On identical harmonic and isochronous periodic oscillations of Lienard type equations

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Abstract

We present in this work some exceptional classes of conservative, quadratic and mixed Lienard type equations with identical exact solutions. We show that, in particular, some of these equations can exhibit the sinusoidal periodic solution of the linear harmonic oscillator. Consequently, they can be used to describe harmonic and isochronous periodic oscillations of dynamical systems.

Keywords: Linear harmonic oscillator, Lienard type equations, exact periodic solutions, sinusoidal solutions, isochronous oscillations.

Introduction

The linear harmonic oscillator

$$\ddot{x} + \omega^2 x = 0 \tag{1}$$

with the solution

$$x(t) = \mu \sin(\omega t + \varphi) \tag{2}$$

where the overdot means differentiation with respect to time, and ω, μ and φ are arbitrary constants, is well known to be the basic model describing oscillations in dynamical systems. As such, the equation (1) could not represent real world dynamical systems when nonlinear phenomena are taken into account. Real world dynamical systems, as it is known, exhibit together geometric, dissipative and inertial nonlinearities so that their description requires to solve complicated nonlinear differential equations [1-4]. However, there is a limited number of nonlinear differential equations that can be solved exactly and explicitly, as one

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can see in the literature. According to [1-4] a general second-order nonlinear differential equation that can adequatly and satisfactorily model dynamical systems can take the form

$$\ddot{x} + u(x)\dot{x}^2 + \vartheta(x)\dot{x} + h(x) = 0 \tag{3}$$

where $u(x)$, $\vartheta(x)$, and $h(x)$ are arbitrary functions of x. As an equation of the type (3) is very difficult to solve, the reduced form

$$\ddot{x} + h(x) = 0 \tag{4}$$

where $h(x)$ is choosen to be nonlinear function of x, to take into account geometric nonlinearity, is widely used in literature to model periodic oscillations of dynamical systems. A famous equation of the type (4) is the conservative cubic Duffing equation

$$\ddot{x} + \alpha x + \beta x^3 = 0 \tag{5}$$

where α and β are arbitrary parameters. This equation has been for a long time used as an oscillator to model a variety of phenomena in physics [5, 6] due to the fact that its general periodic solution is well known [7, 8]. Indeed, the cubic Duffing equation (5) exhibits the Jacobi elliptic functions as general solutions, which are of practical use, since their analytical properties are well known in the literature. However, in [7, 8] the authors have successfully shown that the conservative cubic Duffing equation (5) can also exhibit unbounded tangent periodic solutions such that the feature of oscillator of this equation is not consistent. Another celebrated equation of type (4) is the Ermakov-Pinney equation

$$\ddot{x} + \alpha x + \frac{\gamma}{x^3} = 0 \tag{6}$$

where α and γ are arbitrary constants. As the exact solution of the equation (6) is known, this equation has been used in many fields of science [9]. In [10] Monsia succeeded to calculate a new periodic solution to the Ermakov-Pinney equation (6). Using this solution, Monsia and coworkers in [11] showed the property of two different differential equations to have identical periodic solutions. In this way, the authors [11] presented for the first time a singular quadratic Lienard type equation that exhibits the linear harmonic oscillator solution that is to say, a sinusoidal solution with amplitude independent frequency known as isochronicity property. To take into account dissipative nonlinearity property in dynamical systems, the Lienard type equation

$$\ddot{x} + \vartheta(x)\dot{x} + h(x) = 0 \tag{7}$$

is widely investigated in the literature. In this sense one can consider the modified Emden type equation

$$\ddot{x} + k_1 x\dot{x} + k_2 x^3 = 0 \tag{8}$$

where k_1 and k_2 are arbitrary parameters, studied in several papers [12-14]. In [15] the generalized and modified Emden type equation

$$\ddot{x} + kx\dot{x} + \frac{k^2}{9}x^3 + \lambda x = 0 \tag{9}$$

where k and λ are arbitrary parameters, has been deeply analyzed to secure harmonic and isochronous periodic solution. It is then for the first time such an explicit and exact solution has been obtained for a dissipative equation of the form (7) so that the authors [15] claimed that equation (9) is an unusual Lienard type nonlinear oscillator. In this regard, the equation (9) has become an attractive dynamical system examined in the literature from relevant and different methods. In a recent paper [16] Monsia and coworkers investigated this claimed Lienard type nonlinear oscillator (9) using the first integral method. They found that the equation (9) can exhibit unbounded periodic solution [16]. The authors [16] concluded that the equation (9) is nothing but a pseudo-oscillator. On the other hand, the quadratically dissipative Lienard equation

$$\ddot{x} + u(x)\dot{x}^2 + h(x) = 0 \tag{10}$$

has been also an attractive subject in mathematics and physics. As can be seen, there is a vast literature on the equation (10). In [17] the authors claimed to find a unique oscillator of type (10) exhibiting harmonic periodic solution but with amplitude-dependent frequency. In this regard, the quadratic Lieanrd type equation presented in [17] has been the object of an intensive study from mathematical and physical points of view [18-20] for a long time. Hovewer, recently, Monsia et al. [21] presented a quadratic Lienard type equation that can exhibit sinusoidal periodic oscillations but with amplitude-dependent frequency. In [22] Akande and coworkers presented a class of quadratic Lienard type equations that can exhibit sinusoidal periodic solution but with amplitude-dependent frequency. In other papers [23, 24], the same group calculated the exact and explicit general solution of some quadratic Lienard type equations using the generalized Sudman transformation. In this context, in a recent paper [25] Akande and coworkers presented new exact and explicit general solutions of the so-called Mathews-Lakshmanan oscillator [17]. The result obtained by this group [25] is unambiguous: The Mathews-Lakshmanan quadratically damped equation is a pseudo-oscillator. Indeed, the authors [25] succeeded to show that the Mathews-Lakshmanan equation [17] can exhibit real and complex-valued non-periodic solutions. The same group [25] succeeded also to find the results obtained in [17] using for the first time the direct integration method consisting to reduce the equation to solve to the quadrature. More recently in [26] the authors presented an exceptional quadratically damped Lienard type equation. This equation, indeed, can exhibit the sinusoidal periodic solution of the linear harmonic oscillator that is, with amplitude independent frequency. It was for the first time, such a solution has been found for an equation of type (10). In [11] the authors have also shown the existence of a singular quadratic Lienard type equation that can exhibit the sinusoidal and isochronous periodic oscillations of the linear harmonic oscillator. From the literature, it is possible to notice that the mixed Lienard type equation (3) has been investigated by many authors. In [27] the Lie point symmetries of the equation (3) have been examined. The authors in [28] studied the inverse problem of the mixed Lienard type equation (3) to identify periodic solutions. In [21] Monsia and coworkers investigated the mixed Lienard type equation (3) using the non-point transformation. By going through these works, one easily observes that the problem of calculating periodic solutions arises entirely with acuity. In this regard, Monsia and his group studied recently the equation (3) in [29, 30]. The authors in [29, 30] succeeded to calculate sinusoidal and isochronous periodic solutions of the equation (3) by identifying appropriate choice of functions $u(x)$, $\vartheta(x)$ and $h(x)$ for the first time. The problem to be solved here is to ask whether a mixed Lienard type equation of the form (3) can admit identical periodic solutions with other types of Lienard equations. Consequently, the objective is to show that the mixed Lienard type equation (3) can exhibit identical sinusoidal and isochronous periodic solutions with other types of Lienard equations. To attain the previous objective, we give a brief overview of the first integral under differentiation method introduced in the literature [29, 30] by Monsia and his group (section 2) and show the equations of Lienard of different types with identical sinusoidal and isochronous periodic solutions (section 3). We present a conclusion finally for the work.

2- Brief overview of the theory

According to [29, 30] the second-order mixed Lienard type equation for the present purpose can be read

$$\ddot{x}+\frac{g'(x)}{g(x)}\dot{x}^2+a\,\ell x^{\ell-1}\frac{f(x)}{g(x)}\,\dot{x}-a^2\,x^{2\ell}\,\frac{f'(x)f(x)}{g^2(x)}+ab\,x^{\ell}\,\frac{f'(x)}{g^2(x)}=0 \tag{11}$$

The corresponding first integral is of the form

$$g(x)\,\dot{x}+a\,f(x)x^{\ell}=b \tag{12}$$

where a, b and ℓ are arbitrary parameters, $g(x)\neq 0$, and $f(x)$ are functions of x, and prime means derivative with respect to the argument. We can now establish the Lienard equations with identical sinusoidal and isochronous periodic solutions.

3- Lienard type equations

To establish the Lienard equations of different types with identical periodic solutions, consider $g(x)=f^2(x)\,x^{\ell}$, so that

$$\left[f(x)\,\dot{x}+a\,\right]f(x)x^{\ell}=b \tag{13}$$

In this situation the equation (11) turns into

$$\ddot{x}+\left(\frac{2f'(x)}{f(x)}+\frac{\ell}{x}\right)\dot{x}^2+\frac{a\ell}{f(x)}\,\frac{\dot{x}}{x}-a^2\,\frac{f'(x)}{f^3(x)}+ab\,\frac{f'(x)}{f^4(x)}\,x^{-\ell}=0 \tag{14}$$

By application of $b=0$, the equation (14) reduces to

$$\ddot{x}+\left(\frac{2f'(x)}{f(x)}+\frac{\ell}{x}\right)\dot{x}^2+\frac{a\ell}{f(x)}\,\frac{\dot{x}}{x}-a^2\,\frac{f'(x)}{f^3(x)}=0 \tag{15}$$

and the equation (13) becomes

$$a=-f(x)\,\dot{x} \tag{16}$$

from which we can secure the Lienard equations of interest.

3.1 Mixed Lienard type equation

Setting $f(x)=\left(\mu^2-x^2\right)^{-1/2}$, the equation (15) turns into

$$\ddot{x}+\left(\frac{\ell}{x}+\frac{2x}{\mu^2-x^2}\right)\dot{x}^2+\frac{a\ell}{x}\sqrt{\mu^2-x^2}\,\dot{x}-a^2x=0 \tag{17}$$

The general solution of the equation (17) using the equation (16), is given by the quadrature

$$-a(t+K)=\int f(x)dx \tag{18}$$

which, taking into account, the expression of $f(x)$, becomes

$$-a(t+K)=\int\frac{dx}{\sqrt{\mu^2-x^2}} \tag{19}$$

where K is an integration constant. From the equation (19), one can get by integration, the relation

$$\sin^{-1}\left(\frac{x}{\mu}\right)=-a(t+K) \tag{20}$$

which ensures the solution $x(t)$ in the form

$$x(t)=\mu\sin[-a(t+K)] \tag{21}$$

where $a<0$. The solution (21) is nothing but the exact solution of the linear harmonic oscillator equation (1) where $\omega=-a$, and $\varphi=-aK$. That being so, one can consider the following quadratic Lienard type equation.

3-2 Quadratic Lienard type equation

To get the quadratic Lienard type equation having identical solution with the mixed Lienard equation (17), let $\ell=0$. Then, the equation (17) reduces to

$$\ddot{x}+\frac{2x}{\mu^2-x^2}\dot{x}^2-a^2x=0 \tag{22}$$

One can immediately notice that the solution of (22) is secured by the quadrature (19) that is to say, the equation (22) has the formula (21) as exact and explicit general solution. In this context, the quadratic Lienard type equation (22) can exhibit harmonic and isochronous periodic oscillations as the linear harmonic oscillator. The exact and explicit general solution of the reduced and quadratically damped Lienard type equation [26]

$$\ddot{x} + \frac{f'(x)}{f(x)}\dot{x}^2 = 0 \tag{23}$$

is given by the quadrature (18) so that the equation

$$\ddot{x} + \frac{x}{\mu^2 - x^2}\dot{x}^2 = 0 \tag{24}$$

has the formula (21) as solution and can exhibit sinusoidal and isochronous periodic oscillations.

3-3 Conservative Lienard type equation

It is easy to show that the exact and explicit general solution of the equation

$$\ddot{x} + a^2 \frac{f'(x)}{f^3(x)} = 0 \tag{25}$$

is given by the quadrature (18). In this context where $f(x) = \left(\mu^2 - x^2\right)^{-1/2}$, the equation (25) reduces to

$$\ddot{x} + a^2 x = 0 \tag{26}$$

and its solution is given by the formula (21), that is by the solution of the linear harmonic oscillator equation (1) where $\omega = -a > 0$, and $\varphi = -aK$. Now we can formulate a conclusion of this work.

Conclusion

We have investigated in this paper some classes of Lienard equations of different types. We have shown for the first time that conservative, quadratic and mixed Lienard type equations can exhibit identical solutions, in particular sinusoidal and isochronous periodic solutions.

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