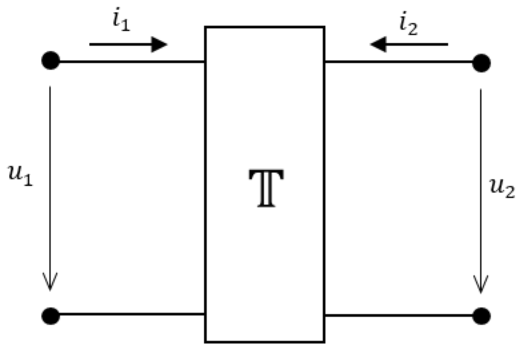

Figure 1

For given currents i_1, i_2 and a given two–port matrix

$$\mathbb{T} := \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

the voltages u_1 and u_2 can be determined from

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbb{T} \cdot \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

We then have the following results:

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Lemma

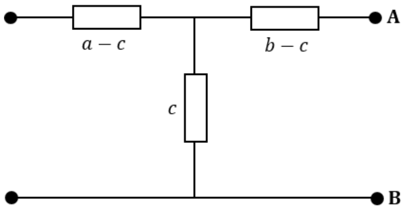

Figure 2

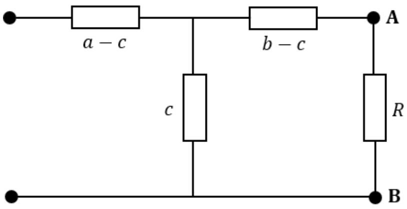

Figure 3

1. The equivalent resistance of the circuit shown in Figure 2 equals

$$R_e = a$$

2. The equivalent resistance of the circuit shown in Figure 3 loaded with resistor R equals

$$\widetilde{R}_e = a - \frac{c^2}{b + R}$$

Proof

1. $R_e = (a - c) + (c) = a \qquad \square$
2. $\widetilde{R}_e = (a - c) + (c) \parallel ((b - c) + R) = (a - c) + \dfrac{c \cdot (b - c + R)}{c + (b - c + R)}$

   $= a - \dfrac{c^2}{b + R} \qquad \square$

From this lemma the following theorem can be derived:

Theorem 1

Let u be the source voltage of a circuit, u_o the open source voltage between the terminals A and B of the unloaded voltage source circuit and R_i the internal resistance of the unloaded voltage source circuit, respectively.

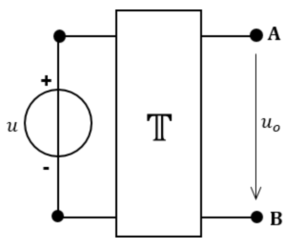

Figure 4

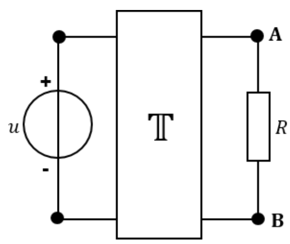

Figure 5

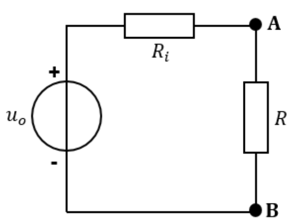

Figure 6

Let p be the electrical power of the unloaded circuit displayed in Figure 4. Further, let $\widetilde{p}$ be the electrical power of the circuit displayed in Figure 5 loaded with resistor R. Then

$$\widetilde{p} - p = \frac{u_o^2}{R_i + R}$$

Proof

Let $\mathbb{T}$ be the two–port matrix of the circuit

$$\mathbb{T} := \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

Using part (1) of the lemma

$$p = \frac{u^2}{R_e} = \frac{u^2}{a}$$

Using part (2) of the lemma

$$\widetilde{p} = \frac{u^2}{\widetilde{R}_e} = \frac{u^2}{a - \dfrac{c^2}{b+R}}$$

From the circuit displayed in Figure 4 it follows

$$u_o = \frac{c}{(a-c)+c} \cdot u = \frac{c}{a} \cdot u$$

and

$$R_i = (b-c) + c \parallel (a-c) = b - c + \frac{c \cdot (a-c)}{c + (a-c)} = \frac{a \cdot b - c^2}{a}$$

Then

$$\frac{u_o^2}{R_i + R} = \frac{\left(\dfrac{c}{a} \cdot u\right)^2}{\dfrac{a \cdot b - c^2}{a} + R} = \frac{c^2}{a \cdot (b+R) - c^2} \cdot \frac{u^2}{a} = \widetilde{p} - p \qquad \square$$

The theorem states that the change in electrical power between the loaded and the unloaded circuit equals the electrical power of the loaded Thévenin circuit:

$$p_{loaded} - p_{unloaded} = p_{Th\acute{e}venin\ loaded}$$

From the previous lemma the following theorem can be derived:

Theorem 2

Let i be the current source of a circuit, u_o the open source voltage between the terminals A and B of the unloaded current source circuit and R_i the internal resistance of the unloaded current source circuit, respectively.

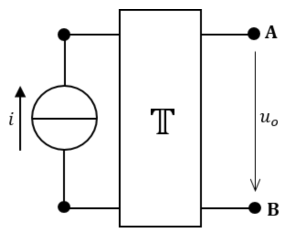

Figure 7

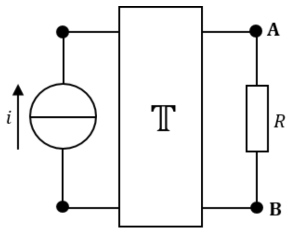

Figure 8

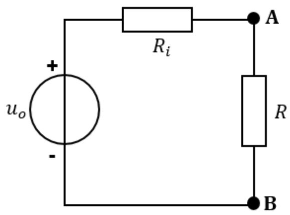

Figure 9

Let p be the electrical power of the unloaded circuit displayed in Figure 7. Further, let $\widetilde{p}$ be the electrical power of the circuit displayed in Figure 8 loaded with resistor R.
Then

$$p - \widetilde{p} = \frac{u_o^2}{R_i + R}$$

Proof

Let $\mathbb{T}$ be the two–port matrix of the circuit

$$\mathbb{T} := \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

Using part (1) of the lemma

$$p = i^2 \cdot R_e = i^2 \cdot a$$

Using part (2) of the lemma

$$\widetilde{p} = i^2 \cdot \widetilde{R}_e = i^2 \cdot \left(a - \frac{c^2}{b + R} \right)$$

From the circuit displayed in Figure 7 it follows that

$$u_o = i \cdot (c) = c \cdot i$$

and

$$R_i = (b - c) + (c) = b$$

Then, it holds that

$$\frac{u_0^2}{R_i + R} = \frac{(c \cdot i)^2}{b + R} = \frac{c^2}{b + R} \cdot i^2 = p - \widetilde{p} \qquad \square$$

Note that the expression $p - \widetilde{p}$ for the change in electrical power for a circuit connected across a current source equals $-(\widetilde{p} - p)$ where $(\widetilde{p} - p)$ expresses the change in electrical power for a voltage source.

The theorem states that the change in electrical power between the unloaded and the loaded circuit equals the electrical power of the loaded Thévenin circuit:

$$p_{unloaded} - p_{loaded} = p_{Thévenin\ loaded}$$

Example 1

Application of the derived theorems to the Wheatstone bridge/twin parallel voltage divider.

Given: $R_1 = 10\ \Omega, R_2 = 40\ \Omega, R_3 = 20\ \Omega, R_4 = 30\ \Omega$.

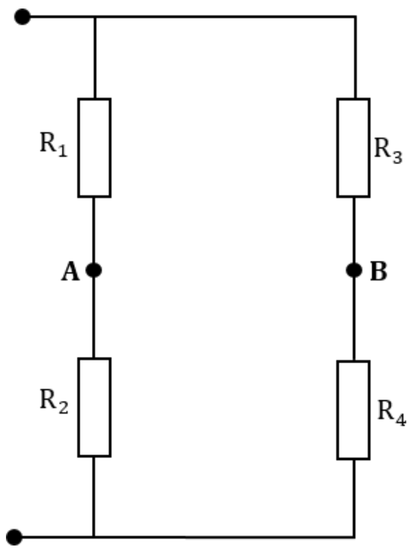

Figure 10

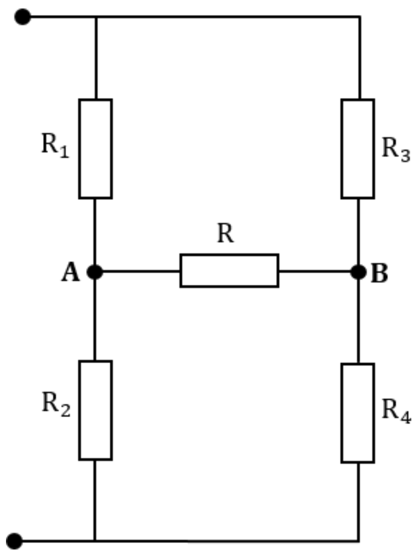

Figure 11

The two–port matrix of this circuit equals

$$\mathbb{T} := \begin{bmatrix} a & c \\ c & b \end{bmatrix} =$$

$$= \frac{1}{R_1 + R_2 + R_3 + R_4} \cdot \begin{bmatrix} (R_1 + R_2) \cdot (R_3 + R_4) & R_2 \cdot R_3 - R_1 \cdot R_4 \\ R_2 \cdot R_3 - R_1 \cdot R_4 & (R_1 + R_3) \cdot (R_2 + R_4) \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 5 \\ 5 & 21 \end{bmatrix}\ \Omega$$

The equivalent resistance of the unloaded circuit equals

$$R_e = a = 25\ \Omega$$

The equivalent resistance of the circuit loaded with resistor $R = 4\ \Omega$ equals

$$\widetilde{R}_e = a - \frac{c^2}{b + R} = 24\ \Omega$$

Given the source voltage $u = 600$ V of the circuit displayed in Figure 12

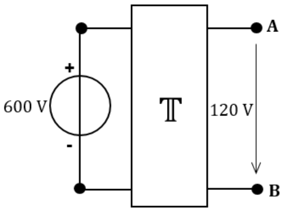

Figure 12

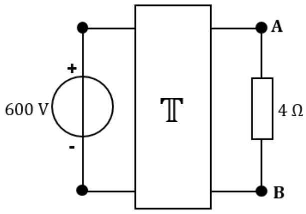

Figure 13

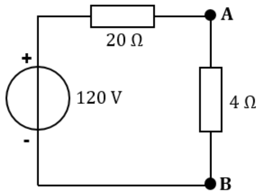

Figure 14

The electrical power of the unloaded circuit equals

$$p = \frac{u^2}{R_e} = 14400 \text{ W}$$

The electrical power of the circuit loaded with resistor $R = 4\ \Omega$ equals

$$\widetilde{p} = \frac{u^2}{\widetilde{R}_e} = 15000 \text{ W}$$

The open source voltage between the terminals A and B equals of the unloaded circuit equals

$$u_0 = \frac{c}{a} \cdot u = 120 \text{ V}$$

The internal resistance of the unloaded circuit

$$R_i = \frac{a \cdot b - c^2}{a} = 20\ \Omega$$

The change in electrical power according to Theorem 1 amounts to

$$\widetilde{p} - p = \frac{u_o^2}{R_i + R} = 600 \text{ W}$$

Example 2

Given the source current $i = 25$ A of the circuit displayed in Figure 15

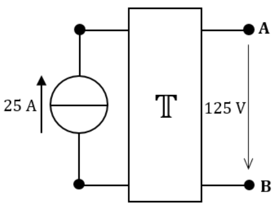

Figure 15

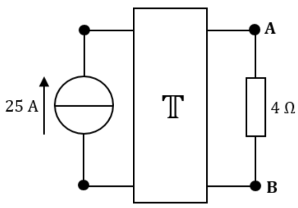

Figure 16

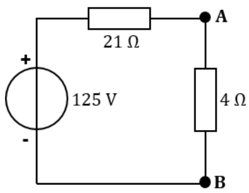

Figure 17

The electrical power of the unloaded circuit equals

$$p = i^2 \cdot R_e = 15625 \text{ W}$$

The electrical power of the circuit loaded with resistor $R = 4\ \Omega$ equals

$$\widetilde{p} = i^2 \cdot \widetilde{R}_e = 15000 \text{ W}$$

The open source voltage between the terminals A and B of the unloaded circuit amounts to

$$u_o = c \cdot i = 125 \text{ V}$$

The internal resistance of the unloaded circuit equals

$$R_i = b = 21\ \Omega$$

The change in electrical power according to Theorem 2 is equal to

$$p - \widetilde{p} = \frac{u_o^2}{R_i + R} = 625 \text{ W}$$

2 Theorems on complex–valued impedances

In the complex plane the power of an impedance can be expressed as

$$P := \frac{|U|^2}{Z^*} = |I|^2 \cdot Z$$

In the complex plane these expressions are no analytic functions of the complex source voltage U or the complex source current I. The electrical powers p and $\widetilde{p}$ are real-valued functions of the source voltage u or the source current i and the entries of the two–port matrix $\mathbb{T}$. The variables u, i and the entries of the two–port matrix $\mathbb{T}$ are real–valued. Yet, the domain of these variables can be extended to the complex plane, to obtain analytic continuations of the functions p and $\widetilde{p}$. For this reason, the theorems stated and derived above, also apply to a complex valued source voltage U or a complex valued source current I and impedances.

Theorem 3

Let U be the source voltage of a circuit and U_o being the open source voltage between the terminals A and B of the unloaded circuit. Let Z_i be the internal impedance of the unloaded circuit.

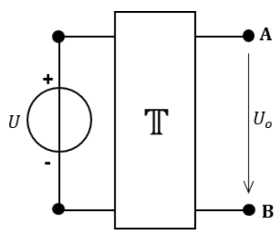

Figure 18

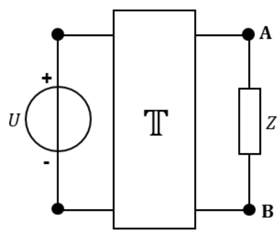

Figure 19

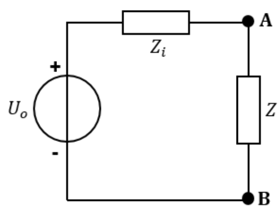

Figure 20

Let P the electrical power of the unloaded circuit displayed in Figure 18 Let $\widetilde{P}$ the electrical power of the circuit displayed in Figure 19 loaded with

impedance Z.
Then

$$\widetilde{P} - P = \left(\frac{U_o^2}{Z_i + Z} \cdot \frac{U^*}{U}\right)^*$$

Remark: note that

$$\widetilde{P} - P \neq \frac{|U_o|^2}{(Z_i + Z)^*}, \ |\widetilde{P} - P| = \left|\frac{|U_o|^2}{(Z_i + Z)^*}\right|$$

Proof

According to Theorem 1

$$\widetilde{p} - p = \frac{U_o^2}{Z_i + Z}$$

The proof proceeds by multiplying this identity with the factor U^*/U followed by conjugating the result:

$$\frac{U^*}{U} \cdot (\widetilde{p} - p) = \frac{U^*}{U} \cdot \left(\frac{U^2}{\widetilde{Z}_e} - \frac{U^2}{Z_e}\right) = \frac{U \cdot U^*}{\widetilde{Z}_e} - \frac{U \cdot U^*}{Z_e} = \frac{|U|^2}{\widetilde{Z}_e} - \frac{|U|^2}{Z_e} = \frac{U_o^2}{Z_i + Z} \cdot \frac{U^*}{U}$$

$$\left(\frac{|U|^2}{\widetilde{Z}_e} - \frac{|U|^2}{Z_e}\right)^* = \frac{|U|^2}{\widetilde{Z}_e^*} - \frac{|U|^2}{Z_e^*} = \widetilde{P} - P = \left(\frac{U_o^2}{Z_i + Z} \cdot \frac{U^*}{U}\right)^* \qquad \square$$

As remarked, the change in electrical power and the electrical power of the Thévenin circuit differ but are equal in magnitude.

Theorem 4

Let I the current source of a circuit. Let U_o be the open source voltage between the terminals A and B of the unloaded circuit. Let Z_i be the internal impedance of the unloaded circuit.

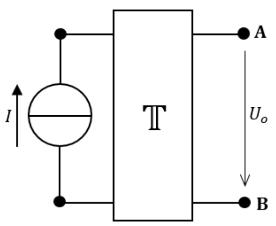

Figure 21

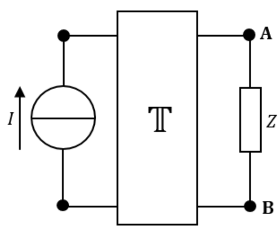

Figure 22

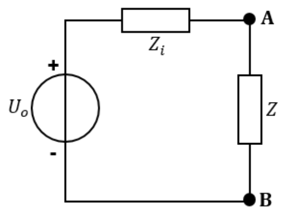

Figure 23

Let P be the electrical power of the unloaded circuit displayed in Figure 21 and $\widetilde{P}$ be the electrical power of the circuit displayed in Figure 22 loaded with resistor Z.
Then

$$P - \widetilde{P} = \frac{U_o^2}{Z_i + Z} \cdot \frac{I^*}{I}$$

Remark. Note that

$$P - \widetilde{P} \neq \frac{|U_o|^2}{(Z_i + Z)^*}, \quad \left|P - \widetilde{P}\right| = \left|\frac{|U_o|^2}{(Z_i + Z)^*}\right|$$

Proof

According to Theorem 2

$$p - \widetilde{p} = \frac{U_o^2}{Z_i + Z}$$

The proof proceeds by multiplying this identity with the factor I^*/I:

$$\frac{I^*}{I} \cdot (p - \widetilde{p}) = \frac{I^*}{I} \cdot \left(I^2 \cdot Z_e - I^2 \cdot \widetilde{Z}_e\right) = I \cdot I^* \cdot Z_e - I \cdot I^* \cdot \widetilde{Z}_e =$$

$$|I|^2 \cdot Z_e - |I|^2 \cdot \widetilde{Z}_e = P - \widetilde{P} = \frac{U_o^2}{Z_i + Z} \cdot \frac{I^*}{I} \qquad \square$$

As has been remarked, the change in electrical power and the electrical power of the Thévenin circuit differ but are equal in magnitude.

Example 3

Given: $Z_1 = -30\jmath\ \Omega, Z_2 = 10\ \Omega$

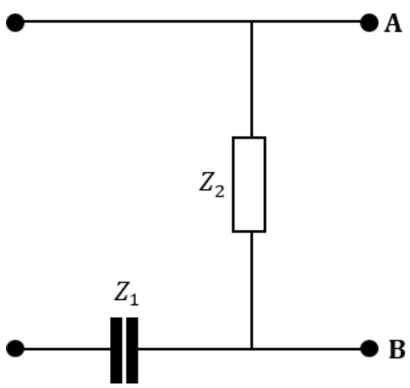

Figure 24

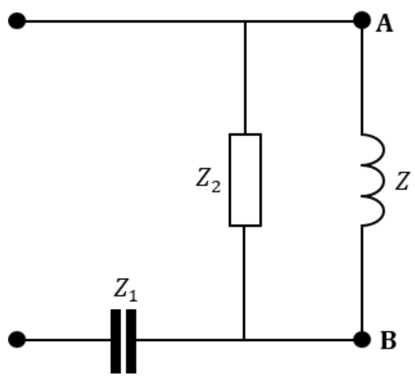

Figure 25

The two–port matrix of this circuit equals

$$\mathbb{T} := \begin{bmatrix} A & C \\ C & B \end{bmatrix} = \begin{bmatrix} Z_1 + Z_2 & Z_2 \\ Z_2 & Z_2 \end{bmatrix} = \begin{bmatrix} 10 - 30\jmath & 10 \\ 10 & 10 \end{bmatrix} \Omega$$

The equivalent impedance of the unloaded circuit equals

$$Z_e = A = 10 - 30\jmath\ \Omega$$

The equivalent impedance of the circuit loaded with impedance $Z = 20\jmath\ \Omega$ equals

$$\widetilde{Z}_e = A - \frac{C^2}{B + Z} = 8 - 26\jmath\ \Omega$$

Given the source voltage $U = 740 + 370\jmath$ V of the circuit displayed in Figure 26

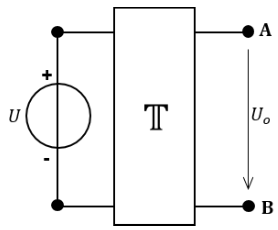

Figure 26

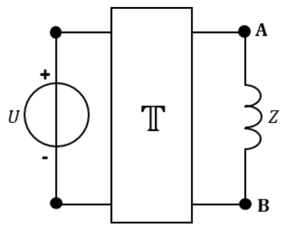

Figure 27

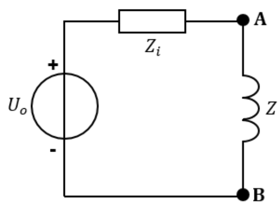

Figure 28

The electrical power of the unloaded circuit equals

$$P = \frac{|U|^2}{Z_e^*} = 6845 - 20535\jmath \text{ VA}$$

The electrical power of the circuit loaded with impedance $Z = 20\jmath\ \Omega$ equals

$$\widetilde{P} = \frac{|U|^2}{\widetilde{Z}_e^*} = 7400 - 24050\jmath \text{ VA}$$

The open source voltage between the terminals A and B of the unloaded circuit equals

$$U_o = \frac{C}{A} \cdot U = -37 + 259\jmath \text{ V}$$

The internal impedance of the unloaded circuit equals

$$Z_i = \frac{A \cdot B - C^2}{A} = 9 - 3\jmath\ \Omega$$

Change in electrical power according to Theorem 3

$$\widetilde{P} - P = \left(\frac{U_o^2}{Z_i + Z} \cdot \frac{U^*}{U} \right)^* = 555 - 3515\jmath \text{ VA}$$

Note that

$$\widetilde{P} - P = 555 - 3515\jmath \text{ VA} \neq \frac{|U_o|^2}{(Z_i + Z)^*} = 1665 + 3145\jmath \text{ VA}$$

whereas

$$\left| \widetilde{P} - P \right| = \left| \frac{|U_o|^2}{(Z_i + Z)^*} \right| = 185\sqrt{370} \text{ VA}$$

Example 4

Given

$$Z_1 = 15\ \Omega, Z_2 = 30\jmath\ \Omega, Z_3 = 40\ \Omega$$

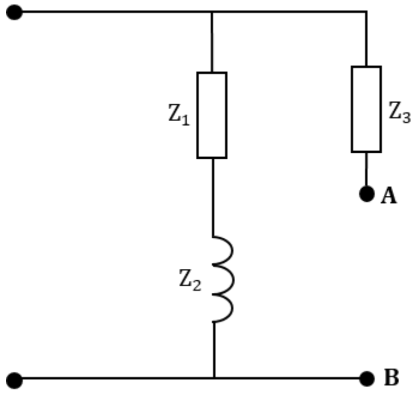

Figure 29: Circuit

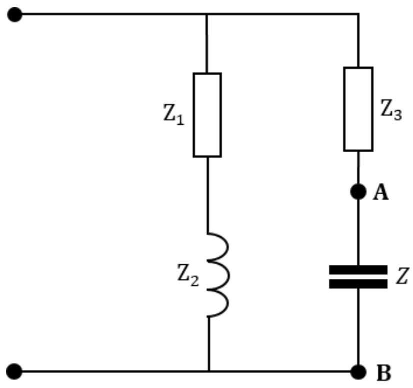

Figure 30: Circuit

The two–port matrix of this circuit equals

$$\mathbb{T} := \begin{bmatrix} A & C \\ C & B \end{bmatrix} = \begin{bmatrix} Z_1 + Z_2 & Z_1 + Z_2 \\ Z_1 + Z_2 & Z_1 + Z_2 + Z_3 \end{bmatrix} = \begin{bmatrix} 15 + 30\jmath & 15 + 30\jmath \\ 15 + 30\jmath & 55 + 30\jmath \end{bmatrix} \; \Omega$$

The equivalent impedance of the unloaded circuit equals

$$Z_e = A = 15 + 30\jmath \; \Omega$$

The equivalent impedance of the circuit loaded with impedance $Z = -20\jmath \; \Omega$ equals

$$\widetilde{Z}_e = A - \frac{C^2}{B + Z} = 24 + 12\jmath \; \Omega$$

Given the source current $I = 100 - 75\jmath$ A of the circuit displayed in Figure 31

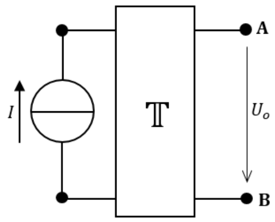

Figure 31

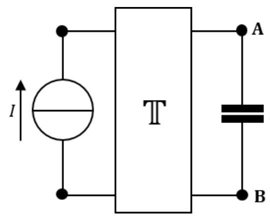

Figure 32

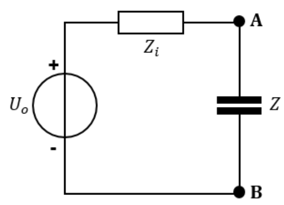

Figure 33

The electrical power of the unloaded circuit equals

$$P = |I|^2 \cdot Z_e = 234375 + 468750\jmath \text{ VA}$$

The electrical power of the circuit loaded with impedance $Z = -20\jmath \; \Omega$ equals

$$\widetilde{P} = |I|^2 \cdot \widetilde{Z}_e = 375000 + 187500\jmath \text{ VA}$$

The open source voltage between the terminals A and B of the unloaded circuit equals

$$U_o = C \cdot I = 3750 + 1875\jmath \text{ VA}$$

The internal impedance of the unloaded circuit equals

$$Z_i = B = 55 + 30\jmath\ \Omega$$

Change in electrical power according to Theorem 4

$$P - \widetilde{P} = \frac{U_o^2}{Z_i + Z} \cdot \frac{I^*}{I} = -140625 + 281250\jmath \text{ VA}$$

Note that

$$P - \widetilde{P} = -140625 + 281250\jmath \text{ VA} \neq \frac{|U_o|^2}{(Z_i + Z)^*} = 309375 + 56250\jmath \text{ VA}$$

whereas

$$\left|P - \widetilde{P}\right| = \left|\frac{|U_o|^2}{(Z_i + Z)^*}\right| = 140625\sqrt{5} \text{ VA}$$

3 Acknowledgement

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