On the zeros of the Riemann zeta function

Jorma Jormakka

Contact by: jorma.o.jormakka@gmail.com

Abstract. The paper proves the Riemann Hypothesis. In Lemma 1 the logarithmic derivative $\frac{d}{ds}\ln\zeta(s)$ of the Riemann zeta function is expanded as $f(s)=\sum_{j=1}^{\infty}h_j(s)$ where $h_j(s)=h_1(js)$. All $h_j(s)$ are continued analytically to $Re\{s\}>0$ by an inductive procedure. The proof of this introductory lemma explains the connection between the zeros of zeta and poles of $f(s)$. The proof of Lemma 2 shows that every nonzero coefficient in the Taylor series of the sum of the poles of $f(s)=\sum_{j=1}^{\infty}h_j(s)$ at $(l,0)$ must decrease at least as $O(x)$ as a function of $x=l^{-1}$ in order for every nonzero coefficient of the Taylor series of $f(s)$ to decrease at least as fast as a negative exponential as a function of x when $l\to\infty$. The proof of Lemma 2 shows that this happens if and only if every zero s_k of zeta in $0<Re\{s\}<0$ fulfills $Re\{s_k\}=\frac{1}{2}$. If every $Re\{s_k\}=\frac{1}{2}$, then the contributions of poles corresponding to the trivial zeros of zeta in $Re\{s\}<0$ and to the pole of zeta at $s=1$ cancel the pole pairs coming from the zeros of zeta in $0<Re\{s\}<1$ for every power $i>1$ of x. Cancellation of the coefficient of the power $i=1$ of x requires special attention.

Key words: Riemann zeta function, Riemann Hypothesis, complex analysis.

1 Definitions

The Riemann zeta function is defined by

$$\zeta(s)=\sum_{n=1}^{\infty}n^{-s} \tag{1}$$

where s is a complex number. The zeta function can be continued analytically to the whole complex plane except for $s = 1$ where the function has a simple pole. The zeta function has trivial zeros at even negative integers. It does not have zeros in $Re\{s\} \geq 1$. The nontrivial zeros lie in the strip $0 < x < 1$, see e.g. [1]. The Riemann Hypothesis claims that nontrivial zeros have $Re\{s\} = \frac{1}{2}$. Let

$$P = \{p_1, p_2, \ldots | p_j \text{ is a prime}, p_{j+1} > p_j > 1, j \geq 1\}$$

be the set of all primes (larger than one). Let $s = x + iy$, $x, y \in \mathbb{R}$ and $x > \frac{1}{2}$. The Riemann zeta function can be expressed as

$$\zeta(s) = \prod_{j=1}^{\infty} (1 - p_j^{-s})^{-1}, \tag{2}$$

This infinite product converges absolutely if $Re\{s\} > 1$.

2 Lemmas and the theorem

Lemma 1. *The functions*

$$h_j(s) = -\sum_{j=1}^{\infty} \ln(p_j) p_j^{-js} \quad , \quad j > 0 \tag{3}$$

are related by $h_j(s) = h_1(js)$. The functions $h_j(s)$ have analytic continuations to $Re\{s\} > 0$ with the exception of isolated first-order poles. The poles of $h_j(s)$ that are not on the x-axis appear in pole pairs: close to s_k, where $Im\{s_k\} > 0$, $h_j(s)$ is of the type

$$h_j(s) = \frac{r}{s - s_k} + f_1(s) \tag{4}$$

and close to s_k^*, where s_k^* is a complex conjugate of s_k, $h_J(s)$ is of the type

$$h_j(s) = \frac{r}{s - s_k^*} + f_2(s)$$

The functions $f_1(s)$ and $f_2(s)$ are analytic close to s_k and s_k^* respectively. If the pole is at the x-axis, there is only one pole of the type (4) with $Im\{s_k\} = 0$.

Proof. The claim

$$h_j(s) = h_1(js) \tag{5}$$

follows directly from (3).

The function $h_1(s)$ converges absolutely if $Re\{s\} > 1$ because

$$\sum_{j=1}^{\infty} p_j^{-s}$$

converges absolutely for $Re\{s\} > 1$ and $|\ln p_j| < |p_j^{\alpha}|$ for any fixed $\alpha > 0$ if j is sufficiently large. Therefore

$$|\ln(p_j)p_j^{-s}| < 2|p_j^{-s+\alpha}|$$

for any fixed $\alpha > 0$ if j is sufficiently large. Therefore, by (5), $h_j(s)$ converges absolutely if $Re\{s\} > \frac{1}{j}$.

From (2) follows

$$\zeta'(s)\zeta(s)^{-1} = \frac{d}{ds}\ln\zeta(s) = \sum_{j=1}^{\infty} h_j(s).$$

The derivative $\zeta'(s)$ is analytic in all points except for $s = 1$. The function $h_1(s)$ is continued analytically to $Re\{s\} > \frac{1}{2}$ by

$$h_1(s) = \zeta(s)^{-1}\zeta'(s) - g(s) \tag{6}$$

where

$$g(s) = \sum_{j=2}^{\infty} h_j(s).$$

The function $\zeta(s)^{-1}$ is analytic except for at points where $\zeta(s)$ has a zero or a pole. The function $g(s)$ is analytic for $Re\{s\} > \frac{1}{2}$ because each $h_j(s)$, $j > 1$, is analytic in $Re\{s\} > \frac{1}{j}$. Thus, the right side of (6) is defined and analytic for $\frac{1}{2} < Re\{s\}$ except for at points where $\zeta(s)$ has a zero or a pole. At those isolated points $h_1(s)$ has a pole.

At a pole s_k of $\zeta(s)$ the zeta function has the expansion

$$\zeta(s) = \frac{C}{(s - s_k)^k} + \text{higher order terms.}$$

If $Re\{s\} > \frac{1}{2}$ the function $h_1(s)$ is of the form

$$h_1(s) = \zeta'(s)\zeta(s)^{-1} - g(s) = \frac{r}{s - s_k} + f_1(s)$$

where $f_1(s)$ is analytic close to s_k and $r = -k < 0$ is an integer. The function $\zeta(s)$ has only one pole, at $s_k = 1 = (1, 0)$, and it is a simple pole, thus $r = -1$.

At a zero s_k of $\zeta(s)$ the zeta function has the expansion

$$\zeta(s) = C(s - s_k)^k + \text{higher order terms.}$$

If $Re\{s\} > \frac{1}{2}$ the function

$$h_1(s) = \zeta'(s)\zeta(s)^{-1} - g(s) = \frac{r}{s - s_k} + f_1(s)$$

where $f_1(s)$ is analytic close to s_k and $r = k > 0$ is an integer. It is known that $\zeta(s)$ has many zeros with $Re\{s_k\} = 1/2$.

Thus, $h_1(s)$ has only first-order poles for $Re\{s\} > \frac{1}{2}$ and therefore $h_j(s)$ has only first-order poles for $Re\{s\} > \frac{1}{2j}$. At every pole of $h_1(s)$ in $Re\{s\} > \frac{1}{2}$ the value of r is an integer.

As $h_1(s)$ is continued to $Re\{s\} > \frac{1}{2}$ by (6), the equation (5) continues $h_j(s)$ to $Re\{s\} > \frac{1}{2j}$. Then (6) continues $h_1(s)$ to $Re\{s\} > \frac{1}{4}$. The function $h_1(s)$ has isolated poles at $Re\{s\} > \frac{1}{4}$. Each pole is a first-order pole, but the value of r at a pole does not need to be an integer.

We can repeat the procedure inductively: If $h_1(s)$ is continued to $Re\{s\} > \frac{1}{2^i}$ by (6), the equation (5) continues $h_j(s)$ to $Re\{s\} > \frac{1}{2^i j}$. Then (6) continues $h_1(s)$ to $Re\{s\} > \frac{1}{2^{i+1}}$. By induction, all $h_j(s)$ are analytically continued to $Re\{s\} > 0$.

In this inductive process $h_1(s)$ gets isolated first-order poles. In these poles s_k the values $r = r_k$ can be positive or negative, and they do not need to be integers. If $h_1(s)$ has a pole

$$h_1(s) = \frac{r}{s - s_k} + f_1(s)$$

(here $f_1(s)$ is analytic close to s_k), then $h_j(s) = h_1(js)$ has a pole at $j^{-1}s_k$ and the r value is $j^{-1}r$ since

$$h_j(s) = h_1(js) = \frac{j^{-1}r}{s - j^{-1}s_k} + f_1(js).$$

The function $h_1(s)$ is symmetric with respect to the real axis. By (4) $h_j(s)$, $j > 1$, is also symmetric with respect to the real axis. Therefore poles of each $h_j(s)$, $j > 0$, appear as pairs s_k and s_k^*. In the special case where s_k is real there is only one pole, not a pair. □

Lemma 2. *All poles s_k of $\sum_{j=1}^{\infty} h_j(s)$ in $Re\}s\} > 0$ satisfy $Re\{s_k\} = \frac{1}{2}$ or $s_k = 1$.*

Proof. Let us consider a function $f(s)$ that has a first-order pole at s_0 and write $z_1 = s - s_0$. The function $f(s)$ does not have a Taylor series at s_0, but the function $z_1 f(z1 + s_0)$ has a Taylor series at $z_1 = 0$ and $f(s)$ can be expressed as

$$f(s) = \frac{c_{-1}}{z_1} + \sum_{k=0}^{\infty} c_k z_1^k. \tag{7}$$

Let us evaluate $f(s)$ at another point at $s_0 + l$, $l > 0$, by first writing $z_1 = l - z_2$ where $|z_1| << 1$, inserting $z_1 = l - z_2$ to the series expression of $f(s)$, and then considering the result when $|z_2| << 1$. The function

$$f_1(z_1) = f(z_1 + s_0) - \frac{c_{-1}}{z_1} \tag{8}$$

has the Taylor series at $z_1 = l - z_2$ where $|z_1| << 1$ as

$$f_1(l - z_2) = \sum_{m=0}^{\infty} c_m (l - z_2)^m$$

$$= \sum_{m=0}^{\infty} \sum_{i=0}^{m} \frac{m!}{i!(m-i)!} l^i (-z_2)^{m-i} c_m$$

$$= \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{(k+i)!}{i!k!} l^i (-1)^k c_{k+i} z_2^k = \sum_{k=0}^{\infty} b_k z_2^k.$$

Thus

$$b_k = \sum_{i=0}^{\infty} \frac{(k+i)!}{i!k!} l^i (-1)^k c_{k+i}.$$

As

$$c_k = \frac{1}{k!} \frac{d^k}{dz_1^k} f_1(s)|_{z_1=0}$$

we can express

$$b_k = \left(\sum_{i=0}^{\infty} \frac{1}{i!} l^i \frac{d^i}{dz_1^i} \right) \frac{1}{k!} (-1)^k \frac{d^k}{dz_1^k} f_1(s)|_{z_1=0}. \tag{9}$$

If there is no pole of $f(s)$ at $s_0 + l$, the function

$$f_1(l - z_2) = \sum_{k=0}^{\infty} b_k z_2^k$$

is analytic and defined by its Taylor series as powers of z_2 where the series converges. The pole of $f(s)$ at c_{-1} can be evaluated as a Taylor series of z_2 at $s_0 + l$ as

$$\frac{c_{-1}}{l - z_2} = \frac{c_{-1}}{l} \frac{1}{1 - z_2 l^{-1}} = \frac{c_{-1}}{l} \sum_{k=0}^{\infty} \left(\frac{z_2}{l} \right)^k.$$

We can subtract a set of first-order poles of $f(s)$ in points $s_j \in A$ and define

$$f_1(z_1) = f(s) - \sum_{j \in A} \frac{r_j}{s - s_j} \tag{11}$$

where $r_j = c_{-1,j}$ and express

$$s - s_k = (s - s_0) - (s_j - s_0) = z_1 - s_j + s_0 = l - z_2 - (s_j - s_0).$$

At the point $s_0 + l$ the set of poles is

$$\sum_{j \in A} \frac{r_j}{s - s_j} = \sum_{j \in A} \frac{r_j}{l - z_2 - (s - s_0)} = x \sum_{j \in A} \frac{r_j}{1 - p_j x} \tag{12}$$

where $p_j = s_j - s_0$ and $x = (l - z_2)^{-1}$. Let us select $s_0 = 0$ for easier notations. Thus, $p_j = Re\{s_j\}$ is the x-coordinate of the pole s_j. Let us consider

$$f(s) = \sum_{k=1}^{k_1} \frac{r_k}{s - s_k} + f_1(s) \tag{13}$$

$$f_1(s) = - \sum_{j=1}^{j_{\max}} \ln(p_j) p_j^{-s}.$$

Let $l >> 1$. The Taylor series of the set of poles points s_k at s_0 in powers of z_1 is

$$- \sum_{i=0}^{\infty} \left(\sum_{k=1}^{k_1} r_k (s_k - s_0)^{-i-1} \right) z_1^i$$

and the Taylor series at $s_0 + l$ in powers of $z_2 = l - z_1$ is

$$\sum_{i=0}^{\infty} \left(\sum_{k=1}^{k_1} r_k (s_0 + l - s_k)^{-i-1} \right) z_2^i.$$

For each k the coefficient of the ith power of z_1 at s_0 is $c_i = r_k(s_k - s_0)^{-i-1}$ while the coefficient of z_2 at $s_0 + l$ is

$$p_i = r_k(s_0 + l - s_k)^{-i-1} = r_k l^{-i-1} + r_k(i+1)(s_k - s_0)l^{-i-2} + \cdots$$

The absolute value of the coefficient p_i of the Taylor series in powers of z_2 at $s_0 + l$ decreases as

$$|\sum_{k=1}^{k_1} r_k| l^{-i-1}$$

as a function of $l >> 1$. The part $f_1(s)$ of $f(s)$ satisfies

$$|f_1(s+l)| = \left| -\sum_{j=1}^{j_{\max}} \ln(p_j) p_j^{-s-l} \right| = \left| -\sum_{j=1}^{j_{\max}} \ln(p_j) p_j^{-s} e^{-l \ln p_j} \right|$$

$$\leq \left| e^{-l \ln 2} \right| \left| -\sum_{j=1}^{j_{\max}} \ln(p_j) p_j^{-s} \right| = e^{-l \ln 2} |f_1(s)|. \tag{13}$$

The absolute value of the coefficient p_i of the Taylor series in powers of z_2 at $s_0 + l$ decreases as

$$|p_i| \leq e^{-l \ln 2} |c_i|.$$

This is negative exponential decrease and much faster than the hyperbolic decrease for the set of poles.

When $l \to \infty$, the contribution from the poles must totally vanish: every nonzero coefficient of the Taylor series of $f(s)$ at $(l, 0)$ when $l \to \infty$ must decrease as a negative exponential of $x = l^{-1}$. The exponent of x grows faster than any power of x, thus the negative exponent of x decreases faster than any negative power of x. For each power of x the coefficient in the power series of the sum of the poles as a function of x must vanish. The coefficient of the power of x from the sum of the poles must go to zero at least as $O(x)$ leaving the negatively exponentially decreasing coefficient from $f_1(s)$ in (11) to dominate.

The sum of the poles clearly decreases as $O(x)$, $x = l^{-1}$, and goes to zero when $x \to 0$ when the x-coordinate of every pole of $f(s)$ is smaller or equal to one, but this kind of convergence to zero is not a sufficient condition for the contribution of the poles to vanish and to leave the contribution of the negative exponential behaviour of $f_1(s)$ to dominate at the limit $l \to \infty$. The condition that the coefficient of a power i of x the sum of poles decreases at least as $O(s)$ means that that the poles of $f(s)$ partially cancel. Poles cannot completely cancel: a pole at s_k with $r = r_k$ can be completely cancelled only by a pole at s_k with $r = -r_k$. The sum of poles has all poles of its terms, but at $l >> 1$ there can be partial cancellation so that the Taylor series coefficients decrease fast as a function of l. This kind of cancellation means that the powers of x separately go to zero. It is a much stronger condition than that the sum of the poles goes to zero when $x \to 0$.

Let us $k_1 \to \infty$ in (13). Then $f(s) = h_1(s)$. If $Re\{s\} = l >> 1$, we are far away of the pole at $s = 1$ and the sum in $h_1(s)$, where k_1 is replaced by infinity, converges absolutely. The absolute values of the Taylor series at $s_0 + l$ for the function $h_1(s)$ must decrease in negative exponential manner as a function of l. The function $h_1(s)$ has the behaviour of the sum of negatively exponential terms when l is very large. It follows that every $h_j(s) = h_1(js)$ also has the behaviour of the sum of negatively exponential terms when l is very large. Consequently the sum of the poles of every $h_j(s) = h_1(js)$ also has the behaviour of the sum of negatively exponential terms when l is very large. Therefore the sum of the poles of

$$f(s) = \frac{d}{ds} \ln \zeta(s) = \sum_{j=1}^{\infty} h_j(s)$$

must vanish in the limit $l \to \infty$. We did not continue $h_j(s)$ to the area $Re\{s\} \le 0$ in Lemma 1, but the function $f(s)$ is analytically continued to $Re\{s\} \le 0$ by

$$f(s) = \frac{d}{ds} \ln \zeta(s)$$

to all points where $\zeta(s) \neq 0$ and we can find all poles of $f(s)$.

The function $f(s)$ has the following poles in $Re\{s\} > 0$:

(i) There is one pole with $r = -1$ at $s = 1$.

(ii) There is a set A of pole pairs of $h_1(s)$ at s_k and s_k^* where s_k has a nonzero imaginary part, and the r-value r_k is positive. All we know of s_k is that the real part of s_k is larger than zero and smaller than one, and that that there are poles s_k with the real part $\frac{1}{2}$.

(iii) There may be a set A_1 of poles $s_{k,1}$ of $h_1(s)$ with $r_{k,1}$ a positive integer and the pole s_k is real, $0 < s_k < 1$. No such pole is known.

The zeros of $\zeta(s)$ in the area $Re\{s\} \leq 0$ are the so called trivial zeros at even negative integers. They come from the formula

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$$

where $B_m = 0$ if $m > 1$ is odd. Zeta does not have a zero at $s = 0$. From the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin(2^{-1}\pi s) \Gamma(1-s) \zeta(1-s) \tag{14}$$

we can deduce that the trivial zeros are zeros of $\sin(2^{-1}\pi s)$ and therefore first-order zeros. Thus, at a point $s_k = -2k$, $k > 0$, the function $f(s)$ has a first-order pole with the r-value 1.

Using the expression (12) instead of (4) for a pole or a pole pair (i.e., $s = s_0 + l$, $s_0 = 0$, $x = l^{-1}$) gives

$$\frac{r_k}{s - s_k} = \frac{x r_k}{1 - p_k x}$$

as a pole on the x-axis. We do not include the analytic function part in (4) but take only the pole at s_k. A pole pair in the positive and negative y-axis can be written as

$$\frac{r_k}{s - s_k} = \frac{x r_k}{1 - (1 + i\alpha_k) p_k x}$$

$$\frac{r}{s-s_k^*}=\frac{xr_k}{1-(1-i\alpha_k)p_kx}.$$

Here $x=(l-z_2)^{-1}>0$ is a small real number if l is large, $p_k=Re\{s_k\}$ and α_k is chosen positive. The number l is the distance from $s_0=0$ to the observation point on the x-axis, $(l,0)$, where the Taylor series with z_2 is evaluated and $|z_2|<<1$. As z_2 is the variable of the Taylor series at $(l,0)$, the expressions are valid for any small z_2 and we select $z_2=0$ for easier notations. Thus, $x=l^{-1}$.

The pole (i) at $s=1$ gives the power series of x where $p_k=1$ and $r=-1$

$$\frac{xr}{1-(p_kx)}=\frac{-x}{1-x}=-x\sum_{m=0}^{\infty}x^m.$$

A pole at $s_k=-2k$, $k>0$, is

$$\frac{r_k}{s-s_k}=\frac{1}{s+2k}.$$

We can evaluate the Taylor series of z_1 at s_0 and the Taylor series of z_2 at s_0+l for any such pole and for a finite sum of such poles:

$$\frac{1}{s_0+z_1+2k}=\frac{1}{s_0+2k}\sum_{i=0}^{\infty}(-1)^i(s_0+2k))^{-1}z_1^i$$

$$\frac{1}{s_0+l-z_2+2k}=\frac{1}{s_0+l+2k}\sum_{i=0}^{\infty}(s_0+l+2k))^{-1}z_2^i$$

but if sum the index k goes to infinity, the series diverges at every finite point s_0+l. We will evaluate the sum of these poles at $s_0=0$, conclude that the contribution is negative, look what happens if the sum of all these poles is evaluated at s_0+l when $l\to\infty$, and finally present a way to move a finite but growing sum of these poles to s_0+l.

First we find out the sign of the infinity of the sum of the poles $s_k=-2k$ at $s_0=0$ and $z_1=0$. Notice that for a point $s_j=-k$ the pole at that point, with the r-value r, when evaluated to a Taylor series at $s_0 = 0$ and $z_1 = 0$ is

$$\frac{r}{s - s_j} = \frac{r}{k}.$$

This is the inverse of a pole with the same r but with $s_j = k$ when evaluated to a Taylor series at $s_0 = 0$ and $z_1 = 0$. As an example, $s_j = 1$ is the pole at $s = 1$ with $r = -1$. When evaluated at $s_0 = z_1 = 0$ it is the inverse of a pole with $r = -1$ but $s_k = -1$. Thus, the pole at $s_k = -2k$ with $r = 1 > 0$ is the same at $s_0 = 0$ as a a pole at $s_k = 2k$ with $r = -1 < 0$. We see that the sum of all poles $s_k = -2k$ gives a negative infinity when evaluated at $s_0 = 0$.

The type of infinity of the sum of the poles $s = -2k$ at $s_0 = 0$ can be calculated. Using the facts that $\zeta(s)$ has a simple pole at $s = 1$

$$\zeta(s) = \frac{a}{s-1} + g(s)$$

where $g(s)$ is analytic at $s = 1$ and that $\lim_{s\to 1}(s-1)\zeta(s) = 1$, so $a = 1$, we can write

$$\zeta(1) = \lim_{s\to 1} \frac{1 + (s-1)f(1)}{s-1} = \lim_{s\to 1} \frac{1}{s-1} = \lim_{s\to 0} \frac{1}{s}$$

This result gives

$$\sum_{k=1}^{\infty} \frac{1}{2k} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k^1} = \frac{1}{2}\zeta(1) = \lim_{s\to 0} \frac{1}{2}\frac{1}{s}.$$

Thus, the sum of the poles at $s_k = -2k$ appears as a simple pole when evaluated at $s_0 = 0$. The pole has a negative r-value with $r = -1$ at $s_0 = 0$. However, it is not a simple pole. A simple pole with $r = -1/2 \lim_{s\to s_0}(-1/2)1/(s - s_0)$ is moved to $s_0 + l$ by writing

$$\lim_{s\to s_0} (-1/2)1/(s - l - s_0) = (-1/2)/l = -x/2$$

where $x = l^{-1}$. Then the pole is finite for every $l > 0$, but the sum of the poles $s_k = -2k$ is infinite at every finite l. This is because the infinity $\lim_{s\to s_0}(-1/2)/(s-s_0)$

is not caused by the pole being physically at s_0, the infinity comes from the sum of the poles. Therefore the infinity stays for every $l > 0$.

Let us calculate the contribution from all poles $s_k = -2k$, $k > 1$ at $s_0 + l$:

$$S_1 = \sum_{k=1}^{N} \frac{1}{2k+l} = \frac{1}{2}\frac{1}{N}\sum_{k=1}^{N} \frac{1}{\frac{k}{N} + \frac{l}{2N}}.$$

A sum converges to an integral as

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(k/N) = \int_0^1 f(y)dy.$$

Assuming that $c = N/l$ is constant, the limit S_1 is

$$S_1 = \frac{1}{2}\int_0^1 \frac{dy}{y + \frac{l}{2}c^{-1}}dy = \frac{1}{2}\ln(1+2c).$$

The contribution from these poles at $s_0 + l$ must be finite so that the the contribution of the sum of all poles vanish. Especially, the contribution must tend to zero as $O(x)$ when $x \to 0$ since the contributions from other poles have this behaviour. This implies that $|2c|$ must be smaller than 1. Then we can expand $\ln(1+2c)$ into a series:

$$S_1 = \sum_{j=1}^{\infty}(-1)^j 2^{j-1}\frac{1}{j}c^j.$$

$$= c - c^2 - \frac{4}{3}c^3 + 2c^4 - \cdots.$$

The condition $|2c| < 1$ implies that l must go to infinity faster than N. Let us select α, $1 > \alpha > 0$, and set $N = l^\alpha$. Then $c = l^{\alpha-1}$ is small for large l and the contribution of the poles $s_k = -2k$ at $s_0 + l = l$, $l \to \infty$, is

$$-x^\alpha + x^{2\alpha} - \frac{4}{3}x^{3\alpha} + 2x^{4\beta} + \cdots.$$

We notice that this contribution is an alternating series of noninteger powers of x. It is not of the correct form for it to cancel the coefficients of a power series of x

created by other poles. Additionally, the power α depends on the relation chosen (without any justification) between N and l. We see that this way is not correct.

There is another problem in adding all poles $s_k = -2k$. The discussion of Taylor functions in the beginning of the proof of Lemma 2 only considered sums of simple poles that can be moved to $s_0 + l$ and are finite when moved. It did not investigate moving a pole of the type created by all poles $s_k = -2k$. This sum is infinite for every finite l. If we subtract all these poles from $f(s)$, then $f_1(s)$is infinite at every point, which would invalidate the method.

This is why only a finite sum of these poles can be subtraced at any given l and when l grows, we can subtract more poles until all are subtracted when $l \to \infty$. The choice of which sums of poles are subtracted for each l cannot influence the result. We will make a convenient choice for these sums. We take a sum of poles $s_k = -2k$ up to $k = N(l)$ and choose a suitable growing function as $N(l)$. A finite sum up to $N(l)$ can be moved to $s_0 + l$ and when $N(l)$ increases with l, all poles $-2k$ are included in the finite sum when $k \leq N(l)$. The tail of the infinite sum that is not in the finite sum up to $N(l)$ goes to zero when $l \to \infty$.

Thus, we take a finite sum

$$\sum_{k=1}^{N(l)} \frac{1}{2k}.$$

As it is a finite sum, it can be moved without creating an infinity. If $N(l)$ is sufficiently large and fixed, and $l = 0$, the sum moves as $-x/2 - \epsilon(l)$. The number $\epsilon(l)$ depends only on $N(l)$ and we can select a function $N(l)$ such that $\epsilon < e^{-l}$, i.e., $\epsilon(l)$ decreases with l faster than ay power of $x = l^{-l}$. The number $N(l)$ increases when l grows, and then the absolute value of the sum grows with l. It gives a function $-xC(l) - \epsilon(l)$. The effect of the sum of all poles must vanish at $l \to \infty$. In the limit $x \to 0$ the function $xC(l)$ must be of the order $O(x)$ because all other poles give contributions of $O(x)$, and as $C(l) \geq 1/2$ is a growing function, the function $-xC(l) - \epsilon$ must converge to $-xC$, where $C > 1/2$ is a finite real number. The number ϵ goes to zero, as it decreases faster than any power of x. We have managed to move the poles $-2k$ to s_0+l. The number C will be determined later in this proof.

The poles (iii) of A_1 sum to a series $x\sum_{m=0}^{\infty} c_m x^m$ where every c_m is nonnegative and $c_{i+1}\neq c_i$ in the limit when $x\to 0$ because all of these poles are in the area $0<s_k<1$ and they are isolated and therefore do not have a concentration point at $s=1$. It follows that they cannot be cancelled when $x\to 0$ by the the sum of poles in $Re\{s\}\leq 0$ giving the contribution $-xC$) and the pole at $s=1$ giving the contribution $-x/(1-x)$. Therefore the poles (iii) could only be cancelled by a set of poles of the type (ii), but the poles of (ii) also yield a power series of x where the coefficient of every x^i is nonnegative. Thus, the poles of A_1 cannot be cancelled in $l\to\infty$ by any set of other poles and therefore the set A_1 must be empty.

For a sum of pole pairs in (ii) the coefficient of the power one of x can be cancelled by sum of the corresponding coefficient -1 of the pole at $s=1$ and the coefficient $-C$ coming from the poles in $Re\{s_k\}\leq 0$. Only the pole at $s=1$ can cancel the higher than power one coefficients of x coming from a sum of pole pairs (ii). Thus, the coefficient of each power $i>1$ of x in the sum of pole pairs (ii) must be cancelled by the corresponding coefficient of x in the pole (i) at least to the degree of $O(x)$.

The two poles (ii) of a pole pair have a real sum:

$$\frac{xr_k}{1-p_k(1+i\alpha_k)x}+\frac{xr_k}{1-p_k(1-i\alpha_k)x}=xr_k\frac{2(1-p_kx)}{1-2p_kx+(1+\alpha_k^2)(p_kx)^2}.$$

We expand the sum S of the poles of a pole pair omitting the multiplier xr_k for simplicity in this calculation up to (16):

$$S=\frac{2(1-p_kx)}{1-2p_kx+\alpha_k^2(p_kx)^2}=\frac{2-2p_kx}{1+\alpha_k^2(p_kx)^2}\frac{1}{1-2p_kx\gamma_k^{-1}}$$

where $\gamma_k = 1 + \alpha_k^2 (p_k x)^2$.

$$= \frac{2 - 2p_k x}{\gamma_k} \sum_{i=0}^{\infty} (2p_k x \gamma_k^{-1})^i.$$

Writing $\beta_{k,i} = (2p_k)^i \gamma_k^{-i-1}$ we get

$$S = 2\sum_{i=0}^{\infty} \beta_{k,i} x^i - 2p_k \sum_{i=0}^{\infty} \beta_{k,i} x^{i+1} = \sum_{i=0}^{\infty} 2\beta_{k,i} x^i - 2p_k \sum_{i=1}^{\infty} \beta_{k,i-1} x^i$$

$$= 2\beta_{k,0} + \sum_{i=1}^{\infty} (2\beta_{k,i} - 2p_k \beta_{k,i-1}) x^i.$$

For $i > 0$

$$2\beta_i - 2p_k \beta_{k,i-1} = 2\frac{(2p_k)^{i-1}}{\gamma_k^i} (2p_k \gamma_k^{-1} - p_k)$$

$$= \frac{(2p_k)^i}{\gamma_k^{i+1}} (2 - \gamma_k) = \beta_{k,i}(2 - \gamma_k).$$

This gives an equation for every $i > 0$

$$2\beta_i - 2p_k \beta_{k,i-1} = 2\beta_{k,i} - \gamma_k \beta_{k,i}.$$

Inserting $\gamma_k = 1 + (\alpha_k p_k x)$ yields for $i > 0$

$$2p_k \beta_{k,i-1} = \gamma_k \beta_{k,i} = \beta_{k,i} + x^2 (\alpha_k p_k)^2 \beta_{k,i}.$$

For every k when $l >> 1$ and therefore $0 < x = l^{-1} << 1$ and $i > 0$ holds

$$2p_k \beta_{k,i-1} = \gamma_k \beta_{k,i} = \beta_{k,i} + O(x^2).$$

The coefficient of the the power x^i, $i > 0$, is

$$2\beta_{k,i} - 2p_k \beta_{k,i-1} = \beta_{k,i} + O(x^2). \tag{16}$$

The coefficient of the power of x^{i+1} in for the pole in $s = 1$ (i.e., in the power series of $x/(1-x)$) is -1 for every $i > 0$. The coefficient of x^{i+1} in the sum of poles (ii) is

$$\sum_{k\in A} r_k(2\beta_{k,i} - 2p_k\beta_{k,i-1})$$

where we have included the multiplier xr_k that was so far omitted. Summing the powers of i from $i = 2$ to $i = i_1 + 1$ and inserting (16) gives the equation where the coefficients of the pole pairs (ii) cancel the coefficients of the pole (i) to the degree of $O(x^2)$:

$$-i_1 = \sum_{i=2}^{i_1+1}(-1) = \sum_{i=2}^{i_1+1}\sum_{k\in A} r_k\beta_{k,i} + O(x^2). \tag{17}$$

For each k, when $x \to 0$ and $i > 0$, holds

$$\beta_{k,i} = 2p_k\beta_{k,i-1}. \tag{18}$$

If every $p_k = \frac{1}{2}$ the recursion equation (18) gives $\beta_{k,i+1} = \beta_{k,i}$ for every k. For every k the power series of x for $i > 1$ is of the form $x\beta_{k,1}(x + x^2 + x^3 + \cdots)$. This is the same form as the power series $x(x + x^2 + x^3 + \cdots)$ for the pole $s = 1$ for $i > 1$. The power series for the poles k for $i > 1$ add to one power series of the type $xb(x + x^2 + x^3 + \cdots)$. We see that if every $p_k = \frac{1}{2}$, the sum of poles (ii) cancel the pole in $s = 1$ when $x \to 0$ and converge to the series of the pole in $s = 1$ with the same $O(x^2)$ speed in every power x^i for $i > 1$.

Assume that one p_k is not $\frac{1}{2}$. The functional equation (14) shows that if there exists a zero $s_0 = x_0 + iy_0$ of $\zeta(s)$ with $0 < x_0 < \frac{1}{2}$ then there exists a zero of $\zeta(s)$ at a symmetric point in $\frac{1}{2} < x < 1$. This implies that we can find $s_{k'}$ such that $2p_{k'} > 1$. The form of (18) for a nonzero x, $i > 0$, is

$$\beta_{k,i} = 2p_k\beta_{k,i-1} + O(x^2). \tag{19}$$

For $p_{k'}$ the recursion (19) gives $\beta_{k',i} = \beta_{k',1}(2p_{k'})^i + O(x^2)$. From (17) we get (2). The right side in (20)

$$-i_1 = \sum_{i=2}^{i_1+1} \sum_{k \in A} r_k \beta_{k,i} + O(x^2) \tag{20}$$

grows at least as fast as $\beta_{k',1}(2p_{k'})^{i_1}$ as a function of i_1 while the left side is linear in i_1. This is a contradiction. Thus, every p_k must be $\frac{1}{2}$.

The claim of Lemma 2 is already proven in (20), but we look at the the sums

$$\beta_i = \sum_{k \in A} r_k \beta_{k,i} \tag{21}$$

to check if they can have the values they get from cancelling the pole at $s = 1$ and if the coefficients in the power series of x vanish sufficiently fast when $l \to \infty$.

By (20) each $p_k = \frac{1}{2}$. Inserting $p_k = 2^{-1}$ to (21) gives

$$\beta_i = \sum_{k \in A} r_k (1 + 2^{-2}(\alpha_k x)^2)^{-i-1}. \tag{22}$$

The recursion equation for $\beta_{k,j}$ is $\beta_{k,i} = (2p_k/\gamma_k)\beta_{k,i-1}$. As $2p_k = 1$ and since $\gamma_k \geq 1$, this implies that $\beta_{k,i-1} \geq \beta_{k,i}$ for all $i > 0$. Recursion (18) for $p_k = \frac{1}{2}$ shows that for every $i > 0$ the value $\beta_{k,i}$ is the same when $x \to 0$. Since $\gamma_k \to 1$ when $x \to 0$, β_i is the same for every $i \geq 0$. Equation (20) implies that $\beta_i = 1$ for every $i > 0$. In the limit $x \to 0$ holds $\beta_{k,0} = \beta_{k,1}$. Therefore also $\beta_i = 1$ when $x \to 0$.

Because $x \to 0$ the values of α_k must grow to infinity with k. The set A is necessarily infinite. We renumber the poles of (ii) so that (α_k) is a growing sequence and the sum $k \in A$ is the sum $k = 1$ to infinity.

Since $p_k = \frac{1}{2}$ by (20) we can evaluate

$$2\beta_{k,i} - 2p_k\beta_{k,i-1} = \beta_{k,i}(2 - \gamma_k)$$

and get

$$\beta_{k,i} = \beta_{k,i-1}\left(1 - \frac{(0.5\alpha_k x)^2}{1 + (0.5\alpha_k x)^2}\right).$$

Let $l >> 1$ be fixed. If $\alpha_k >> l = x^{-1}$, then

$$\frac{(0.5\alpha_k x)^2}{1 + (0.5\alpha_k x)^2}$$

is close to one and $\beta_{k,i}$ is close to zero. This means that large values of α_k contribute very little to the Taylor series at $s_0 + l$. The sum in (22) can be finite and there is no reason why it could not be one. The value of every β_i, $i \geq 0$, must be one because of (20).

The contributions of all poles must vanish in the limit $l \to \infty$. The contribution from the poles at $s_k = -2k$ is $-xC$, from the pole at $s = 1$ it is $-x/(1-x)$, from the poles of A_1 it is 0, and the contribution of the pole pairs of A is approaching the series $x(2+x+x^2+x^3+\cdots)$ when $x \to 0$ as $O(x^2)$ separately for the coefficient of each power i of x^i. The sum of these contributions when $l \to \infty$ is

$$-Cx - \frac{x}{1-x} + 0 + x(2 + x + x^2 + x^3 + \cdots) = (-3/2 + 2\beta_0)x = (1-C)x. \quad (24)$$

The sum (24) must be zero, thus $C = 1$.

The convergence of the coefficients of the powers of x to zero in (24) when x grows is $O(x^2)$ for the coefficient of each power $i > 1$ of x^i separately, which fulfills the convergence criterion.

The convergence criterium does not apply to the power one of x because when l grows new poles $-2k$ are added. Epsilon (see the text after (15)) converges as $O(e^{-l})$, therefore the sum to $N(l)$ closely approximates the sum to infinity. The sum at infinity must give $C = 1$ because of (24), but the convergence of the coefficient of the power one od x to zero is not fully clarified by these convergences. It is not necessary to check the convergence of the coefficient of the power one of

x: the proof of the claim of Lemma 2, i.e., that each $p_k = \frac{1}{2}$ for $s_k \in A$, is already in (20).

The proof of Lemma 2 is complete. □

Theorem 1. *The Riemann Hypothesis is true.*

Proof. By Lemma 2 every $p_k = Re\{s_k\} = \frac{1}{2}$ for poles of the type (ii) in the set of pole pairs A and the set A_1 is empty. □

Notice that the value $Re\{s_k\} = p_k = \frac{1}{2}$ comes because the recursion (19) must yield the same form of the power series of x as the pole at $s = 1$ when $x \to 0$ in order for the the set of pole pairs to cancel the pole at $s = 1$ in the limit $l \to \infty$. Equation (19) arises from the expansion of a pole pair. Thus, ultimately $Re\{s_k\} = \frac{1}{2}$ because pole pairs cancel a pole at $s = 1$ and therefore the real part of the poles in the pole pairs must be the half of one. □

All known facts of the Riemann zeta function that are used in this proof can be found in [1]. The history and background of the Riemann Hypothesis are well described in the book [2]. As the problem is still open, recently published results do not add so much to the topic and as they are not needed in this proof, they are not referred to.

References

1. E. T. Whittaker and G. N. Watson, A Course in Modern Analysis, Cambridge, University Press, 1952.
2. K. Sabbagh, The Riemann Hypothesis, the greatest unsolved problem in mathematics, Farrar, Strauss and Giroux, New York, 2002.