The Two Relativistic Rydberg Formulas of Suto and Haug: Further Comments

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Abstract

In a recent paper, we discussed [1] that Suto [2] has pointed out an interesting relativistic extension of Rydberg's formula. In that paper, we had slightly misunderstood Suto's approach, something we will comment on further here. The relativistic Suto formula is actually derived from a theory where the standard relativistic momentum relation is changed. The relativistic Rydberg formula we presented and mistakenly thought was the same as his formula of Suto is, on the other hand, derived to be fully consistent with the standard relativistic momentum relation. Here we will point out the differences between the formulas and correct some errors in our previous paper.

Key Words: Rydberg's formula, relativistic extension, Compton wavelength.

1 Introduction

Rydberg's [3] formula is given by

$$\frac{1}{\lambda} = R_\infty Z^2 \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right) \tag{1}$$

where R_∞ is the Rydberg's constant, which has a value of 10973731.568160(21) m^{-1} (NIST CODATA value). Even though the formula is very simple, it is hard to gain much intuition from it. The Rydberg constant can be rewritten as

$$\begin{aligned}
R_\infty &= \frac{m_e e^4}{8\epsilon_0^2 h^3 c} \\
R_\infty &= \frac{\frac{\hbar}{\bar{\lambda}_e}\frac{1}{c}\left(\sqrt{\frac{\hbar}{c}}\sqrt{\alpha}\sqrt{10^7}\right)^4}{8\epsilon_0^2 h^3 c} \\
R_\infty &= \frac{\frac{\hbar^3}{\bar{\lambda}_e}\frac{1}{c^3}\alpha^2(10^7)^2}{8\left(\frac{1}{4\pi c^2 10^{-7}}\right)^2 h^3 c} \\
R_\infty &= \frac{\frac{\hbar^3}{\bar{\lambda}_e}\frac{1}{c^3}\alpha^2}{8\frac{1}{16\pi^2 c^4}h^3 c} \\
R_\infty &= \frac{1}{2}\frac{\hbar}{h}\frac{1}{\bar{\lambda}_e}\alpha^2 \\
R_\infty &= \frac{1}{2}\frac{\frac{h}{2\pi}}{h}\frac{1}{\bar{\lambda}_e}\alpha^2 \\
R_\infty &= \frac{\alpha^2}{2\lambda_e}
\end{aligned} \tag{2}$$

And this we can rewrite as

$$R_\infty = \frac{\alpha^2}{2\frac{h}{m_e c}} = \frac{\alpha^2 m_e c}{2h} \tag{3}$$

(In the former paper, we had incorrectly used the Compton wavelength rather than the reduced Compton wavelength in the beginning of this derivation and therefore incorrectly got $R_\infty = \frac{\alpha^2}{2\frac{h}{m_e c}} = \frac{\alpha^2 m_e c}{4\pi\hbar}$. However, we made another error further down that canceled this error out and therefore we obtained the right result with respect to the Rydberg formula.)

This is well known, so we have shown nothing new so far. Let us now replace this in Rydberg's formula, which gives

$$\begin{aligned} \frac{1}{\lambda} &= \frac{\alpha^2 m_e c}{2h} Z^2 \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right) \\ h\frac{c}{\lambda} &= Z^2 \left(\frac{1}{2} m_e \frac{\alpha^2 c^2}{n_1^2} - \frac{1}{2} m_e \frac{\alpha^2 c^2}{n_2^2} \right) \end{aligned} \tag{4}$$

where $\frac{\alpha^2 c^2}{n_1^2}$ can be seen as v_1^2 and $\frac{\alpha^2 c^2}{n_2^2}$ as v_2^2. In other words, we can write this as

$$E = h\frac{c}{\lambda} = \frac{Z^2}{2\pi} \left(\frac{1}{2} m_e v_1^2 - \frac{1}{2} m_e v_2^2 \right) \tag{5}$$

Since $\frac{1}{2}mv^2$ is the well-known approximation of the kinetic energy when $v << c$ (the first term of a Taylor series approximation), the Rydberg formula is clearly non-relativistic. Even though this is known, we have not seen any relativistic extension of the formula before the paper of Suto [2]. However, before we discuss his formula, we will briefly show how we arrived at our relativistic version of the Rydberg formula. Since $\frac{1}{2}mv^2$ is the approximation for $v << c$, we simply replaced this approximation by the full relativistic kinetic energy $E_k = mc^2\gamma - mc^2$, where, as usual, $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$. This gives

$$E = \left(\frac{m_e c^2}{\sqrt{1 - v_1^2/c^2}} - mc^2 - \frac{m_e c^2}{\sqrt{1 - v_2^2/c^2}} - mc^2 \right) = \left(\frac{m_e c^2}{\sqrt{1 - v_1^2/c^2}} - \frac{m_e c^2}{\sqrt{1 - v_2^2/c^2}} \right) \tag{6}$$

where $v_1 = z\alpha c/n_1$ and $v_2 = z\alpha c/n_2$, and we also have that $m_e = \frac{h}{\lambda_e}\frac{1}{c}$, where λ_e is the Compton [4] wavelength of the electron. The equation can then be rewritten as

$$\begin{aligned} h\frac{c}{\lambda} &= \left(\frac{\frac{h}{\lambda_e} c}{\sqrt{1 - \frac{z^2\alpha^2}{n_1^2}}} - \frac{\frac{h}{\lambda_e} c}{\sqrt{1 - \frac{z^2\alpha^2}{n_2^2}}} \right) \\ \frac{1}{\lambda} &= \left(\frac{1}{\lambda_e \sqrt{1 - \frac{z^2\alpha^2}{n_1^2}}} - \frac{1}{\lambda_e \sqrt{1 - \frac{z^2\alpha^2}{n_2^2}}} \right) \end{aligned} \tag{7}$$

We mistakenly thought that the formula we presented in the last paper was the same as Suto's relativistic Rydberg formula. However, Suto's [2] relativistic formula is (his equation 48)

$$\frac{1}{\lambda} = \left(\frac{1}{\lambda_e \sqrt{1 + \frac{\alpha^2}{n_2^2}}} - \frac{1}{\lambda_e \sqrt{1 + \frac{\alpha^2}{n_1^2}}} \right) \tag{8}$$

where λ is the photon wavelength, and λ_e is the Compton wavelength of the electron.

While our new relativistic formula should be consistent with the standard energy momentum relation $E^2 = p^2c^2 + m^2c^4$, where p is the momentum, the Suto formula is not consistent with this, but it is consistent with the modified energy momentum relation that he presented in the same paper as $m_e^2c^4 - p_n^2c^2 = m_n^2c^4$. The Taylor expansion of our relativistic formula is

$$\frac{1}{\bar{\lambda}} = \frac{1}{\bar{\lambda}_e} \left(\left(1 - \frac{z^2\alpha^2}{2n_1^2} + \frac{3z^4\alpha^4}{8n_1^4} + \frac{5z^6\alpha^6}{16n_1^6} + \cdots \right) - \left(1 - \frac{z^2\alpha^2}{2n_2^2} + \frac{3z^4\alpha^4}{8n_2^4} + \frac{5z^6\alpha^6}{16n_2^6} + \cdots \right) \right) \tag{9}$$

And the Taylor series expansion of the Suto formula is

$$\frac{1}{\bar{\lambda}} = \frac{1}{\bar{\lambda}_e} \left(\left(1 - \frac{\alpha^2}{2n_1^2} - \frac{3\alpha^4}{8n_1^4} + \frac{5\alpha^6}{16n_1^6} + \cdots \right) - \left(1 - \frac{\alpha^2}{2n_2^2} - \frac{3\alpha^4}{8n_2^4} + \frac{5\alpha^6}{16n_2^6} + \cdots \right) \right) \tag{10}$$

We incorrectly pointed out that Suto had possibly made a mistake and there was a sign error in his series expansion, but this is actually not the case. This was because we thought his formula was identical to the one that we had derived[1].

[1] We apologize for missing this, but we are taking steps to correct that here.

The Suto formula, when expanded to hold for any atom, based on his energy momentum assumption must likely be:

$$\frac{1}{\lambda} = \left(\frac{1}{\lambda_e \sqrt{1 + \frac{z^2\alpha^2}{n_2^2}}} - \frac{1}{\lambda_e \sqrt{1 + \frac{z^2\alpha^2}{n_1^2}}} \right) \tag{11}$$

while our relativistic extension of the Rydberg formula is

$$\frac{1}{\lambda} = \left(\frac{1}{\lambda_e \sqrt{1 - \frac{z^2\alpha^2}{n_1^2}}} - \frac{1}{\lambda_e \sqrt{1 - \frac{z^2\alpha^2}{n_2^2}}} \right) \tag{12}$$

In other words, we have recently gotten two relativistic Rydberg formulas, one consistent with the standard relativistic energy momentum relation (the Haug formula) and one consistent with what we can call a somewhat alternative theory of Suto.

In general, we would think the formula that is consistent with the standard relativistic energy momentum relation is more correct and consistent. However, it is not necessarily easy to test out which one is best, as the hydrogen atom is known to be best described by the relativistic Dirac [5] wave equation.

2 Table Calculations

In Table 1 we look at the Lyman series. This is for a hydrogen atom where we hold $n_1 = 1$ and in this table let n_2 vary from 2 to 7. We see that the Haug formula predicts a slightly shorter wavelength than the non-relativistic Rydberg formula and that the Suto formula predicts a slightly longer wavelength than the non-relativistic formula. Table 2 show predictions from the three formulas for the Balmer series, where we have $n_1 = 2$ and let n_2 vary from 3 to 7. The first column in Table 2 are real observations. The real observations in this case have been done in air, so we have adjusted all the three formulas by the refraction index in air; this simply means we need to divide the formulas by the refraction index in air, which is about 1.00029. One can find observation studies done in a vacuum and in air; when comparing theoretical predictions against observations it is naturally important to know whether the observations have been done in a vacuum or in air.

Since Table 1 and Table 2 are covering hydrogen atoms, we have also adjusted all of the formulas by multiplying by the adjusted mass $\frac{m_P}{m_P+m_e}$, where m_P is the proton mass, and m_e is the electron mass. First of all, observations for they hydrogen atom are likely not accurate enough to distinguish between the non-relativistic and relativistic formulas, but we leave that to other experts to decide. For the hydrogen atom, the Lyman series is where the differences between the three formulas are the biggest, so there is no reason to look at the Paschen, Brackett, Humphreys, or Pfund series in addition, as the differences between non-relativistic and relativistic predictions would be even smaller. To test out the formulas, one would need to look at much heavier hydrogen-like atoms, as the electrons in general move much faster and therefore relativistic effects would play a bigger role. Still, there is the issue that one likely needs a relativistic wave equation to include all necessary adjustments.

n_2	Non-Relativistic	Haug	Difference	Suto	Difference
2	121.568	121.562	-0.0050%	121.575	0.0050%
3	102.573	102.569	-0.0044%	102.578	0.0044%
4	97.255	97.251	-0.0042%	97.259	0.0042%
5	94.975	94.971	-0.0042%	94.979	0.0042%
6	93.781	93.778	-0.0041%	93.785	0.0041%
7	93.076	93.072	-0.0041%	93.080	0.0041%

Table 1: The table shows the Lyman series calculated from the non-relativistic formula, the Haug relativistic formula, and the Suto relativistic formula. The difference-column shows the difference in percent between the relativistic formula predictions and the non- relativistic formula predictions for the Haug and Suto formulas. The Haug relativistic formula predicts a slightly shorter wavelength than the non-relativistic formula, and the Suto formula predicts a slightly longer wavelength. The Haug formula seems to be consistent with relativistic length contraction (also of waves).

In our original table, there was a typo in the spreadsheet that resulted in incorrect values from our relativistic Rydberg formula. Below we present the corrected tables; here we have not adjusted the formulas based on the reduced mass, so in any of the formulas the difference in values will be the same, even if we multiply each predicted wavelength with the reduced mass: $\frac{m_P}{m_P+m_e}$.

3 Conclusion

We have shown that the relativistic extensions of the Rydberg formula given by Suto and Haug are two different formulas. The Haug relativistic Rydberg formula is consistent with the standard relativistic energy momentum relation, and the Suto formula is based on a somewhat alternative theory, with a modified relativistic energy

n_2	Observed in air	Non-relativistic in air	Difference	Haug in air	Difference	Suto in air	Difference
3	656.28	656.279	-0.0001%	656.270	-0.0016%	656.289	0.0013%
4	486.13	486.133	0.0006%	486.127	-0.0007%	486.139	0.0018%
5	434.05	434.047	-0.0007%	434.042	-0.0018%	434.052	0.0005%
6	410.17	410.175	0.0011%	410.170	0.0000%	410.179	0.0022%
7	397.005	397.008	0.0009%	397.004	-0.0002%	397.013	0.0020%

Table 2: The table shows the Balmer series calculated from the non-relativistic formula, the Haug relativistic formula, and the Suto relativistic formula. The difference column is the difference in percent between the relativistic formula predictions and the non-relativistic formula predictions for the Haug and Suto formulas. The Haug relativistic formula predicts a slightly shorter wavelength than the non-relativistic formula, and the Suto formula predicts a slightly longer wavelength. The Haug formula seems to be consistent with relativistic length contraction (also of waves). The observations are from the Atomic Spectra NIST Standard Reference Database 78 Version 5.7, are done in air, so we have made an adjustment based on refraction index in all formulas based on air. If this adjustment is not done, our prediction is far off, as expected.

momentum formula. It is too early to say whether or not these new relativistic extensions of the Rydberg formula can tell us anything new that is consistent with observations. We encourage others to look further into this, and we hope to do so some time in the future as well.

References

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Atomic #	Rydberg formula	Relativistic formula	Diff.	Diff. %	Atomic #	Rydberg formula	Relativistic formula	Diff.	Diff. %
1	121.5023	121.4962	(0.0061)	-0.005%	71	0.0241	0.0181	(0.0060)	-32.9%
2	30.3756	30.3695	(0.0061)	-0.020%	72	0.0234	0.0175	(0.0060)	-34.2%
3	13.5003	13.4942	(0.0061)	-0.045%	73	0.0228	0.0168	(0.0060)	-35.4%
4	7.5939	7.5878	(0.0061)	-0.080%	74	0.0222	0.0162	(0.0060)	-36.8%
5	4.8601	4.8540	(0.0061)	-0.125%	75	0.0216	0.0156	(0.0060)	-38.1%
6	3.3751	3.3690	(0.0061)	-0.180%	76	0.0210	0.0151	(0.0060)	-39.5%
7	2.4796	2.4736	(0.0061)	-0.245%	77	0.0205	0.0145	(0.0060)	-41.0%
8	1.8985	1.8924	(0.0061)	-0.320%	78	0.0200	0.0140	(0.0060)	-42.5%
9	1.5000	1.4940	(0.0061)	-0.406%	79	0.0195	0.0135	(0.0060)	-44.0%
10	1.2150	1.2090	(0.0061)	-0.502%	80	0.0190	0.0130	(0.0059)	-45.6%
11	1.0042	0.9981	(0.0061)	-0.607%	81	0.0185	0.0126	(0.0059)	-47.3%
12	0.8438	0.8377	(0.0061)	-0.724%	82	0.0181	0.0121	(0.0059)	-49.0%
13	0.7189	0.7129	(0.0061)	-0.9%	83	0.0176	0.0117	(0.0059)	-50.8%
14	0.6199	0.6138	(0.0061)	-1.0%	84	0.0172	0.0113	(0.0059)	-52.7%
15	0.5400	0.5339	(0.0061)	-1.1%	85	0.0168	0.0109	(0.0059)	-54.6%
16	0.4746	0.4686	(0.0061)	-1.3%	86	0.0164	0.0105	(0.0059)	-56.6%
17	0.4204	0.4144	(0.0061)	-1.5%	87	0.0161	0.0101	(0.0059)	-58.6%
18	0.3750	0.3689	(0.0061)	-1.6%	88	0.0157	0.0098	(0.0059)	-60.8%
19	0.3366	0.3305	(0.0061)	-1.8%	89	0.0153	0.0094	(0.0059)	-63.0%
20	0.3038	0.2977	(0.0061)	-2.0%	90	0.0150	0.0091	(0.0059)	-65.3%
21	0.2755	0.2695	(0.0061)	-2.2%	91	0.0147	0.0087	(0.0059)	-67.7%
22	0.2510	0.2450	(0.0061)	-2.5%	92	0.0144	0.0084	(0.0059)	-70.2%
23	0.2297	0.2236	(0.0061)	-2.7%	93	0.0140	0.0081	(0.0059)	-72.8%
24	0.2109	0.2049	(0.0061)	-3.0%	94	0.0138	0.0078	(0.0059)	-75.5%
25	0.1944	0.1884	(0.0061)	-3.2%	95	0.0135	0.0075	(0.0059)	-78.4%
26	0.1797	0.1737	(0.0061)	-3.5%	96	0.0132	0.0073	(0.0059)	-81.3%
27	0.1667	0.1606	(0.0061)	-3.8%	97	0.0129	0.0070	(0.0059)	-84.4%
28	0.1550	0.1489	(0.0061)	-4.1%	98	0.0127	0.0067	(0.0059)	-87.6%
29	0.1445	0.1384	(0.0060)	-4.4%	99	0.0124	0.0065	(0.0059)	-91.0%
30	0.1350	0.1290	(0.0060)	-4.7%	100	0.0122	0.0062	(0.0059)	-94.6%
31	0.1264	0.1204	(0.0060)	-5.0%	101	0.0119	0.0060	(0.0059)	-98.3%
32	0.1187	0.1126	(0.0060)	-5.4%	102	0.0117	0.0058	(0.0059)	-102.2%
33	0.1116	0.1055	(0.0060)	-5.7%	103	0.0115	0.0056	(0.0059)	-106.3%
34	0.1051	0.0991	(0.0060)	-6.1%	104	0.0112	0.0053	(0.0059)	-110.6%
35	0.0992	0.0931	(0.0060)	-6.5%	105	0.0110	0.0051	(0.0059)	-115.1%
36	0.0938	0.0877	(0.0060)	-6.9%	106	0.0108	0.0049	(0.0059)	-119.9%
37	0.0888	0.0827	(0.0060)	-7.3%	107	0.0106	0.0047	(0.0059)	-125.0%
38	0.0841	0.0781	(0.0060)	-7.7%	108	0.0104	0.0045	(0.0059)	-130.4%
39	0.0799	0.0738	(0.0060)	-8.2%	109	0.0102	0.0043	(0.0059)	-136.1%
40	0.0759	0.0699	(0.0060)	-8.6%	110	0.0100	0.0041	(0.0059)	-142.1%
41	0.0723	0.0662	(0.0060)	-9.1%	111	0.0099	0.0040	(0.0059)	-148.6%
42	0.0689	0.0628	(0.0060)	-9.6%	112	0.0097	0.0038	(0.0059)	-155.4%
43	0.0657	0.0597	(0.0060)	-10.1%	113	0.0095	0.0036	(0.0059)	-162.8%
44	0.0628	0.0567	(0.0060)	-10.6%	114	0.0093	0.0035	(0.0059)	-170.7%
45	0.0600	0.0540	(0.0060)	-11.2%	115	0.0092	0.0033	(0.0059)	-179.1%
46	0.0574	0.0514	(0.0060)	-11.7%	116	0.0090	0.0031	(0.0059)	-188.3%
47	0.0550	0.0490	(0.0060)	-12.3%	117	0.0089	0.0030	(0.0059)	-198.1%
48	0.0527	0.0467	(0.0060)	-12.9%	118	0.0087	0.0028	(0.0059)	-208.9%
49	0.0506	0.0446	(0.0060)	-13.5%	119	0.0086	0.0027	(0.0059)	-220.6%
50	0.0486	0.0426	(0.0060)	-14.1%	120	0.0084	0.0025	(0.0059)	-233.4%
51	0.0467	0.0407	(0.0060)	-14.8%	121	0.0083	0.0024	(0.0059)	-247.4%
52	0.0449	0.0389	(0.0060)	-15.4%	122	0.0082	0.0022	(0.0059)	-263.0%
53	0.0433	0.0372	(0.0060)	-16.1%	123	0.0080	0.0021	(0.0059)	-280.4%
54	0.0417	0.0357	(0.0060)	-16.9%	124	0.0079	0.0020	(0.0059)	-299.9%
55	0.0402	0.0342	(0.0060)	-17.6%	125	0.0078	0.0018	(0.0059)	-321.9%
56	0.0387	0.0327	(0.0060)	-18.3%	126	0.0077	0.0017	(0.0059)	-347.1%
57	0.0374	0.0314	(0.0060)	-19.1%	127	0.0075	0.0016	(0.0060)	-376.2%
58	0.0361	0.0301	(0.0060)	-19.9%	128	0.0074	0.0015	(0.0060)	-410.2%
59	0.0349	0.0289	(0.0060)	-20.8%	129	0.0073	0.0013	(0.0060)	-450.8%
60	0.0338	0.0278	(0.0060)	-21.6%	130	0.0072	0.0012	(0.0060)	-500.2%
61	0.0327	0.0267	(0.0060)	-22.5%	131	0.0071	0.0011	(0.0060)	-562.0%
62	0.0316	0.0256	(0.0060)	-23.4%	132	0.0070	0.0009	(0.0060)	-642.0%
63	0.0306	0.0246	(0.0060)	-24.3%	133	0.0069	0.0008	(0.0061)	-751.3%
64	0.0297	0.0237	(0.0060)	-25.3%	134	0.0068	0.0007	(0.0061)	-912.6%
65	0.0288	0.0228	(0.0060)	-26.3%	135	0.0067	0.0005	(0.0061)	-1184.2%
66	0.0279	0.0219	(0.0060)	-27.3%	136	0.0066	0.0003	(0.0062)	-1794.2%
67	0.0271	0.0211	(0.0060)	-28.4%	137	0.0065	0.0001	(0.0064)	-11232.7%
68	0.0263	0.0203	(0.0060)	-29.5%					
69	0.0255	0.0195	(0.0060)	-30.6%					
70	0.0248	0.0188	(0.0060)	-31.7%					

Table 3: The table shows the Rydberg formula predictions and the relativistic predictions for the first 137 elements. As we can see, the difference increases between the two models. Here we are just looking at the case $n_1 = 1$ and $n_2 = 2$.

Atomic #	Rydberg formula	Relativistic formula	Diff.	Diff. %	Atomic #	Rydberg formula	Relativistic formula	Diff.	Diff. %
1	121.5023	121.4962	-0.0061	-0.0050%	71	0.0241	0.0144	-0.0097	-67.6%
2	30.3756	26.0315	-4.3440	-16.7%	72	0.0234	0.0139	-0.0096	-68.9%
3	13.5003	11.0414	-2.4589	-22.3%	73	0.0228	0.0134	-0.0094	-70.2%
4	7.5939	6.0710	-1.5229	-25.1%	74	0.0222	0.0129	-0.0093	-71.6%
5	4.8601	3.8329	-1.0272	-26.8%	75	0.0216	0.0125	-0.0091	-73.0%
6	3.3751	2.6374	-0.7377	-28.0%	76	0.0210	0.0121	-0.0090	-74.5%
7	2.4796	1.9247	-0.5549	-28.8%	77	0.0205	0.0116	-0.0088	-76.0%
8	1.8985	1.4659	-0.4326	-29.5%	78	0.0200	0.0112	-0.0087	-77.6%
9	1.5000	1.1533	-0.3467	-30.1%	79	0.0195	0.0109	-0.0086	-79.2%
10	1.2150	0.9308	-0.2842	-30.5%	80	0.0190	0.0105	-0.0085	-80.9%
11	1.0042	0.7668	-0.2373	-31.0%	81	0.0185	0.0101	-0.0084	-82.6%
12	0.8438	0.6425	-0.2013	-31.3%	82	0.0181	0.0098	-0.0083	-84.4%
13	0.7189	0.5460	-0.1729	-31.7%	83	0.0176	0.0095	-0.0082	-86.2%
14	0.6199	0.4696	-0.1503	-32.0%	84	0.0172	0.0092	-0.0081	-88.1%
15	0.5400	0.4081	-0.1319	-32.3%	85	0.0168	0.0088	-0.0080	-90.1%
16	0.4746	0.3579	-0.1168	-32.6%	86	0.0164	0.0085	-0.0079	-92.2%
17	0.4204	0.3163	-0.1042	-32.9%	87	0.0161	0.0083	-0.0078	-94.3%
18	0.3750	0.2815	-0.0935	-33.2%	88	0.0157	0.0080	-0.0077	-96.5%
19	0.3366	0.2521	-0.0845	-33.5%	89	0.0153	0.0077	-0.0076	-98.8%
20	0.3038	0.2270	-0.0768	-33.8%	90	0.0150	0.0075	-0.0075	-101.2%
21	0.2755	0.2054	-0.0701	-34.1%	91	0.0147	0.0072	-0.0075	-103.7%
22	0.2510	0.1867	-0.0643	-34.5%	92	0.0144	0.0070	-0.0074	-106.3%
23	0.2297	0.1704	-0.0593	-34.8%	93	0.0140	0.0067	-0.0073	-109.0%
24	0.2109	0.1561	-0.0548	-35.1%	94	0.0138	0.0065	-0.0073	-111.8%
25	0.1944	0.1436	-0.0509	-35.4%	95	0.0135	0.0063	-0.0072	-114.7%
26	0.1797	0.1324	-0.0473	-35.8%	96	0.0132	0.0061	-0.0071	-117.7%
27	0.1667	0.1225	-0.0442	-36.1%	97	0.0129	0.0058	-0.0071	-120.9%
28	0.1550	0.1136	-0.0414	-36.5%	98	0.0127	0.0056	-0.0070	-124.2%
29	0.1445	0.1056	-0.0389	-36.8%	99	0.0124	0.0054	-0.0070	-127.7%
30	0.1350	0.0984	-0.0366	-37.2%	100	0.0122	0.0053	-0.0069	-131.3%
31	0.1264	0.0919	-0.0346	-37.6%	101	0.0119	0.0051	-0.0068	-135.1%
32	0.1187	0.0860	-0.0327	-38.0%	102	0.0117	0.0049	-0.0068	-139.1%
33	0.1116	0.0806	-0.0310	-38.4%	103	0.0115	0.0047	-0.0067	-143.3%
34	0.1051	0.0757	-0.0294	-38.8%	104	0.0112	0.0045	-0.0067	-147.7%
35	0.0992	0.0712	-0.0280	-39.3%	105	0.0110	0.0044	-0.0067	-152.3%
36	0.0938	0.0671	-0.0267	-39.7%	106	0.0108	0.0042	-0.0066	-157.2%
37	0.0888	0.0633	-0.0254	-40.2%	107	0.0106	0.0040	-0.0066	-162.4%
38	0.0841	0.0598	-0.0243	-40.7%	108	0.0104	0.0039	-0.0065	-167.9%
39	0.0799	0.0566	-0.0233	-41.2%	109	0.0102	0.0037	-0.0065	-173.7%
40	0.0759	0.0536	-0.0223	-41.7%	110	0.0100	0.0036	-0.0065	-179.8%
41	0.0723	0.0508	-0.0214	-42.2%	111	0.0099	0.0034	-0.0064	-186.3%
42	0.0689	0.0483	-0.0206	-42.7%	112	0.0097	0.0033	-0.0064	-193.3%
43	0.0657	0.0459	-0.0199	-43.3%	113	0.0095	0.0032	-0.0064	-200.8%
44	0.0628	0.0436	-0.0191	-43.9%	114	0.0093	0.0030	-0.0063	-208.8%
45	0.0600	0.0415	-0.0185	-44.4%	115	0.0092	0.0029	-0.0063	-217.3%
46	0.0574	0.0396	-0.0178	-45.1%	116	0.0090	0.0028	-0.0063	-226.6%
47	0.0550	0.0378	-0.0172	-45.7%	117	0.0089	0.0026	-0.0062	-236.6%
48	0.0527	0.0360	-0.0167	-46.3%	118	0.0087	0.0025	-0.0062	-247.4%
49	0.0506	0.0344	-0.0162	-47.0%	119	0.0086	0.0024	-0.0062	-259.2%
50	0.0486	0.0329	-0.0157	-47.7%	120	0.0084	0.0023	-0.0062	-272.1%
51	0.0467	0.0315	-0.0152	-48.4%	121	0.0083	0.0021	-0.0062	-286.3%
52	0.0449	0.0301	-0.0148	-49.1%	122	0.0082	0.0020	-0.0061	-302.0%
53	0.0433	0.0289	-0.0144	-49.8%	123	0.0080	0.0019	-0.0061	-319.5%
54	0.0417	0.0277	-0.0140	-50.6%	124	0.0079	0.0018	-0.0061	-339.1%
55	0.0402	0.0265	-0.0136	-51.4%	125	0.0078	0.0017	-0.0061	-361.3%
56	0.0387	0.0255	-0.0133	-52.2%	126	0.0077	0.0016	-0.0061	-386.6%
57	0.0374	0.0244	-0.0130	-53.0%	127	0.0075	0.0015	-0.0061	-415.8%
58	0.0361	0.0235	-0.0126	-53.8%	128	0.0074	0.0013	-0.0061	-450.0%
59	0.0349	0.0226	-0.0123	-54.7%	129	0.0073	0.0012	-0.0061	-490.7%
60	0.0338	0.0217	-0.0121	-55.6%	130	0.0072	0.0011	-0.0061	-540.2%
61	0.0327	0.0209	-0.0118	-56.6%	131	0.0071	0.0010	-0.0061	-602.1%
62	0.0316	0.0201	-0.0115	-57.5%	132	0.0070	0.0009	-0.0061	-682.3%
63	0.0306	0.0193	-0.0113	-58.5%	133	0.0069	0.0008	-0.0061	-791.8%
64	0.0297	0.0186	-0.0111	-59.5%	134	0.0068	0.0006	-0.0061	-953.2%
65	0.0288	0.0179	-0.0108	-60.6%	135	0.0067	0.0005	-0.0062	-1224.9%
66	0.0279	0.0173	-0.0106	-61.7%	136	0.0066	0.0003	-0.0062	-1835.0%
67	0.0271	0.0166	-0.0104	-62.8%	137	0.0065	0.0001	-0.0064	-11273.7%
68	0.0263	0.0160	-0.0102	-63.9%					
69	0.0255	0.0155	-0.0101	-65.1%					
70	0.0248	0.0149	-0.0099	-66.3%					

Table 4: The table shows the Rydberg formula predictions and the relativistic predictions for the first 137 elements. As we can see, the difference increases between the two models. Here we are just looking at the case $n_1 = 1$ and $n_2 = 2$.