Assuming $c < rad^2 abc$, A New Proof of the *abc* Conjecture

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Abstract In this paper, we consider the *abc* conjecture. Assuming that $c < rad^2(abc)$ is true, we give a new proof of the *abc* conjecture, by proceeding with the contradiction of the definition of the *abc* conjecture, for $\epsilon \geq 1$, then for $\epsilon \in]0, 1[$.

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To the memory of my Father who taught me arithmetic, To my wife, my daughter and my son

1. *Introduction and notations*

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a). \prod_i a_i^{\alpha_i - 1} \tag{1.1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a . rad(a) \tag{1.2}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [Wal13]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 1.1 (***abc* Conjecture**): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{1.3}$$

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where K is a constant depending only of ϵ.

The idea to try to write a paper about this conjecture was born after the publication in September 2018, of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [Kre18]. The difficulty to find a proof of the *abc* conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$. So, I will give a simple proof that can be understood by undergraduate students.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [Wal13]. A conjecture was proposed that $c < rad^2(abc)$ [Mih14]. It is the key to resolve the *abc* conjecture. In my paper, I assume that the conjecture $c < rad^2(abc)$ holds, I propose an elementary proof of the *abc* conjecture. The paper is organized as follows: in the second section, we give the proof of the *abc* conjecture.

2. *The Proof of the abc conjecture*

We note $R = rad(abc)$ in the case $c = a + b$ or $R = rad(ac)$ in the case $c = a + 1$. We assume that $c < R^2$ is true. We recall the following proposition [Nit96]:

Proposition 2.1 Let $\epsilon \longrightarrow K(\epsilon)$ the application verifying the *abc* conjecture, then:

$$lim_{\epsilon \to 0} K(\epsilon) = +\infty \tag{2.1}$$

2.1. Case : $\epsilon \geq 1$

Assuming that $c < R^2$ is true, we have $\forall \epsilon \geq 1$:

$$c < R^2 \leq R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad with \;\; K(\epsilon) = e, \;\epsilon \geq 1 \tag{2.2}$$

Then the *abc* conjecture is true.

2.2. Case: $\epsilon < 1$

2.2.1. Case: $c < R$. In this case, we can write :

$$c < R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad with \;\; K(\epsilon) = e > 1, \;\epsilon < 1 \tag{2.3}$$

Then the *abc* conjecture is true.

2.2.2. Case: $c > R$. From the statement of the *abc* conjecture 1.1, we want to give a proof that $c < K(\epsilon)R^{1+\epsilon} \Longleftrightarrow Logc < LogK(\epsilon) + (1+\epsilon)LogR \Longleftrightarrow LogK(\epsilon) + (1+$

$\epsilon)LogR - Logc > 0$. For our proof, we proceed by contradiction of the abc conjecture, so we assume that the conjecture is false:

$$\exists\, \epsilon_0 \in]0,1[, \forall\, K(\epsilon) > 0, \quad \exists\, c_0 = a_0 + b_0 \quad \text{so that} \;\; c_0 > K(\epsilon_0)R_0^{1+\epsilon_0} \Longrightarrow c_0 \text{ not a prime} \tag{2.4}$$

We choose the constant $K(\epsilon) = e^{\frac{1}{\epsilon^2}}$. Let :

$$Y_{c_0}(\epsilon) = \frac{1}{\epsilon^2} + (1+\epsilon)LogR_0 - Logc_0, \epsilon \in]0,1[\tag{2.5}$$

From the above explications, if we will obtain $\forall \epsilon \in]0,1[, Y_{c_0}(\epsilon) > 0 \Longrightarrow Y_{c_0}(\epsilon_0) > 0$, then the contradiction with (2.4).
About the function Y_{c_0}, we have $lim_{\epsilon \longrightarrow 1} Y_{c_0}(\epsilon) = 1 + Log(R_0^2/c_0) > 0$ and $lim_{\epsilon \longrightarrow 0} Y_{c_0}(\epsilon) = +\infty$. The function $Y_{c_0}(\epsilon)$ has a derivative for $\forall \epsilon \in]0,1[$, we obtain:

$$Y'_{c_0}(\epsilon) = -\frac{2}{\epsilon^3} + LogR_0 = \frac{\epsilon^3 LogR_0 - 2}{\epsilon^3} \tag{2.6}$$

$Y'_{c_0}(\epsilon) = 0 \Longrightarrow \epsilon = \epsilon' = \sqrt[3]{\dfrac{2}{LogR_0}} \in]0,1[$.

Discussion:

- If $Y_{c_0}(\epsilon') \geq 0$, it follows that $\forall \epsilon \in]0,1[, Y_{c_0}(\epsilon) \geq 0$, then the contradiction with $Y_{c_0}(\epsilon_0) < 0 \Longrightarrow c_0 > K(\epsilon_0)R_0^{1+\epsilon_0}$. Hence the *abc* conjecture is true for $\epsilon \in]0,1[$.

- If $Y_{c_0}(\epsilon') < 0 \Longrightarrow \exists\, 0 < \epsilon_1 < \epsilon' < \epsilon_2 < 1$, so that $Y_{c_0}(\epsilon_1) = Y_{c_0}(\epsilon_2) = 0$. Then we obtain $c_0 = K(\epsilon_1)R_0^{1+\epsilon_1} = K(\epsilon_2)R_0^{1+\epsilon_2}$. We recall the following definition:

Definition 2.1. The number ξ is called algebraic number if there is at least one polynomial:

$$l(x) = l_0 + l_1 x + \cdots + a_m x^m, \quad a_m \neq 0 \tag{2.7}$$

with integral coefficients such that $l(\xi) = 0$, and it is called transcendental if no such polynomial exists.

We consider the equality :

$$c_0 = K(\epsilon_1)R_0^{1+\epsilon_1} \Longrightarrow \frac{c_0}{R} = \frac{\mu_c}{rad(ab)} = e^{\frac{1}{\epsilon_1^2}} R_0^{\epsilon_1} \tag{2.8}$$

i) - We suppose that $\epsilon_1 = \beta_1$ is an algebraic number then $\beta_0 = 1/\epsilon_1^2$ and $R_0 = \alpha_1$ are also algebraic numbers. We obtain:

$$\frac{\mu_c}{rad(ab)} = e^{\frac{1}{\epsilon_1^2}} R_0^{\epsilon_1} = e^{\beta_0}.\alpha_1^{\beta_1} \tag{2.9}$$

From the theorem (see theorem 3, page 196 in [Bak71]):

Theorem 2.2. *$e^{\beta_0}\alpha_1^{\beta_1}\ldots\alpha_n^{\beta_n}$ is transcendental for any nonzero algebraic numbers $\alpha_1,\ldots,\alpha_n,\beta_0,\ldots,\beta_n$.*

we deduce that the right member $e^{\beta_0}.\alpha_1^{\beta_1}$ of (2.9) is transcendental, but the term $\dfrac{\mu_c}{rad(ab)}$ is an algebraic number, then the contradiction and the *abc* conjecture is true.

ii) - We suppose that ϵ_1 is transcendental, in this case there is also a contradiction, and the *abc* conjecture is true.

Then the proof of the *abc* conjecture is finished. We obtain that $\forall \epsilon > 0$, $\exists K(\epsilon) > 0$, if $c = a + b$ with a, b, c positive integers relatively coprime, then :

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{2.10}$$

and the constant $K(\epsilon)$ depends only of ϵ.

Q.E.D

Ouf, end of the mystery!

3. *Conclusion*

Assuming $c < R^2$ is true, we have given an elementary proof of the *abc* conjecture. We can announce the important theorem:

Theorem 3.1. *For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, and assuming $c < rad^2(abc)$ holds, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{3.1}$$

where K is a constant depending of ϵ.

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