On sinusoidal and isochronous periodic solution of dissipative Lienard type equations

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Abstract

We present in this paper an exceptional dissipative Lienard type equation. The equation can exhibit the sine function as exact and general solution and be made isochronous by an appropriate choice of model parameters. As a result, such a dissipative Lienard equation and the linear harmonic oscillator have identical solutions.

Keywords: Nonlinear dissipative Lienard type equation, harmonic periodic solution, isochronous property, exact solution.

Introduction

The nonlinear dissipative Lienard type differential equation

$$\ddot{x}+u(x)\dot{x}+\vartheta(x)=0 \tag{1}$$

where $u(x)$ and $\vartheta(x)$ are arbitrary functions of x, has been the subject of intensive study in pure and applied mathematics [1-5]. When $u(x)=\lambda$, and $\vartheta(x)=\omega_0^2 x$, the equation becomes

$$\ddot{x}+\lambda\dot{x}+\omega_0^2 x=0 \tag{2}$$

where λ and ω_0 are arbitrary constants, which is the prototype of damped oscillations equations. Unlike this property where the presence of dissipative term leads to damped oscillations, it has been suspected that the equation (1) can exhibit periodic solutions when the coefficient of $\dot{x}$, that is $u(x)$, is variable. In this sense the generalized and modified Emden type equation

$$\ddot{x}+\alpha x\dot{x}+\beta x^3+\omega_0^2 x=0 \tag{3}$$

where overdot means derivative with respect to time and α and β are arbitrary parameters, has been deeply investigated in the literature [2-4]. In [2]

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Chandrasekar and coworkers succeeded to obtain for the first time harmonic and isochronous periodic solution for the equation (3). Later in [3] the authors showed that such harmonic and isochronous periodic solution can be obtained for a more generalized equation (3). After the result of [2], a vast literature has been carried out on the equation (3) following different methods of analytical investigation [5-7]. Recently, Doutètien et al. [4] investigated the equation (3) and found that, in contrast to the periodic solution exhibited by the authors in [2], the equation (3) can exhibit unbounded periodic solutions such that this equation is in fact a pseudo-oscillator. This feature of pseudo-oscillator has been recently observed for many nonlinear differential equations presented to represent conservative nonlinear oscillators in the literature [8, 9]. In [10] Monsia and coworkers have shown the existence of an equation of Lienard type (1) which can exhibit isochronous periodic solution. In view of the above, it is reasonable to ask whether there are oscillators of Lienard type (1) which can exhibit harmonic and isochronous periodic solutions. The objective in this paper is to show the existence of such equations. In this way, we briefly review the theory of nonlinear dissipative Lienard type differential equations (section 2) introduced recently in literature by Monsia and his group [4], and establish the dissipative Lienard type equation of interest and calculate its sinusoidal and isochronous periodic solutions (section 3). We carry out a conclusion finally for the work.

2- Review of the theory

Let

$$g(x)\dot{x} + a\, f(x)x^{\ell} = b \tag{4}$$

be the first integral introduced in [4], where a, b and ℓ are arbitrary parameters and $f(x)$ and $g(x)$ are arbitrary functions of x. By differentiation with respect to time, one can secure immediately the general class of dissipative Lienard type equations

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 + a\ell x^{\ell-1}\frac{f(x)}{g(x)}\dot{x} + a\,x^{\ell}\,\frac{f'(x)}{g(x)}\dot{x} = 0 \tag{5}$$

which can become, using the equation (4), the general class of mixed Lienard type equations

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 + a\,\ell x^{\ell-1}\frac{f(x)}{g(x)}\,\dot{x} + ab\,x^{\ell}\,\frac{f'(x)}{g^2(x)} - a^2\,x^{2\ell}\,\frac{f'(x)f(x)}{g^2(x)} = 0 \tag{6}$$

The equation (6) can take the form

$$\ddot{x}+a\ell\, f(x)\,x^{\ell-1}\,\dot{x}-a^2x^{2\ell}\, f'(x)\,f(x)+ab\,x^{\ell}\, f'(x)=0 \qquad (7)$$

where $g(x)=1$. The general solution of the dissipative Lienard type equation (7) is given then by the quadrature defined by

$$t+K=\int\frac{g(x)dx}{b-a\, f(x)x^{\ell}} \qquad (8)$$

that is

$$t+K=\int\frac{dx}{b-a\, f(x)x^{\ell}} \qquad (9)$$

as $g(x)=1$. K is a constant of integration. Now we can study the dissipative Lienard type equation of interest in the following section.

3- The equation and its solution

3.1 The equation

Let us consider the equation (7). Then, setting $\ell=1$, and $b=0$, leads to get

$$\ddot{x}+a\, f(x)\,\dot{x}-a^2x^2\, f'(x)\,f(x)=0 \qquad (10)$$

Substituting $f(x)=\sqrt{k_1+\frac{k_2}{x^2}}$, where k_1 and k_2 are arbitrary parameters, into the equation (10), yields the desired Lienard type oscillator equation

$$\ddot{x}+a\sqrt{\left(k_1x^2+k_2\right)}\,\frac{\dot{x}}{x}+\frac{a^2k_2}{x}=0 \qquad (11)$$

3.2 The solution

By application of $\ell=1$, and $b=0$, the equation (9) reduces to

$$-a\left(t+K\right)=\int\frac{dx}{f(x)x} \qquad (12)$$

which, for $f(x)=\sqrt{k_1+\frac{k_2}{x^2}}$, becomes

$$-a(t+K)=\int \frac{dx}{\sqrt{k_1x^2+k_2}} \tag{13}$$

The evaluation of this integral allows one to obtain [11]

$$\sin^{-1}\left(x\sqrt{-\frac{k_1}{k_2}}\right)=-a\sqrt{-k_1}(t+K) \tag{14}$$

from which one can secure the general periodic solution

$$x(t)=\frac{\sqrt{-k_1k_2}}{k_1}\sin\left[-a\sqrt{-k_1}(t+K)\right] \tag{15}$$

where $k_1<0$ and $k_2>0$. The sinusoidal solution (15) becomes isochronous when $k_1=-1$, such that one can get

$$x(t)=\sqrt{k_2}\sin[a(t+K)] \tag{16}$$

where $a>0$. As can be seen, the solution (16) is identical to the solution of the linear harmonic oscillator

$$\ddot{x}+a^2x=0 \tag{17}$$

where the amplitude of oscillations is taken as $\sqrt{k_2}$. Now we can formulate a conclusion for this work.

Conclusion

We have presented in this paper an exceptional dissipative Lienard type nonlinear oscillator equation. We have shown for the first time that such a type of equation can exhibit sinusoidal and isochronous periodic solution, that is can have the linear harmonic oscillator solution.

References

[1] S. N. Pandey, P. S. Bindu, M. Senthilvelan, and M. Lakshmanan, A group theoretical identification of integrable cases of the Liénard-type equation $\ddot{x}+f(x)\dot{x}+g(x)=0$. I. Equations having nonmaximal number of Lie point symmetries, Journal of Mathematical Physics 50, (2009) 082702.

[2] V. K. Chandrasekar, M. Senthilvelan, and M. Lakshmanan, Unusual Liénard-type nonlinear oscillator, Physical Review E 72, (2005) 066203

[3] R. Iacono and F. Russo, Class of solvable nonlinear oscillators with isochronous orbits, Physical Review E, 83(2011), 027601-1-027601-4.

[4] E. A. Doutètien, A. B. Yessoufou, Y. J. F. Kpomahou and M. D. Monsia, Unbounded periodic solution of a modified Emden type equation, Math.Phys.,viXra.org/2101.0002v1.pdf (2021).

[5] R.V. Chithiika, M. Senthilvelan and M. Lakshmanan, Exact quantization of a PT-symmetric (reversible) Lienard type nonlinear oscillator, J. Phys. A: Math. Theor. 45, 382002 (2012).

[6] R. G. Pradeep, V. K. Chandrasekar, M. Senthilvelan, and M. Lakshmanan, Non-standard conserved Hamiltonian structures in dissipative/ damped systems: Nonlinear generalizations of damped harmonic oscillator, arXiv: 0810.1819v2 (2009) 1-22.

[7] M. Lakshmanan and V. K. Chandrasekar, Generating finite dimensional integrable nonlinear dynamical systems, arXiv: 1307.0273v1 [nlin.SI] 2013.

[8] M. Gadella and L. P. Lara, on the solutions of a nonlinear pseudo-oscillator equation, Phys. Scr. 89 105205 (2014).

[9] J. Akande, K. K. D. Adjaï, L.H. Koudahoun and M.D. Monsia, Existence of non-periodic real-valued and complex-valued solutions of Mathews-Lakshmanan oscillator equation, Math.Phys.,viXra.org/ 2012.0104v1.pdf (2020).

[10] Y. J. F. Kpomahou, M. Nonti, A. B. Yessoufou and M. D. Monsia, On isochronous periodic solution of a generalized Emden type equation, Math.Phys.,viXra.org/2101.0024v1.pdf (2021).

[11] I.S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series and Products. ed.Elsevier, Alan Jeffrey and Daniel Zwillinger, USA (2007).