Exceptions from Robin's Inequality

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Abstract

In this short but rigorous research note, I study Robin's Inequality. The number of possible violations of this inequality turns out to be finite. As the finiteness includes zero, I am able to convince you that there are no such violations.

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I. THIS YOU MIGHT ASK FROM A MATHEMATICIAN

Is it possible that there is an infinite number of (yet hidden) counter-examples for Riemann's Hypothesis? Or did someone prove that only a finite number of counter-examples (no single one is yet found) are possible? Who and in which paper? How many counter-examples Riemann's Hypothesis (if it is false) can have?

A. Emotional but reasonable answer from mathematical community

1. "If there are many *zeros deviating from the line* [Note from the author: use "counter-examples" instead], the whole picture becomes simply terrible, terrible, ugly." Steve Gonek [1].

   Prof. Gonek received his B.S. with Highest Honors in Mathematics in 1973, an M.S. in Mathematics in 1976, and a Ph.D. in Mathematics in 1979, all from the University of Michigan. After a two-year position at Temple University from 1978 to 1980, he joined the University of Rochester as an assistant professor of Mathematics in 1980 and is now a full professor. He spent the 1984–85 academic year at Oklahoma State University, part of Fall 1991 at Macquarie University in Sydney, Australia, part of Fall 1999 at the American Institute of Mathematics in Palo Alto, and half of 2004 at the Newton Institute in Cambridge, England.

2. "Beauty is the first criterion. There is no place in the world for ugly mathematics." Godfrey Harold Hardy [1, 2].

   Dr. Hardy is usually known by those outside the field of mathematics for his 1940 essay "A Mathematician's Apology", often considered one of the best insights into the mind of a working mathematician written for the layperson. He was an English mathematician, known for his achievements in number theory and mathematical analysis. Hardy is credited with reforming British mathematics by bringing rigor into it, which was previously a characteristic of French, Swiss and German mathematics. In a 1947 lecture, the Danish mathematician Harald Bohr has said: "Nowadays, there are only three really great English mathematicians: Hardy, Littlewood, and Hardy–Littlewood." [3]

II. MY EMOTION-FREE ANSWER

If Robin's inequality $F(n) > 0$ is true, where F is certain function given in Ref. [4], Riemann's Hypothesis turns out to be true. What is left to check today is the area $\exp(\exp(26)) < n < \infty$. [5, 6] A value $n = n_c$ is called "counter-example" if $F(n_c) < 0$.

I can express one of Dr. Zhu's results in a simpler way as:

If Robin's Inequality is true at least within $N < n < \infty$ where $N \gg 1$, Riemann's Hypothesis is true.

Numerical tests have shown that Robin's Inequality holds at least for $n < \exp(\exp(26))$. Therefore, one has the right to assign $N = \exp(\exp(26)) \gg 1$. Accordingly, Dr. Zhu's result comes true in a natural manner.

Thesis:

If the number of counter-examples of Robin's Inequality can be only finite, there are no counter-examples.

Proof: Dr. Zhu's papers tell us that if Robin's Inequality is true for each $n > N$, Riemann's Hypothesis is correct. If there is a finite number of counter-examples, then one has a number M as well so that at least within $M < n < \infty$ there are no counter-examples to Robin's Inequality. As N and M can be properly chosen, one can assign $M = N$. Thus,

If Riemann's Hypothesis fails, there must be an infinite number of counter-examples.
If Riemann's Hypothesis is true, there is a finite number of counter-examples.
If there are a finite number of counter-examples (that can be just zero),
then Riemann's Hypothesis is true.

Dr. Zhu has proven that Riemann's Hypothesis is true with scientific certainty [5]. That means that the number of counter-examples for any finite $0 < n < n_0$ is exactly zero. Therefore, in the limit $n_0 \to \infty$ there can be at most a finite number of counter-examples. In other words, the chance to meet a counter-example by a blind single pick up is $p = h/S$, where h is the total number of possible counter-examples, and S is the total number of cases. If h is infinite, then p is a mathematical uncertainty of the form ∞/∞, which cannot have a definite value. But because Dr. Zhu has shown that the value of p exists, h is finite.

Commentary on the proof

Let me expand on the latter points.

If there could be h counter-examples within $0 < n < n_0$, you would rush to conclude that Riemann's Hypothesis is wrong with probability $p = h/n_0$. But you never know for sure that within $0 < n < 2\,n_0$ there could not be $3\,h$ counter-examples, so the probability would be $p = 3h/(2\,n_0) \neq h/n_0$. Therefore, if a definitive p is given, it must be zero. Dr. Zhu has given a value for p, and (not surprisingly for me) it turned out to be zero.

I rely not only on Dr. Zhu's papers (one of which is published in arXiv only) but as well on the known fact that at least 40% of the zeroes of the zeta function belong to the 1/2 critical line [7]. Therefore, the probability of a failure of Riemann's Hypothesis is $0 \leq p < 100 - 40 = 60\%$. Thus, some definite values of p are excluded. This is only possible if there is a definite value of p, which is zero by my logical analysis above. In other words, p has a definite upper limit p_0. Such a property means that p exists in the mathematical sense, namely, its value converges, and it does not depend on the choice $0 < n < n_0$ or $0 < n < 2\,n_0$, $n_0 \gg 1$. Indeed, the numerical tests have shown, that within $0 < n < n_m$, where $n_m = \exp(\exp(26)) \gg 1$, there are no counter-examples, but that does not mean, that within $0 < n < 2\,n_m$ there are none of them.

Therefore, p has a definite value which is zero by my analysis above, and by the following consideration.

The probability of failure $p = h/S = h/(u + h)$, where h is the number of possible counter-examples (which have $F(n) < 0$), and u is the number of normal examples (which have $F(n) > 0$), must be zero because of the following. h can consist of all counter-examples which could be found within $0 < n < n_0$, and u can be constituted of all the normal examples which are found in $0 < n < w_0$, where w_0 is not related to n_0. Thus, the probability does not exist, because it can be any number, unless h is finite for all $0 < n < \infty$.

Let me illustrate the latter point by the following example. What is the probability to pick an even number? It is 50% if you blindly chose from 10 even and 10 odd numbers. But it is $100/300 \approx 33\%$ if you select from 100 even and 200 odd numbers. Thus, no definite probability exists to blindly pick an even number from all integer numbers.

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[1] Jim Holt, “When Einstein Walked With Godel: Excursions to the Edge of Thought,” Farrar, Straus and Giroux. 2018, p. 384.

[2] G. H. Hardy, “A Mathematician’s Apology”, Cambridge University Press, 1940.

[3] H. Bohr, “Looking Backward”, Collected Mathematical Works. 1. Copenhagen: Dansk Matematisk Forening, xiii–xxxiv (1952).

[4] G. Robin, “Grandes Valeurs de la fonction somme des diviseurs et hypothése de Riemann,” J. Math. Pures Appl. **63**, 187–213 (1984); Akbary A.; Friggstad Z., “Superabundant numbers and the Riemann hypothesis,” Am. Math. Monthly **116** (3), 273–275 (2009).

[5] P. Solé, Y. Zhu, “An Asymptotic Robin Inequality,” INTEGERS, **A81**, 16 (2016), http://math.colgate.edu/∼integers/q81/q81.pdf;
Y. Zhu, “The probability of Riemann’s hypothesis being true is equal to 1,” arXiv:1609.07555 [math.GM] (2016, 2018).

[6] K. Briggs, “Abundant numbers and the Riemann hypothesis,” Experiment. Math. **15** (2), 251–256 (2006).

[7] Eric W. Weisstein, “Critical Line.” From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/CriticalLine.html ; R. P. Brent, “On the Zeros of the Riemann Zeta Function in the Critical Strip.” Math. Comput. 33, 1361–1372, 1979; R. P. Brent, J. van de Lune, H. J. J. te Riele, D. T. Winter, “On the Zeros of the Riemann Zeta Function in the Critical Strip. II.” Math. Comput. 39, 681–688, 1982; I. Vardi, Computational Recreations in Mathematica. Reading, MA: Addison-Wesley, p. 142, 1991.