25 Zeno Dichotomies

Draft chapter of the book *Infinity Put to the Test* by Antonio León (next publication)

Abstract.-This chapter analyzes Zeno Dichotomies from the perspective of the w-order of the natural order of precedence of the natural numbers.

Keywords: Zeno dichotomy I and II, w-order, actual infinity, potential infinity.

Introductory definitions

P1 This chapter introduces a formalized version of Zeno's Dichotomy in its two variants (here referred to as Dichotomy I and II) based on the successiveness and discontinuity of ω-order (Dichotomy I) and of ω^*-order (Dichotomy II). Each of these formalized versions leads to a contradiction pointing to the inconsistency of the hypothesis of the actual infinity (the existence of the '*totality of finite cardinal numbers*", in Cantor's words [5, p. 103]) from which the first transfinite ordinal number ω is deduced [5, §15, Theorem A, p. 169].

P2 In the second half of the XX century, several solutions to some of Zeno's paradoxes were proposed with the aid of Cantor's transfinite arithmetic, topology, measure theory and more recently internal set theory (a branch of non-standard analysis) [7, 8, 25, 9, 11, 10, 17, 16]. It is also worth noting the solutions proposed by P. Lynds [13, 14] within classical and quantum mechanics frameworks. Some of these solutions, however, have been contested. And in most cases, the proposed solutions do not explain where Zeno's arguments fail. Moreover, some of the proposed solutions gave rise to a new collection of problems so exciting as Zeno's paradoxes [18, 1, 19, 20, 12, 21]. In the discussion that follows I propose a new way of discussing Zeno's Dichotomies based on the notion of ω-order, the order type of the well order sets whose ordinal number is ω, the least transfinite ordinal [5, §15, Theorem A, p.160]. The set $\mathbb{N}$ of the natural numbers in their natural order of precedence is an example of ω-ordered set.

P3 A sequence $\langle a_i \rangle$ indexed by the ω-ordered set $\mathbb{N}$ of the natural numbers in their natural order of precedence is also ω-ordered by the relation of precedence of their indexes (Theorem of the indexed collections; Chapter

4. See end note), which can be the same, or not, as their natural precedence if any. As is well known, in an ω-ordered sequence there is a first element but not a last one, and each element has an immediate successor and an immediate predecessor, except the first one, which has no predecessor. So, assuming the set of natural numbers exist as a complete infinite totality (hypothesis of the actual infinity subsumed into the Axiom of Infinity) means that any ω-ordered sequence also exists as a complete infinite totality, despite the fact that no last element completes the sequence.

P4 An ω^*-ordered sequence is one in which there exists a last element but not a first one, and each element has an immediate predecessor and an immediate successor, except the last one that has no successor. Since there is not a first element these sequences are non-well-ordered. From the same infinitist perspective, ω^*-ordered sequences are also complete infinite totalities, in spite of the fact that there is not a first element to begin with. The *increasing* sequence of negative integers, $\mathbb{Z}^* = \ldots$, -3, -2, -1, is an example of ω^*-ordered sequence.

P5 That said, let us consider a point particle P moving through the X axis from the point -1 to the point 2 at a constant finite velocity v (Figure 25.1). Assume P is in the point 0 just at the precise instant t_0. At instant $t_1 = t_0 + 1/v$ it will be exactly in the point 1. Consider now the following ω^*-ordered sequence of Z*-points $\langle z_i^* \rangle$ within the real interval $(0, 1)$, defined by [22]

$$z_{n*}^* = \frac{1}{2^n}, \ \forall n \in \mathbb{N} \tag{1}$$

where z_{n*}^* stands for the last but $n-1$ element of the ω^*-ordered sequence $\langle z_i^* \rangle$ of Z*-points. Consider also the sequence of Z-points $\langle z_i \rangle$ within the real interval $(0, 1)$ defined by:

$$z_n = \frac{2^n - 1}{2^n}, \ \forall n \in \mathbb{N} \tag{2}$$

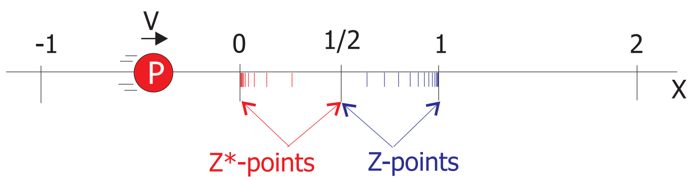

Figura 25.1 – Z-points and Z^*-points.

P6 Although the points of the X axis are densely ordered (between any two of its points other points do exist), Z*-points and Z-points are not. Between any two successive Z*-points $z^*_{(n+1)*}$, z_{n*} there is no other Z*-point (ω^*-discontinuity), and a distance greater than zero $z^*_{n*} - z^*_{(n+1)*} > 0$ always exists. Because of ω^*-discontinuity, Z*-points can only be traversed in a successive way, one at a time, one after the other, and in such a way that between any two successive Z*-points, a distance greater than zero $z^*_{n*} - z^*_{(n+1)*} > 0$ must always be traversed. The traversal will take a time greater than zero if it is traversed at a finite velocity. The same applies to Z-points, which exhibit ω-discontinuity.

P7 As P passes over the points of the closed real interval $[0, 1]$ it must traverse the successive Z*-points and the successive Z-points. It makes no sense to wonder about the instant at which the successive Z*-points begin to be traversed because there is not a first Z*-point to be traversed. The same can be said on the instant at which the traversal of the Z-points ends, in this case because there is not a last Z-point to be traversed. For this reason, we will focus our attention on the number of Z*-points that have already been traversed and on the number of Z-points that still remain to be traversed at any instant t within the closed real interval $[t_o, t_1]$.

P8 In this sense, and being t any instant within $[t_o, t_1]$, let $Z^*(t)$ be the number of Z*-points P has traversed just at instant t. And let $Z(t)$ be the number of Z-points to be traversed by P at instant t. The discussion that follows examines the evolution of $Z^*(t)$ and $Z(t)$ as P moves from the point 0 to the point 1. Both discussions are formalized versions of Zeno's Dichotomy II and I respectively. See, for instance, [3, 4, 23, 20, 12, 24, 6, 15].

P9 The strategy of pairing off the Z*-points (or the Z-points) with the successive instants of an strictly increasing infinite sequence of instants was firstly used (in a broad sense) by Aristotle [2, Books-III-VI] when trying to solve Zeno's dichotomies. Although Aristotle ended up by rejecting his original strategy, it is still the preferred to solve both paradoxes. As we will see, however, the discontinuity and separation of Z*-points and Z-points leads to a conflicting conclusion.

Zeno's Dichotomy II

P10 Let us begin by analyzing the way P passes over the Z*-points. Since the sequence of Z*-points is ω^*-ordered, its first point does not exist, and consequently its first n points, for any finite number n, do not exist either. Thus, and taking into account that P is in the point 0 at t_0 and in the point 1 at t_1, it holds:

$$\forall t \in [t_0, t_1] \begin{cases} t = t_0 : & Z^*(t) = 0 \\ t > t_0 : & Z^*(t) = \aleph_o \end{cases} \tag{3}$$

According to (3), no instant t exists within $[t_0, t_1]$ at which $Z^*(t) = n$, whatever be the finite number n, otherwise there would exist the impossible first n elements of an ω^*-ordered sequence. Notice $Z^*(t)$ is well defined in the whole interval $[t_0, t_1]$. Thus, equation (3) represents a dichotomy, ω^*-dichotomy: $Z^*(t)$ can only take two values along the whole closed interval $[t_0, t_1]$: 0 and $\aleph_o$.

P11 In agreement with P10 and regarding the number of traversed Z*-points, P can only have two successive states: the state $P^*(0)$ at which it has traversed zero Z*-points, and the state $P^*(\aleph_o)$ at which it has traversed aleph-null Z*-points. The number of traversed Z*-points change directly from zero to $\aleph_o$ (ω^*-dichotomy), without finite intermediate states at which only a finite number of Z*-points had been traversed.

P12 Taking into account the ω^*-discontinuity of Z*-points and the fact that between any two successive Z*-points a distance greater than zero always exists, to traverse two successive Z*-points $z^*_{(n+1)*}, z^*_{n*}$, whatsoever they be, means to traverse a distance greater than zero: $z^*_{n*} - z^*_{(n+1)*} > 0, \forall n \in \mathbb{N}$. In consequence, to traverse $\aleph_o$ of such successive Z*-points means to traverse a distance greater than zero. And to traverse a distance greater than zero at the finite velocity v of P means the traversal has to last a time greater than zero.

P13 Although it is impossible to calculate neither the exact duration of the transition $P^*(0) \rightarrow P^*(\aleph_o)$ nor the distance P must traverse while performing such a transition (there is neither a first instant nor a first point at which the transition begins), we have proved in P12 that, indeterminable as they might be, that duration and that distance must be greater than zero. It will now be proved they cannot be greater than zero.

P14 Let d be any real number greater than zero and consider the real interval $(0, d)$. According to ω^*-dichotomy (3), at any point x within $(0, d)$ our point-particle P have already traversed aleph-null Z*-points. In consequence the distance P must traverse while performing the transition $P^*(0) \rightarrow P^*(\aleph_o)$ is less than d. And since d is any real number greater than zero, we must conclude the distance P must traverse while performing the transition $P^*(0) \rightarrow P^*(\aleph_o)$ is less than any real number greater than zero.

P15 So then, according to P12, the distance P must traverse while performing the transition $P^*(0) \to P^*(\aleph_o)$ is greater than zero. And according to P14 that distance must be less than any number greater than zero. But there is no real number greater than zero and less than any real number greater than zero. So, it is impossible for the distance P must traverse wile performing the transition $P^*(0) \to P^*(\aleph_o)$ to be greater than zero. The same conclusion, and for the same reasons, applies to the time elapsed while performing the transition $P^*(0) \to P^*(\aleph_o)$.

P16 In line with P12 and P14, the point particle P needs to traverse a distance greater than zero for a time greater than zero to perform the transition $P^*(0) \to P^*(\aleph_o)$, but neither that distance nor that time can be greater than zero. Note this is not a question of indeterminacy but of impossibility. If it were a question of indeterminacy there would exist a nonempty set of possible solutions, although we could not determine which of them is the correct one. In our case the set of possible solutions is the empty set, because the set of real numbers greater than zero and less than any real number greater than zero is the empty set.

P17 In short:

A) According to the actual infinity hypothesis, the transition $P^*(0) \to P^*(\aleph_o)$ takes place.

B) The transition $P^*(0) \to P^*(\aleph_o)$ can only take place along a distance and a time greater than zero, because the of ω^*-discontinuity and to the distance greater than zero between any two Z^*-points.

C) The transition $P^*(0) \to P^*(\aleph_o)$ cannot take place along a distance and a time greater than zero, because of ω^*-dichotomy, and because no real number greater than zero is less than all real numbers greater than zero.

Zeno's Dichotomy I

P18 We will now examine the way P traverses the Z-points between the point 0 and the point 1. Being $Z(t)$ the number of Z-points to be traversed by P at the precise instant t in $[t_0, t_1]$, that number can only take two values: $\aleph_o$ and 0. In fact, assume that at any instant t within $[t_0, t_1]$ the number of Z-points to be traversed by P is a finite number $n > 0$. This would imply the impossible existence of the last n points of an ω-ordered sequence of points. Thus, we have a new dichotomy:

$$\forall t \in [t_0, t_1] \begin{cases} t < t_1 : & Z(t) = \aleph_o \\ t = t_1 : & Z(t) = 0 \end{cases} \tag{4}$$

Therefore, no instant t exists at which $Z(t) = n$, whatever be the finite number n. Notice $Z(t)$ is well defined in the whole interval $[t_0, t_1]$. Thus, equation (4) expresses a new dichotomy, ω-dichotomy: $Z(t)$ can only take two values: $\aleph_o$ and 0.

P19 In accord with P18 and regarding the number of Z-points to be traversed, P can only have two successive states: the state $P(\aleph_o)$ at which that number is $\aleph_o$, and the state $P(0)$ at which that number is 0. The number of Z-points to be traversed by P decreases directly from $\aleph_o$ to 0, without finite intermediate states at which only a finite number of Z-points remain to be traversed.

P20 Taking into account the ω-discontinuity of Z-points and the fact that between any two successive Z-points a distance greater than zero always exists, to traverse two successive Z-points, whatsoever they be, means to traverse a distance greater than zero: $z_{n+1} - z_n > 0, \forall n \in \mathbb{N}$. In consequence, to traverse $\aleph_o$ of such successive Z-points means to traverse a distance greater than zero. And to traverse a distance greater than zero at the finite velocity v of P means the traversal has to last a time greater than zero.

P21 Although it is impossible to calculate neither the exact duration of the transition $P(\aleph_o) \rightarrow P(0)$ nor the distance P must traverse while performing such a transition (there is neither a last instant nor a last point at which the transition ends), we have proved in P12 that, indeterminable as they might be, that duration and that distance must be greater than zero. It will now be proved they cannot be greater than zero.

P22 Let τ be any real number greater than zero, and consider the real interval $(0, \tau)$. According to ω-dichotomy (4), for any instant t within $(0, \tau)$ the number of Z-points to be traversed at instant $t_1 - t$ is $\aleph_o$. In consequence, the time P needs to perform the transition $P(\aleph_o) \rightarrow P(0)$ is less than τ. And since τ is any real number greater than zero, we must conclude the time P needs to perform the transition $P(\aleph_o) \rightarrow P(0)$ is less than any real number greater than zero.

P23 So then, according to P20, the time P needs to perform the transition $P(\aleph_o) \rightarrow P(0)$ is greater than zero. And according to P22 that time must be less than any real number greater than zero. But there is no real number greater than zero and less than any real number greater than zero. So, it is impossible for the transition $P(\aleph_o) \rightarrow P(0)$ to last a time greater than zero. The same conclusion, and for the same reasons, applies to the distance P must traverse while performing the transition $P(\aleph_o) \rightarrow P(0)$.

P24 In line with P20 and P22, P needs to traverse a distance greater than zero during a time greater than zero to perform the transition $P(\aleph_o) \rightarrow P(0)$, but neither that distance nor that time can be greater than zero. Note this is not a question of indeterminacy but of impossibility. If it were a question of indeterminacy there would exist a nonempty set of possible solutions, although we could not determine which of them is the correct one. In our case the set of possible solutions is the empty set because the set of real numbers greater than zero and less than any real number greater than zero is, in fact, the empty set.

P25 In short:

A) According to the actual infinity hypothesis, the transition $P(\aleph_o) \rightarrow P(0)$ takes place.

B) The transition $P(\aleph_o) \rightarrow P(0)$ can only take place along a distance and a time greater than zero, because of ω-discontinuity and of the distance greater than zero between any two Z^*-points.

C) The transition $P(\aleph_o) \rightarrow P(0)$ cannot take place along a distance and a time greater than zero (because of ω-dichotomy), and because no real number greater than zero is less than all real numbers greater than zero..

Conclusion

P26 According to the hypothesis of the actual infinity, the set of Z-points and the set of Z*-points do exist as complete totalities. Therefore the transitions $P^*(0) \rightarrow P^*(\aleph_o)$ and $P(\aleph_o) \rightarrow P(0)$ take place while P moves from the point 0 to the point 1. Now then, the transitions $P^*(0) \rightarrow P^*(\aleph_o)$ and $P(\aleph_o) \rightarrow P(0)$ can only take place along a distance and a time greater than zero. The problem is that they cannot take place along a distance and a time greater than zero because that time and that distance is less than any real number greater than zero, and no real number greater than zero and less than any real number greater than zero do exist.

P27 The above contradictions are direct consequences of assuming that ω-ordered and ω^*-ordered sets, as the sets of Z-points and of Z*-points, exist as complete infinite totalities, which in turn is a consequence of assuming the existence of all finite cardinals as an infinite totality [5, §14, p. 152], or in other words of assuming the existence of all finite natural numbers as a complete infinite totality, which is the hypothesis of the actual infinity subsumed into the Axiom of Infinity in modern set theories. An hypothesis that, consequently, should be put to the test.

End note.

Theorem of the indexed collections.-*An indexed collection and its collection of indexes have the same cardinal and the same ordinal if they are equipotent, and the relation of precedence is the same in both the collection of indexes and the indexed collection.*

Chapter References

[1] Joseph S. Alper and Mark Bridger, *Mathematics, Models and Zeno's Paradoxes*, Synthese **110** (1997), 143 – 166.

[2] Aristóteles, *Física*, Gredos, Madrid, 1998.

[3] Florian Cajori, *The History of Zeno's Arguments on Motion*, American Mathematical Monthly **XXII** (1915), 1–6, 38–47, 77–82, 109–115, 143–149, 179,–186, 215–220, 253–258, 292–297, http://www.matedu.cinvestav.mx/librosydocelec/Cajori.pdf.

[4] ______, *The Purpose of Zeno's Arguments on Motion*, Isis **III** (1920-21), 7–20.

[5] Georg Cantor, *Contributions to the founding of the theory of transfinite numbers*, Dover, New York, 1955.

[6] Giorgio Colli, *Zenón de Elea*, Sexto Piso, Madrid, 2006.

[7] Adolf Grünbaum, *Modern Science and Refutation of the Paradoxes of Zeno*, The Scientific Monthly **LXXXI** (1955), 234–239.

[8] ______, *Modern Science and Zeno's Paradoxes*, George Allen And Unwin Ltd, London, 1967.

[9] ______, *Modern Science and Refutation of the Paradoxes of Zeno*, Zeno's Paradoxes (Wesley C. Salmon, ed.), Hackett Publishing Company, Inc, Indianapolis/Cambridge, 2001, pp. 164 – 175.

[10] ______, *Modern Science and Zeno's Paradoxes of Motion*, Zeno's Paradoxes (Wesley C. Salmon, ed.), Hackett Publishing Company, Inc, Indianapolis/Cambridge, 2001, pp. 200 – 250.

[11] ______, *Zeno's Metrical Paradox of Extension*, Zeno's Paradoxes (Wesley C. Salmon, ed.), Hackett Publishing Company, Inc, Indianapolis/Cambridge, 2001, pp. 176 – 199.

[12] Nick Huggett, *Zeno's Paradoxes*, The Stanford Encyclopaedia of Philosophy(Summer 2004 Edition) (Edward N. Zalta(ed.), ed.), Stanford University, ¡http://plato.stanford.edu/archives/sum2004/entries/paradox-zeno/¿, 2004.

[13] Peter Lynds, *Time and Classical and Quantum Mechanics: Indeterminacy vs. Discontinuity*, Foundations of Physics Letters **16** (2003), 343 – 355.

[14] Peter Lynds, *Zeno's Paradoxes: A Timely Solution*, philsci-archives (3003), 1 – 9, http://philsci-archives.pitt.edu/archive/00001197.

[15] Joseph Mazur, *The Motion Paradox*, Dutton, 2007.

[16] William I. McLaughlin, *Una resolución de las paradojas de Zenón*, Investigación y Ciencia (Scientifc American) (1995), no. 220, 62 – 68.

[17] William I. McLaughlin and Silvia L. Miller, *An Epistemological Use of non-standard Analysis to Answer Zeno's Objections Against Motion*, Synthese **92** (1992), no. 3, 371 – 384.

[18] Alba Papa-Grimaldi, *Why mathematical solutions of Zeno's paradoxes miss the point: Zeno's one and many relation and Parmenides prohibition*, The Revew of Metaphysics **50** (1996), 299–314.

[19] Jon Pérez Laraudogoitia, *Supertasks*, The Stanford Encyclopaedia of Philosophy (E. N. Zaltax, ed.), Standford University, URL = http://plato.stanford.edu, 2001.

[20] Wesley C. Salmon, *Introduction*, Zeno's Paradoxes (Wesley C. Salmon, ed.), Hackett Publishing Company, Inc, Indianapolis, Cambridge, 2001, pp. 5 – 44.

[21] Z. K. Silagadze, *Zeno meets modern science*, Philsci-archieve (2005), 1–40.

[22] Gregory Vlastos, *Zeno's Race Course*, Journal of the History of Philosophy **IV** (1966), 95–108.

[23] ______, *Zeno of Elea*, The Encyclopaedia of Philosophy (Paul Edwards, ed.), McMillan and Free Press, New York, 1967.

[24] David Foster Wallace, *Everything and more. Acompact history of infinity*, Orion Books Ltd., London, 2005.

[25] Mark Zangari, *Zeno, Zero and Indeterminate Forms: Instants in the Logic of Motion*, Australasian Journal of Philosophy **72** (1994), 187–204.