

Set theory INC[#] based on intuitionistic logic with restricted modus ponens rule.

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Abstract:

In this article Russell's paradox resolved successfully using intuitionistic logic with restricted modus ponens rule.

1. Introduction

Considering only pure sets, the naive set comprehension principle says, for any condition, that there is a set containing all and only the sets satisfying this condition. In first-order logic, this can be formulated as the following schematic principle, where ϕ may be any formula in which x does not occur freely:

$$\exists y \forall x (x \in y \leftrightarrow \phi). \quad (1.1)$$

Russell's paradox shows that the instance obtained by letting ϕ be $x \notin x$ is inconsistent in classical logic. One response to the paradox is to restrict naive set comprehension by ruling out this and other problematic instances: only for each of some special conditions is it claimed there is a set containing all and only the sets satisfying the condition. Many well known set theories can be understood as instances of this generic response, differing in how they understand special. For example, the axiom schema of separation (1.1) in Zermelo-Fraenkel set theory (ZF) restricts set comprehension to conditions which contain, as a conjunct, the condition of being a member of some given set:

$$\exists y \forall x (x \in y \leftrightarrow \phi \wedge x \in z). \quad (1.2)$$

Similarly, in Quine's New Foundations (NF) set comprehension is restricted to conditions which are stratified, where ϕ is stratified just in case there is a mapping f from individual variables to natural numbers such that for each subformula of ϕ of the form $x \in y$, $f(y) = f(x) + 1$ and for each subformula of ϕ of the form $x = y$, $f(x) = f(y)$. Both of

these responses block Russell's paradox by ruling out the condition $x \notin x$. Must every restriction of naive comprehension take the form of simply ruling out certain instances? In this article, I have suggest and explore a different approach. As we have seen, standard set comprehension axioms restrict attention to some special conditions: for each of these special conditions, they provide for the existence of a set containing all and only the sets which satisfy it.

Instead of restricting the conditions one is allowed to consider, we propose restricting the way in which the sets in question satisfy a given condition: for every condition, our comprehension axiom will assert the existence of a set containing all and only the sets satisfying that condition in a special way using intuitionistic first-order logic with restricted modus ponens rule.

2. Russell's paradox resolution using intuitionistic first-order logic with restricted modus ponens rule.

2.1. Russell's paradox

The comprehension principle (1.1) for the condition $x \notin x$ gives

$$\exists \mathfrak{R} \forall x (x \in \mathfrak{R} \Leftrightarrow x \notin x). \quad (2.1)$$

Thus \mathfrak{R} is the set whose members are exactly those sets that are not members of themselves. It follows from (2.1)

$$\mathfrak{R} \in \mathfrak{R} \Leftrightarrow \mathfrak{R} \notin \mathfrak{R}. \quad (2.2) \quad \text{Is } \mathfrak{R} \text{ a member of itself?}$$

If it is, i.e. $\mathfrak{R} \in \mathfrak{R}$ then it must satisfy the condition of not being a member of itself and so it is not, i.e. $\mathfrak{R} \notin \mathfrak{R}$. If it is not, then it must not satisfy the condition of not being a member of itself, and so it must be a member of itself. Since by classical logic only one case or the other one must hold – either \mathfrak{R} is a member of itself or it is not – it follows that the theory implies a contradiction.

Remark 2.1. Remind classical logic mandates that any contradiction trivializes a theory by making every sentence of the theory provable. This is because, in classical logic, the following is a theorem:

$$\text{Ex Falso Quadlibet} : A \Rightarrow (\neg A \Rightarrow B). \quad (2.3)$$

Remark 2.2. Now, virtually the only way to avoid EFQ is to give up disjunctive syllogism

also known as disjunction elimination :

$$\frac{P \vee \neg Q}{Q} \quad (2.4)$$

that is, given the usual definitions of the connectives, modus ponens! So altering basic sentential logic in this way is radical indeed – but possible.

Remark 2.3. Unfortunately, even giving up EFQ is not enough to retain a semblance of naive Cantor set theory (NC). One also has to give up the following additional theorem of basic sentential logic:

$$\text{Contraction: } (A \supset (A \supset B)) \supset (A \supset B). \quad (2.5)$$

It can then be argued that NC leads directly, not merely to an isolated contradiction, but to triviality. (For the argument that this is so, see the entry on Curry's paradox [1].

Thus it seems that the woes of NC are not confined to Russell's paradox but also include a negation-free paradox due to Curry.

Remark 2.4. Another suggestion might be to conclude that the paradox depends upon an instance of the principle of Excluded Middle, that either \mathfrak{R} is a member of \mathfrak{R} or it is not. This is a principle that is rejected by some non-classical approaches to logic, including intuitionism [2],[3].

Remind that in classical logic, we often discuss the truth values that a formula can take. The values are usually chosen as the members of a Boolean algebra. The meet and join operations in the Boolean algebra are identified with the \wedge and \vee logical connectives, so that the value of a formula of the form $A \wedge B$ is the meet of the value of A and the value of B in the Boolean algebra. Then we have the useful theorem that a formula is a valid proposition of classical logic if and only if its value is 1 for every valuation—that is, for any assignment of values to its variables. A corresponding theorem is true for intuitionistic logic, but instead of assigning each formula a value from a Boolean algebra, one uses values from an Heyting algebra, of which Boolean algebras are a special case. A formula is valid (or holds) in intuitionistic logic if and only if it receives the value of the top element for any valuation on any Heyting algebra. It can be shown that to recognize valid formulas, it is sufficient to consider a single Heyting algebra whose elements are the open subsets of the real line \mathbb{R} [2]. In this algebra we have: (1) $\mathbf{Value}[\perp] = \emptyset$, (2) $\mathbf{Value}[\top] = \mathbb{R}$, (3) $\mathbf{Value}[A \wedge B] = \mathbf{Value}[A] \cap \mathbf{Value}[B]$, (4) $\mathbf{Value}[A \vee B] = \mathbf{Value}[A] \cup \mathbf{Value}[B]$,

(5) $\mathbf{Value}[A \Rightarrow B] = \mathbf{Int}(\mathbf{Value}[A]^c \cup \mathbf{Value}[B])$, (6) $\mathbf{Value}[\neg A] = \mathbf{Int}(\mathbf{Value}[A]^c)$, where $\mathbf{Int}(X)$ is the interior of X and X^c its complement.

Remark 2.5. With these assignments (1)-(6), intuitionistically valid formulas are precisely

those that are assigned the value of the entire line [2]. For example, the formula $\neg(A \wedge \neg A)$ is valid, since $\mathbf{Value}[\neg(A \wedge \neg A)] = \mathbb{R}$. So the valuation of this formula is true, and indeed the formula is valid. But the law of the excluded middle, $A \vee \neg A$, can be easily

shown to be invalid by using a specific value of the set of positive real numbers for A : $\mathbf{Value}[A] = \{x|x > 0\} = \mathbb{R}_+$. For such A one obtains $\mathbf{Value}[\neg(A \wedge \neg A)] \neq \mathbb{R}$.

We do now as follows:

Case I. Assume now that: (a) $\mathfrak{R} \in \mathfrak{R}$ holds, i.e. $\mathbf{Value}[\mathfrak{R} \in \mathfrak{R}] = \mathbb{R}$ and therefore $\mathfrak{R} \notin \mathfrak{R}$ is not holds, since $\mathbf{Value}[\mathfrak{R} \notin \mathfrak{R}] = \emptyset$.

From (2.2) it follows that (b) $\mathfrak{R} \in \mathfrak{R} \Rightarrow \mathfrak{R} \notin \mathfrak{R}$. From (a) and (b) by modus ponens rule it follows that

$$\mathfrak{R} \in \mathfrak{R}, \mathfrak{R} \in \mathfrak{R} \Rightarrow \mathfrak{R} \notin \mathfrak{R} \vdash \mathfrak{R} \notin \mathfrak{R}. \quad (2.6)$$

From (2.6) and (a) one obtains the following formula $\mathfrak{R} \in \mathfrak{R} \wedge \mathfrak{R} \notin \mathfrak{R}$. But by the Law of

Non-contradiction we know that $\neg(\mathfrak{R} \in \mathfrak{R} \wedge \mathfrak{R} \notin \mathfrak{R})$. Thus we obtain a contradiction and

therefore $\mathfrak{R} \in \mathfrak{R}$ is not holds.

Case II. Assume now that:

(a) $\mathfrak{R} \notin \mathfrak{R}$ holds, i.e. $\mathbf{Value}[\mathfrak{R} \notin \mathfrak{R}] = \mathbb{R}$ and therefore $\mathfrak{R} \in \mathfrak{R}$ is not holds, since $\mathbf{Value}[\mathfrak{R} \in \mathfrak{R}] = \emptyset$.

From (2.2) it follows that (b) $\mathfrak{R} \notin \mathfrak{R} \Rightarrow \mathfrak{R} \in \mathfrak{R}$. From (a) and (b) by modus ponens rule it follows that

$$\mathfrak{R} \notin \mathfrak{R}, \mathfrak{R} \notin \mathfrak{R} \Rightarrow \mathfrak{R} \in \mathfrak{R} \vdash \mathfrak{R} \in \mathfrak{R}. \quad (2.7)$$

From (2.7) and (b) one obtains the following formula $\mathfrak{R} \in \mathfrak{R} \wedge \mathfrak{R} \notin \mathfrak{R}$. But by the Law of

Non-contradiction we know that $\neg(\mathfrak{R} \in \mathfrak{R} \wedge \mathfrak{R} \notin \mathfrak{R})$. Thus we obtain a contradiction and

therefore $\mathfrak{R} \notin \mathfrak{R}$ is not holds. Thus both $\mathfrak{R} \in \mathfrak{R}$ and $\mathfrak{R} \notin \mathfrak{R}$ is not holds, but by absent

the Excluded Middle but by absent the law Excluded Middle this does not pose any problems.

Remark 2.6. However it well known that it is possible to derive the contradiction only from

the statement (2.2) i.e., $\mathfrak{R} \in \mathfrak{R} \Leftrightarrow \mathfrak{R} \notin \mathfrak{R}$. We do so as follows:

Assume now that: $\mathfrak{R} \in \mathfrak{R} \Leftrightarrow \mathfrak{R} \notin \mathfrak{R}$ holds and therefore $\mathfrak{R} \in \mathfrak{R} \Rightarrow \mathfrak{R} \notin \mathfrak{R}$. But we also know that $\mathfrak{R} \in \mathfrak{R} \Rightarrow \mathfrak{R} \in \mathfrak{R}$. So $\mathfrak{R} \in \mathfrak{R} \Rightarrow \mathfrak{R} \in \mathfrak{R} \wedge \mathfrak{R} \notin \mathfrak{R}$.

But by the Law of Non-contradiction we know that $\neg(\mathfrak{R} \in \mathfrak{R} \wedge \mathfrak{R} \notin \mathfrak{R})$

So by modus tollens we conclude that $\mathfrak{R} \notin \mathfrak{R}$.

At the same time we also know that $\mathfrak{R} \notin \mathfrak{R} \Rightarrow \mathfrak{R} \in \mathfrak{R}$, and thus by modus ponens we conclude that $\mathfrak{R} \in \mathfrak{R}$.

So we can deduce both $\mathfrak{R} \in \mathfrak{R}$ and its negation $\mathfrak{R} \notin \mathfrak{R}$ using only intuitionistically acceptable methods.

Remark 2.7. Another suggestion might be to conclude that the paradox depends upon an instance of the Law of Non-contradiction, that $\neg(\mathfrak{R} \in \mathfrak{R} \wedge \mathfrak{R} \notin \mathfrak{R})$. This is a principle

that is rejected by some non-classical approaches to logic, including paraconsistent logic

[3]. Nevertheless even paraconsistent logic can not safe *NC* from a triviality [3].

3. Russell's paradox resolution using intuitionistic first-order logic with restricted modus ponens rule.

3.1. The intuitionistic propositional calculus $\mathbf{Pp}^\#$ with restricted modus ponens rule.

The first step in the metamathematical study of any part of logic or mathematics is to specify a formal language \mathcal{L} . For propositional or sentential logic, the standard language has denumerably many distinct proposition letters P_0, P_1, P_2, \dots and symbols $\&, \vee, \rightarrow, \neg, \perp$ for the propositional connectives "and," "or," "if ...then," and "not" respectively, with left and right parentheses $(,)$ (sometimes written "[,]" for ease of reading). Classical logic actually needs only two connectives (since classical \vee and \rightarrow can be defined in terms of $\&$ and \neg), but the four intuitionistic connectives are independent. The classical language is thus properly contained in the intuitionistic, which is more expressive. The most

important tool of metamathematics is generalized induction, a method Brouwer endorsed. The class of wff's (well-formed formulas) of the language $\mathcal{L}(\mathbf{Pp}^\#)$ of $\mathbf{Pp}^\#$ is defined inductively by the rules:

- (i) Each proposition letter is a (prime) formula.
- (ii) If A, B are formulas so are $(A \& B), (A \vee B), (A \rightarrow B)$ and $(\neg A)$.
- (iii) Nothing is a formula except as required by (i) and (ii).
- (iv) The class of wff's of the language $\mathcal{L}(\mathbf{Pp}^\#)$ we will denote by $\mathcal{F}_{\text{wff}}(\mathbf{Pp}^\#)$.

As in classical logic, $(A \leftrightarrow B)$ abbreviates $((A \rightarrow B) \& (B \rightarrow A))$.

The axioms are all formulas of the following forms:

- Pp[#] 1.** $A \rightarrow (B \rightarrow A)$.
- Pp[#] 2.** $(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C))$.
- Pp[#] 3.** $A \rightarrow (B \rightarrow A \& B)$.
- Pp[#] 4.** $A \& B \rightarrow A$.
- Pp[#] 5.** $A \& B \rightarrow B$.
- Pp[#] 6.** $A \rightarrow A \vee B$.
- Pp[#] 7.** $B \rightarrow A \vee B$.
- Pp[#] 8.** $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$.
- Pp[#] 9.** $(A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)$.
- Pp[#] 10.** $\neg A \rightarrow (A \rightarrow B)$.
- Pp[#] 11.** $\perp \rightarrow A$.

Remark 3.1. The system of classical logic is obtained by adding any one of the following axioms: 1. $\phi \vee \neg \phi$ (Law of the excluded middle. May also be formulated as

- $(\phi \rightarrow \chi) \rightarrow ((\neg \phi \rightarrow \chi) \rightarrow \chi)$
- 2. $\neg \neg \phi \rightarrow \phi$ (Double negation elimination)
- $((\phi \rightarrow \chi) \rightarrow \phi) \rightarrow \phi$ (Peirce's law)
- $(\neg \phi \rightarrow \neg \chi) \rightarrow (\chi \rightarrow \phi)$ (Law of contraposition)

The rule of inference of **Pp[#]** is

R[#]1. RMP (Restricted Modus Ponens).

From A and $A \rightarrow B$, conclude B iff $A \notin \Delta_1$ and $(A \rightarrow B) \notin \Delta_2$, where $\Delta_1, \Delta_2 \subseteq \mathcal{F}_{\text{wff}}(\mathbf{Pp}^\#)$

We abbreviate by $A, A \rightarrow B \vdash_{\text{RMP}} B$.

R[#]2. MT (Modus Tollens)

$P \rightarrow Q, \neg Q \vdash \neg P$.

If Γ is any collection of formulas and E_1, \dots, E_k any finite sequence of formulas each of which is a member of Γ , an axiom, or an immediate consequence by RMP of two preceding formulas, then E_1, \dots, E_k is a derivation in **Pp[#]** of its last formula E_k from the assumptions Γ . We write $\Gamma \vdash_{\text{Pp}^\#} E$ to denote that such a derivation exists with $E_k = E$. The following theorem is proved by induction over the definition of a derivation; its converse follows from **R[#]1**.

Deduction Theorem. If Γ is any collection of formulas and A, B are any formulas such that $\Gamma \cup \{A\} \vdash_{\text{RMP}} B$, then also $\Gamma \vdash_{\text{RMP}} (A \rightarrow B)$.

3.2. The intuitionistic first-order predicate calculus **Pd[#]** with restricted modus ponens rule.

The pure firstorder language $\mathcal{L}(\mathbf{Pd}^\#)$ has individual variables a_1, a_2, a_3, \dots , and countably infinitely many distinct predicate letters $P_1(\dots), P_2(\dots), P_3(\dots), \dots$ of arity n for each $n = 0, 1, 2, \dots$, including the 0-ary proposition letters. There are two new logical

symbols \forall (“for all”) and \exists (“there exists”). The terms of the language $\mathcal{L}(\mathbf{Pd}^\#)$ of $\mathbf{Pd}^\#$ are the individual variables. The well formed formulas are defined by the rules:

- (i) If $P(\dots)$ is an n -ary predicate letter and t_1, \dots, t_n are terms then $P(t_1, \dots, t_n)$ is a (prime) formula.
- (ii) If A, B are formulas so are $(A \& B), (A \vee B), (A \rightarrow B)$ and $(\neg A)$.
- (iii) If A is a formula and x an individual variable, then $(\forall x A)$ and $(\exists x A)$ are formulas.
- (iv) Nothing else is a formula.
- (v) The class of wff’s of the language $\mathcal{L}(\mathbf{Pd}^\#)$ we will denote by $\mathcal{F}_{\text{wff}}(\mathbf{Pd}^\#)$

We use $x, y, z, w, x_1, y_1, \dots$ and $A, B, C, \dots, A(x), A(x, y), \dots$ as metavariables for variables and formulas, respectively. Anticipating applications (e.g. to arithmetic), s, t, s_1, t_1, \dots vary over terms. In omitting parentheses, $\forall x$ and $\exists x$ are treated like \neg . The scope of a quantifier, and free and bound occurrences of a variable in a formula, are defined as usual. A formula in which every variable is bound is a sentence or closed formula.

If x is a variable, t a term, and $A(x)$ a formula which may or may not contain x free, then $A(t)$ denotes the result of substituting an occurrence of t for each free occurrence of x in $A(x)$. The substitution is free if no free occurrence in t of any variable becomes bound in $A(t)$; in this case we say t is free for x in $A(x)$.

In addition to **Pp1** - **Pp11**, $\mathbf{Pd}^\#$ has two new axiom schemas, where $A(x)$ may be any formula and t any term free for x in $A(x)$:

Pd[#]12. $\forall x A(x) \rightarrow A(t)$.

Pd[#]13. $A(t) \rightarrow \exists x A(x)$.

The rules of inference are:

R[#]1.RMP (Restricted Modus Ponens).

From A and $A \rightarrow B$, conclude B iff $A \notin \Delta_1$ and $(A \rightarrow B) \notin \Delta_2$, where $\Delta_1, \Delta_2 \subseteq \mathcal{F}_{\text{wff}}(\mathbf{Pd}^\#)$

We abbreviate **R[#]1** by $A, A \rightarrow B \vdash_{\text{RMP}} B$.

R[#]2.MT (Modus Tollens)

$P \rightarrow Q, \neg Q \vdash \neg P$.

R[#]3. From $C \rightarrow A(x)$ where x does not occur free in C , conclude $C \rightarrow \forall x A(x)$.

R[#]4. From $A(x) \rightarrow C$ where x does not occur free in C , conclude $\exists x A(x) \rightarrow C$.

A deduction (or derivation) in $\mathbf{Pd}^\#$ of a formula E from a collection Γ of assumption formulas is a finite sequence of formulas, each of which is an axiom by **Pd[#]1** - **Pd[#]13**, or a member of Γ , or follows immediately by **R[#]1**, **R[#]2** or **R[#]3** from one or two formulas occurring earlier in the sequence. A proof is a deduction from no assumptions. If Γ is a collection of sentences and E a formula, the notation $\Gamma \vdash_{\text{RMP}} E$ means that a deduction of E from Γ exists. If Γ is a collection of formulas, we write $\Gamma \vdash_{\text{RMP}} E$ only if there is a deduction of E from Γ in which neither **R[#]2** nor **R[#]3** is used with respect to any variable free in Γ . With this restriction, the deduction theorem extends to $\mathbf{Pd}^\#$: If $\Gamma \cup \{A\} \vdash_{\text{RMP}} B$ then $\Gamma \vdash_{\text{RMP}} (A \rightarrow B)$. such that $\Gamma \cup \{A\} \vdash_{\text{RMP}} B$, then also $\Gamma \vdash_{\text{RMP}} (A \rightarrow B)$.

3.2. Russell’s paradox resolution using first-order predicate calculus $\mathbf{Pd}^\#$ with restricted modus ponens rule.

Assume now that: $\mathfrak{R} \in \mathfrak{R} \Leftrightarrow \mathfrak{R} \notin \mathfrak{R}$ holds and therefore $\mathfrak{R} \in \mathfrak{R} \Rightarrow \mathfrak{R} \notin \mathfrak{R}$.

Remark 3.2. We set now $(\mathfrak{R} \notin \mathfrak{R}) \in \Delta_1$ and $(\mathfrak{R} \notin \mathfrak{R} \Rightarrow \mathfrak{R} \in \mathfrak{R}) \in \Delta_2$.

We also know that $\mathfrak{R} \in \mathfrak{R} \Rightarrow \mathfrak{R} \in \mathfrak{R}$. So $\mathfrak{R} \in \mathfrak{R} \Rightarrow \mathfrak{R} \in \mathfrak{R} \wedge \mathfrak{R} \notin \mathfrak{R}$.

But by the Law of Non-contradiction we know that $\neg(\mathfrak{R} \in \mathfrak{R} \wedge \mathfrak{R} \notin \mathfrak{R})$

So by modus tollens we conclude that $\mathfrak{R} \notin \mathfrak{R}$.

At the same time we also know that $\mathfrak{R} \notin \mathfrak{R} \Rightarrow \mathfrak{R} \in \mathfrak{R}$, and thus by restricted modus ponens we can not conclude that $\mathfrak{R} \in \mathfrak{R}$.

From $\mathfrak{R} \in \mathfrak{R} \Leftrightarrow \mathfrak{R} \notin \mathfrak{R}$ we obtain $\mathfrak{R} \notin \mathfrak{R} \Rightarrow \mathfrak{R} \in \mathfrak{R}$. We also know that

$\mathfrak{R} \notin \mathfrak{R} \Rightarrow \mathfrak{R} \notin \mathfrak{R}$. So $\mathfrak{R} \notin \mathfrak{R} \Rightarrow \mathfrak{R} \in \mathfrak{R} \wedge \mathfrak{R} \notin \mathfrak{R}$. But by the Law of Non-contradiction we know that $\neg(\mathfrak{R} \in \mathfrak{R} \wedge \mathfrak{R} \notin \mathfrak{R})$

So by modus tollens we conclude that $\neg(\mathfrak{R} \notin \mathfrak{R})$. Note that $\mathfrak{R} \in \mathfrak{R} \Rightarrow \neg(\mathfrak{R} \notin \mathfrak{R})$.

Thus by using calculus $\mathbf{Pd}^\#$ with restricted modus ponens rule Russell's paradox dissappears.

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