Unbounded periodic solution of a modified Emden type equation

E. A. Doutètien[1], A .B. Yessoufou[1], Y. J. F. Kpomahou[2], M.D. Monsia[1*]

1-Department of Physics, University of Abomey-Calavi, Abomey-Calavi, 01.B P.526, Cotonou, BENIN

2- Department of Industrial and Technical Sciences, ENSET-Lokossa, UNSTIM-Abomey, Abomey, BENIN

Abstract

We investigate a modified Emden type equation known as a Lienard type nonlinear oscillator. We show the existence of unbounded periodic solution of the equation. As a result, such an equation may exhibit bounded and unbounded periodic solutions for the same numerical values of model parameters.

Keywords: Modified Emden type equation, Lienard type nonlinear oscillator, exact periodic solution, unbounded solution.

Introduction

The Lienard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{1}$$

where overdot denotes derivative with respect to time, as one can see in the literature, contains several nonlinear differential equations of physical importance. Among then one can consider the modified Emden type equation [1-7]

$$\ddot{x} + \alpha x^s \dot{x} + \beta x^{2s+1} + \lambda x^s = 0 \tag{2}$$

where α, β, λ and s are arbitrary parameters, which has been investigated from various solution methods of different complexity. When $s = 1$, and $\lambda = 0$, the equation (2) turns into the well-known modified Emden equation

$$\ddot{x} + \alpha x\dot{x} + \beta x^3 = 0 \tag{3}$$

* Corresponding author : E-mail: monsiadelphin@yahoo.fr

which has been intensively analyzed in [8-10]. This equation is also known as the Painlevé-Ince equation in the literature. The choice $\lambda = 0$, reduces the equation (2) to the form

$$\ddot{x} + \alpha x^s \dot{x} + \beta x^{2s+1} = 0 \tag{4}$$

Using the Hamiltonian approach, the authors in [11] succeeded to calculate exact solutions of (4). In [2] the exact analytical solutions of the equation (4) are given in terms of quadrature by means of nonlocal transformation of the damped linear harmonic oscillator equation. The application of $s = 1$, leads to obtain the interesting equation

$$\ddot{x} + \alpha x\dot{x} + \beta x^3 + \lambda x = 0 \tag{5}$$

which has been widely examined by several authors from different point of view when $\alpha = k$ and $\beta = \frac{k^2}{9}$. The Lie point symmetries of this equation have been studied in [10, 12]. Lagrangian analysis has been performed in [10, 13]. In [14] the authors investigated this equation from the Jacobi last multiplier. The quantization of this equation is also carried out in [15, 16]. In [17], Chandrasekar and coworkers presented the equation (5) as an unusual Lienard type oscillator with the isochronous properties. The authors [17] were able to calculate exact sinusoidal periodic solutions to such a dissipative nonlinear differential equation using the modified Prelle-Singer method. Such a solution has been obtained later from different approach in [18]. However, as can be seen in the literature, many nonlinear differential equations can exhibit different types of solutions. Despite of the existence of periodic solutions, these nonlinear equations can also exhibit real non-periodic solutions or complex-valued solutions for the same selected numerical parameters. It was the case of a truly nonlinear differential equation widely studied in the literature by several authors using various analytical techniques. Recently in [19, 20] it has been shown that such a nonlinear differential equation exhibits non-periodic solutions. It was also, more recently, the case of the celebrated Mathews-Lakshmanan equation presented in 1974 [21] as a unique quadratic Lienard type equation with sinusoidal periodic solution but having amplitude dependent frequency. Monsia and coworkers in [22] showed that this equation is in fact a pseudo-oscillator as it can exhibit real non-periodic solutions for the same values of model parameters for which the authors [21] in 1974 derived harmonic periodic solutions. Other differential equations assumed to be truly nonlinear oscillators have been found to have non-periodic solutions, in the literature. In this perspective, it is convenient to ask whether the Lienard type nonlinear differential equation (5) has only bounded sinusoidal periodic solutions for $\alpha = k$, $\beta = \frac{k^2}{9}$, and λ positive. We predict in this paper that the equation (5) can exhibit unbounded periodic solutions. To do so, we show that the equation (5) belongs to a general class of Lienard type nonlinear differential equations which may be reduced to quadrature (section 2) from which one can secure the unbounded periodic solution and the real non-periodic solution of the equation (5) (section 3). A conclusion is finally addressed for the work.

2- General class of Lienard type equations

Let us consider the first order differential equation introduced recently in [4, 23, 24, 25]

$$g(x)\dot{x} + a\,f(x)x^{\ell} = b \tag{6}$$

where a, b and ℓ are arbitrary parameters. By differentiation with respect to time, one may secure immediately the mixed Lienard type equation

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 + a\ell x^{\ell-1}\frac{f(x)}{g(x)}\dot{x} + a\,x^{\ell}\frac{f'(x)}{g(x)}\dot{x} = 0 \tag{7}$$

which may be, using equation (6), reduced to the nonlinear equation

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 + a\,\ell x^{\ell-1}\frac{f(x)}{g(x)}\dot{x} + abx^{\ell}\frac{f'(x)}{g(x)} - a^2x^{2\ell}\frac{f'(x)f(x)}{g(x)} = 0 \tag{8}$$

which becomes

$$\ddot{x} + a\,\ell\,f(x)x^{\ell-1}\dot{x} - a^2x^{2\ell}f'(x)f(x) + ab\,x^{\ell}f'(x) = 0 \tag{9}$$

where $g(x) = 1$. The general class of dissipative Lienard type equation (9) may then be reduced to the quadrature defined by

$$t + \gamma = \int \frac{g(x)dx}{b - a\,f(x)x^{\ell}} \tag{10}$$

that is

$$t + \gamma = \int \frac{dx}{b - a\,f(x)x^{\ell}} \tag{11}$$

where $g(x)=1$, and γ an arbitrary parameter. Subtituting $f(x)=\frac{1}{x}$, into the equation (9), yields the dissipative Lienard type nonlinear differential equation

$$\ddot{x}+a\,\ell x^{\ell-2}\dot{x}+a^2\,x^{2\ell-3}-ab\,x^{\ell-2}=0 \tag{12}$$

where the equation (11) becomes

$$t+\gamma=\int\frac{dx}{b-a\,x^{\ell-1}} \tag{13}$$

Making $\ell=2m+3$, where m is an arbitrary parameter, the equation (12) transforms into the form

$$\ddot{x}+(2m+3)a\,x^{2m+1}\dot{x}+a^2\,x^{4m+3}-ab\,x^{2m+1}=0 \tag{14}$$

The application of $\ell=3$, that is $m=0$, reduces the equation (12) or (14) to

$$\ddot{x}+3a\,x\,\dot{x}+a^2x^3-ab\,x=0 \tag{15}$$

which becomes the equation (5) of interest, when $\alpha=3a$, $\beta=a^2$, and , $\lambda=-ab$. The equation (15) is identical to the equation considered in [18] when $a=-b=1$. But the generalized form solved in [18] is different from (14). In this context the equation (13) takes the form

$$t+\gamma=\int\frac{dx}{b-a\,x^2} \tag{16}$$

Now we may integrate easily the integral in the equation (16) to ensure the general solutions of the equation (5).

3- General solution of (5)

3.1 Non-periodic hyperbolic solution of (5)

In this case $a>0$, $b>0$, and $\lambda<0$. In this context the evaluation of the integral in the equation (16) allows one to obtain

$$\tanh^{-1}\left(\frac{a}{\sqrt{ab}}x\right)=\sqrt{ab}\,(t+\gamma) \tag{17}$$

that is

$$x(t)=\frac{\sqrt{ab}}{a}\tanh\left[\sqrt{ab}(t+\gamma)\right] \tag{18}$$

which may be rewritten as

$$x(t) = \frac{3\sqrt{-\lambda}}{\alpha}\tanh\left[\sqrt{-\lambda}\left(t+\gamma\right)\right] \tag{19}$$

3.2 Unbounded periodic solution of (5)

In the present case we consider $a > 0$, and $b < 0$, that is $\lambda > 0$. Thus the evaluation of the integral in the equation (16) leads to obtain

$$\tan^{-1}\left(\frac{a}{\sqrt{-ab}}x\right) = \left[-\sqrt{-ab}\left(t+\gamma\right)\right] \tag{20}$$

from which one may get

$$x(t) = -\frac{\sqrt{-ab}}{a}\tan\left[\sqrt{-ab}\left(t+\gamma\right)\right] \tag{21}$$

which may be rearranged as

$$x(t) = -\frac{3\sqrt{\lambda}}{\alpha}\tan\left[\sqrt{\lambda}\left(t+\gamma\right)\right] \tag{22}$$

This solution becomes

$$x(t) = -\tan\left[\left(t+\gamma\right)\right] \tag{23}$$

in the case $a = -b = 1$. The solution (21) as (22) is unbounded periodic solution for any value of $\alpha > 0$, and $\lambda > 0$. It is worth to note that the formula (21) is also the solution of differential equations [24, 26]

$$\ddot{x} + 2a\,x\,\dot{x} = 0 \tag{24}$$

and

$$\ddot{x} - 2a^2 x^3 + 2ab\,x = 0 \tag{25}$$

Therefore the equations (5) and (24) and the cubic Duffing equation (25) have identical non-sinusoidal periodic solution.

Conclusion

We have investigated in this paper a modified Emden type equation known as a Lienard type nonlinear oscillator. Although this nonlinear oscillator has harmonic periodic solution, we have shown that it can also exhibit unbounded periodic solution, that is non-sinusoidal periodic solution, for the same numerical value of model parameters.

References

[1] J. Akande, D. K. K. Adjaï , L. H. Koudahoun, Y. J. F. Kpomahou and M. D. Monsia, Generalized Modified Emden Equation Mapped into the Linear Harmonic Oscillator Equation , Math.Phys.,viXra.org/1609.0260v1.pdf (2016).

[2] M. D. Monsia, J. Akande, L. H. Koudahoun, D. K. K. Adjaï and Y. J. F. Kpomahou, Generalized Duffing-van der Pol and Modified Emden Type Equations as Limiting Cases of the Monsia et al. [2] Nonlinear Oscillator Equation, Math.Phys.,viXra.org/1609.0029v1.pdf (2016).

[3] M. D. Monsia, J. Akande, L. H. Koudahoun, D. K. K. Adjaï and Y. J. F. Kpomahou, Exact Solutions of a Class of Duffing-van der Pol and Modified Emden Type Equations via Nonlocal Transformation, Math.Phys.,viXra.org/1609.0036v1.pdf (2016).

[4] M. D. Monsia, J. Akande, D. K. K. Adjaï, L. H. Koudahoun and Y. J. F. Kpomahou, A theory of exactly integrable quadratic Liénard type equations, Int. J. Adv. Appl. Math. And Mech. 4 (1) 2016, 21-25.

[5] V. K. Chandrasekar, J.H. Sheeba, R. G. Pradeep, R.S. Divy and M. Lakshmanan, A class of solvable coupled nonlinear oscillators with amplitude independent frequencie, arXiv: 1204.6166v1 (2012) 1-18.

[6] V. K. Chandrasekar, S. N. Pandey, M. Senthilvelan, and M. Lakshmanan, A simple and unified approach to identify integrable nonlinear oscillators and systems, Journal of Mathematical Physics 45, (2006) 023508-1-023508-37.

[7] S. Thakar and A. K. Kulkarni, Integrability analysis of generalizes modified Emden type equation, Indian Journal of Mathematics, (2018) 181-206.

[8] V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, On the general solution for the modified Emden type equation $\ddot{x}+\alpha x\dot{x}+\beta x^3=0$, arXiv: 0611043v1 (2006) 1-12.

[9] R. Lacono, Comment on “ On the general solution for the modified Emden-type equation”, J. Phys. A: Math. Theor. 41, 068001 (2008).

[10] Aparna Saha and Benoy Talukdar, Lagrangian representations and solutions of modified Emden-type equations Mechanics, Materials Sciences & Engineering, (2016), DOI: 10.13140/RG.2.1.2683.0323.

[11] R. G. Pradeep, V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, Non-standard conserved Hamiltonian structures in dissipative damped systems: Nonlinear generalizations of damped harmonic oscillator, arXiv:0810.1819v2 (2009) 1-22.

[12] A. K. Tiwari, S. N. Pandey, M. Senthilvelan and M. Lakshmanan, Classification of Lie point symmetries for quadratic Lienard type equation $\ddot{x} + f(x)\dot{x}^2 + g(x) = 0$, Journal of Mathematical Physics 54, 053506 (2013).

[13] F. Jose Carinena, Manual F. Ranada and Mariano Santander, Lagrangian formalism for nonlinear second-order Riccati systems: one dimentional integrability and two dimentional superintegrability, J. Math. Phys., 46(6), Sr. N0-062703, (2005), DOI: 10.1063/1.1920287.

[14]M. C. Nucci and P. G. L. Leach, Jacobi's Last multiplier and the complete symmetry group of the Ermakov-Pinney equation, Journal of Nonlinear Mathematical Physics, Vol. 12, 2 (2005), 305-320.

[15] R.V. Chithiika, M. Senthilvelan and M. Lakshmanan, Exact quantization of a PT-symmetric (reversible) Lienard type nonlinear oscillator, J. Phys. A: Math. Theor. 45, 382002 (2012).

[16] A. G. Choudhury and P. Guha, Quantization of the Lienard II equation and Jacobi last multiplier, J. Phys. A: Math. Theor. 46, 165202 (2013).

[17] V. K. Chandrasekar, M. Senthilvelan, and M. Lakshmanan, Unusual Liénard-type nonlinear oscillator. Phys. Rev. E **72**, 066203 (2005)

[18] R. Iacono and F. Russo, Class of solvable nonlinear oscillators with isochronous orbits, Physical Review E, 83(2011), 027601-1-027601-4.

[19] M. Gadella and L. P. Lara, on the solutions of a nonlinear pseudo-oscillator equation, Phys. Scr. 89 105205 (2014).

[20]] E. A. Doutetien, A. R. Yehossou, P. Mallick, B. Rath, M. D. Monsia, On the General Solutions of a Nonlinear Pseudo-Oscillator Equation and Related Quadratic Lienard Systems, PINSA(2020), Doi:10.16943/ptinsa/2020/154987

[21] P. M. Mathews and M. Lakshmanan, On a unique nonlinear oscillator, Quarterly of Applied Mathematics, 32 (1974) 215-218, Doi: 10.1090/qam/430422.

[22] J. Akande, K. K. D. Adjaï, L. H. Koudahoun, and M.D. Monsia, Existence of non-periodic real-valued and complex-valued solutions of Mathews-Lakshmanan oscillator equation, , Math.Phys.,viXra.org/2012.0104v1.pdf (2020).

[23] J. Akande, K. K. D. Adjaï, L. H. Koudahoun, Y. J. F. Kpomahou and M.D. Monsia, Analytical and Classical Mechanics of Integrable Mixed and Quadratic Liénard Type Oscillator Equations, Math.Phys.,viXra.org/1608.0181v2.pdf (2016).

[24] K. K. D. Adjaï, J. Akande, P. Mallick, B. Rath and M. D. Monsia, Trigonometric and hyperbolic functions for general solutions of Duffing equation, Math.Phys.,viXra.org/1910.0510v1.pdf (2019).

[25] M. Nonti, J. Akande, P. Mallick, K. K. D. Adjaï, B. Rath and M.D. Monsia, First Integrals and Lagrangian Analysis of Nonlinear Differential Equations, Math.Phys.,viXra.org/1909.0177v1.pdf (2019)

[26] M. D. Monsia, Exact and non-sinusoidal periodic solutions of Lienard equations, Math.Phys.,viXra.org/2011.0059v3.pdf (2020)