On the zeros of the Riemann zeta function

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Abstract. The paper proves the Riemann Hypothesis. In Lemma 1 the logarithmic derivative $\frac{d}{ds}\ln\zeta(s)$ of the Riemann zeta function is expanded as $\sum_{j=1}^{\infty} h_j(s)$ where $h_j(s) = h_1(js)$. All $h_j(s)$ are continued analytically to $Re\{s\} > 0$ by an inductive procedure. In Lemma 2 it is shown that if $h_1(s)$ has a first-order pole at some point s_k, then there must be an infinite number of poles in $h_1(s)$ and in each $h_j(s)$. Lemma 2 presents a pole cancellation procedure where poles of $h_j(s)$ cancel poles of other $h_m(s)$. This procedure leaves uncancelled only a subset of poles. In Lemma 3 it is shown that the total contribution of these uncancelled poles of $\sum_{j=1}^{\infty} h_j(s)$ must vanish at a point $(l, 0)$ where $l > 0$ goes to infinity. This yields an equation which shows that poles s_k of with $Re\{s\} > 0$ of $h_1(s)$ that are not on the x-axis must have $p_k = Re\{s_k\} = \frac{1}{2}$. It is shown that there are no zeros of zeta on the x-axis with $Re\{s\} > 0$. The effect of the zeros of zeta for $Re\{s\} \leq 0$ is calculated and it contributes to the cancellation of poles when $l \to \infty$.

Key words: Riemann zeta function, Riemann Hypothesis, complex analysis.

1 Definitions

The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \tag{1}$$

where s is a complex number. The zeta function can be continued analytically to the whole complex plane except for $s = 1$ where the function has a simple pole.

The zeta function has trivial zeros at even negative integers. It does not have zeros in $Re\{s\} \geq 1$. The nontrivial zeros lie in the strip $0 < x < 1$, see e.g. [1]. The Riemann Hypothesis claims that all these zeros have $Re\{s\} = \frac{1}{2}$. Let

$$P = \{p_1, p_2, \ldots | p_j \text{ is a prime}, p_{j+1} > p_j > 1, j \geq 1\}$$

be the set of all primes (larger than one). Let $s = x + iy$, $x, y \in \mathbb{R}$ and $x > \frac{1}{2}$. The Riemann zeta function can be expressed as

$$\zeta(s) = \prod_{j=1}^{\infty} (1 - p_j^{-s})^{-1}, \tag{2}$$

This infinite product converges absolutely if $Re\{s\} > 1$.

2 Lemmas and the theorem

Lemma 1. *The functions*

$$h_j(s) = -\sum_{j=1}^{\infty} \ln(p_j) p_j^{-js} \quad , \quad j > 0 \tag{3}$$

are related by $h_j(s) = h_1(js)$. *The functions* $h_j(s)$ *have analytic continuations to* $Re\{s\} > 0$ *with the exception of isolated first-order poles. The poles of* $h_j(s)$ *that are not on the x-axis appear in pole pairs: close to* s_k, *where* $Im\{s_k\} > 0$, $h_j(s)$ *is of the type*

$$h_j(s) = \frac{r}{s - s_k} + f_1(s) \tag{4}$$

and close to s_k^*, *where* s_k^* *is a complex conjugate of* s_k, $h_J(s)$ *is of the type*

$$h_j(s) = \frac{r}{s - s_k^*} + f_2(s)$$

The functions $f_1(s)$ *and* $f_2(s)$ *are analytic close to* s_k *and* s_k^* *respectively. If the pole is at the x-axis, there is only one pole of the type (4) with* $Im\{s_k\} = 0$.

Proof. The claim

$$h_j(s) = h_1(js) \tag{5}$$

follows directly from (3).

The function $h_1(s)$ converges absolutely if $Re\{s\} > 1$ because

$$\sum_{j=1}^{\infty} p_j^{-s}$$

converges absolutely for $Re\{s\} > 1$ and $|\ln p_j| < |p_j^{\alpha}|$ for any fixed $\alpha > 0$ if j is sufficiently large. Therefore

$$|\ln(p_j)p_j^{-s}| < 2|p_j^{-s+\alpha}|$$

for any fixed $\alpha > 0$ if j is sufficiently large. Therefore, by (5), $h_j(s)$ converges absolutely if $Re\{s\} > \frac{1}{j}$.

From (2) follows

$$\zeta'(s)\zeta(s)^{-1} = \frac{d}{ds}\ln\zeta(s) = \sum_{j=1}^{\infty} h_j(s).$$

The derivative $\zeta'(s)$ is analytic in all points except for $s = 1$. The function $h_1(s)$ is continued analytically to $Re\{s\} > \frac{1}{2}$ by

$$h_1(s) = \zeta(s)^{-1}\zeta'(s) - g(s) \tag{6}$$

where

$$g(s) = \sum_{j=2}^{\infty} h_j(s).$$

The function $\zeta(s)^{-1}$ is analytic except for at points where $\zeta(s)$ has a zero or a pole. The function $g(s)$ is analytic for $Re\{s\} > \frac{1}{2}$ because each $h_j(s)$, $j > 1$, is analytic in $Re\{s\} > \frac{1}{j}$. Thus, the right side of (6) is defined and analytic for

$\frac{1}{2} < Re\{s\}$ except for at points where $\zeta(s)$ has a zero or a pole. At those isolated points $h_1(s)$ has a pole.

At a pole s_k of $\zeta(s)$ the zeta function has the expansion

$$\zeta(s) = \frac{C}{(s - s_k)^k} + \text{higher order terms.}$$

If $Re\{s\} > \frac{1}{2}$ the function $h_1(s)$ is of the form

$$h_1(s) = \zeta'(s)\zeta(s)^{-1} - g(s) = \frac{r}{s - s_k} + f_1(s)$$

where $f_1(s)$ is analytic close to s_k and $r = -k < 0$ is an integer. The function $\zeta(s)$ has only one pole, at $s_k = 1 = (1, 0)$, and it is a simple pole, thus $r = -1$.

At a zero s_k of $\zeta(s)$ the zeta function has the expansion

$$\zeta(s) = C(s - s_k)^k + \text{higher order terms.}$$

If $Re\{s\} > \frac{1}{2}$ the function

$$h_1(s) = \zeta'(s)\zeta(s)^{-1} - g(s) = \frac{r}{s - s_k} + f_1(s)$$

where $f_1(s)$ is analytic close to s_k and $r = k > 0$ is an integer. It is known that $\zeta(s)$ has many zeros with $Re\{s_k\} = 1/2$.

Thus, $h_1(s)$ has only first-order poles for $Re\{s\} > \frac{1}{2}$ and therefore $h_j(s)$ has only first-order poles for $Re\{s\} > \frac{1}{2j}$. At every pole of $h_1(s)$ in $Re\{s\} > \frac{1}{2}$ the value of r is an integer.

As $h_1(s)$ is continued to $Re\{s\} > \frac{1}{2}$ by (6), the equation (5) continues $h_j(s)$ to $Re\{s\} > \frac{1}{2j}$. Then (6) continues $h_1(s)$ to $Re\{s\} > \frac{1}{4}$. The function $h_1(s)$ has isolated poles at $Re\{s\} > \frac{1}{4}$. Each pole is a first-order pole, but the value of r at a pole does not need to be an integer.

We can repeat the procedure inductively: If $h_1(s)$ is continued to $Re\{s\} > \frac{1}{2^i}$ by (6), the equation (5) continues $h_j(s)$ to $Re\{s\} > \frac{1}{2^i j}$. Then (6) continues $h_1(s)$ to $Re\{s\} > \frac{1}{2^{i+1}}$. By induction, all $h_j(s)$ are analytically continued to $Re\{s\} > 0$.

In this inductive process $h_1(s)$ gets isolated first-order poles. In these poles s_k the values $r = r_k$ can be positive or negative, and they do not need to be integers. If $h_1(s)$ has a pole

$$h_1(s) = \frac{r}{s - s_k} + f_1(s)$$

(here $f_1(s)$ is analytic close to s_k), then $h_j(s) = h_1(js)$ has a pole at $j^{-1}s_k$ and the r value is $j^{-1}r$ since

$$h_j(s) = h_1(js) = \frac{j^{-1}r}{s - j^{-1}s_k} + f_1(js).$$

The function $h_1(s)$ is symmetric with respect to the real axis. By (4) $h_j(s)$, $j > 1$, is also symmetric with respect to the real axis. Therefore poles of each $h_j(s)$, $j > 0$, appear as pairs s_k and s_k^*. In the special case where s_k is real there is only one pole, not a pair. □

Lemma 2. *The following two claims hold: (i) The only poles of the sum*

$$\sum_{j=1}^{\infty} h_j(s)$$

that remain after cancellations of poles of $h_j(s)$ by poles of other $h_m(s)$ are poles of $h_j(s)$ at points s_k of the type

$$\frac{r_k}{s - s_k} + f_k(s)$$

where $f_k(s)$ is analytic close to s_k. The number $r \neq 0$ is an integer. Only for one s_k the number r_k is negative and has the value -1. It is the pole of $\zeta(s)$ at $s_k = 1$. For the other poles s_k, $k > 0$, the value r_k is a positive integer. (ii) The function $h_1(s)$ has an infinite number of poles in $Re\{s\} > 0$.

Proof. Claim (i) follows directly. All poles of every $h_j(s)$, $j > 0$, that have a noninteger value of r must be cancelled or partially cancelled by poles of other $h_m(s)$, $m > 0$, because at a pole of $\zeta(s)$ and a zero of $\zeta(s)$ the value r is always an integer. Additionally, there cannot be any other poles of $\zeta(s)$ than the one at $s = 1$. Thus, in the sum of $h_j(s)$ all negative values of r sum to zero or to a positive integer value except for $h_1(s)$ in the pole $s = 1$. There $r = -1$.

For the claim (ii) we give one possible cancellation process and then notice that every cancellation process has the same features leading to (ii).

Let the set of natural numbers $\{1, 2, 3, 4, \ldots\}$ be divided into disjoints sets $C_j = \{t_j, 2t_j, \ldots, 2^k t_j, \ldots\}$, where 2 does not divide t_j and $t_{j+1} > t_j$. Thus, $t_1 = 1$, $t_2 = 3$, $t_3 = 5$, and so on.

Let us take the sum of two pairs of poles of $h_1(s)$

$$\frac{r_k}{s - s_k} + \frac{r_k}{s - s_k^*}$$

$$-\frac{2^{-1} r_k}{s - 2^{-1} s_k} - \frac{2^{-1} r_k}{s - 2^{-1} s_k^*}. \tag{7}$$

Because of (5) there must be corresponding poles at $h_m(s)$

$$\frac{\frac{1}{m} r_k}{s - \frac{1}{m} s_k} + \frac{\frac{1}{m} r_k}{s - \frac{1}{m} s_k^*}$$

$$-\frac{\frac{1}{m} 2^{-1} r_k}{s - \frac{1}{m} 2^{-1} s_k} - \frac{\frac{1}{m} 2^{-1} r_k}{s - \frac{1}{m} 2^{-1} s_k^*}. \tag{8}$$

In C_j there are corresponding poles for each $m = 2^n t_j$.

Let us sum these poles over C_j. We see that most terms cancel

$$\frac{\frac{1}{t_j} r_k}{s - \frac{1}{t_j} s_k} + \frac{\frac{1}{t_j} r_k}{s - \frac{1}{t_j} s_k^*} \tag{9}$$

$$-\frac{\frac{1}{t_j} 2^{-1} r_k}{s - \frac{1}{t_j} 2^{-1} s_k} - \frac{\frac{1}{t_j} 2^{-1} r_k}{s - \frac{1}{t_j} 2^{-1} s_k^*}.$$

$$+\frac{\frac{1}{t_j}2^{-1}r_k}{s-\frac{1}{t_j}2^{-1}s_k}+\frac{\frac{1}{t_j}2^{-1}r_k}{s-\frac{1}{t_j}2^{-1}s_k^*}$$

$$-\frac{\frac{1}{t_j}2^{-2}r_k}{s-\frac{1}{t_j}2^{-2}s_k}-\frac{\frac{1}{t_j}2^{-2}r_k}{s-\frac{1}{t_j}2^{-2}s_k^*}$$

$$\dots$$

$$+\frac{\frac{1}{t_j}2^{-i}r_k}{s-\frac{1}{t_j}2^{-i}s_k}+\frac{\frac{1}{t_j}2^{-i}r_k}{s-\frac{1}{t_j}2^{-i}s_k^*}$$

$$-\frac{\frac{1}{t_j}2^{-i-1}r_k}{s-\frac{1}{t_j}2^{-i-1}s_k}-\frac{\frac{1}{t_j}2^{-i-1}r_k}{s-\frac{1}{t_j}2^{-i-1}s_k^*}.$$

$$+\frac{\frac{1}{t_j}2^{-i-1}r_k}{s-\frac{1}{t_j}2^{-i-1}s_k}+\frac{\frac{1}{t_j}2^{-i-1}r_k}{s-\frac{1}{t_j}2^{-i-1}s_k^*}$$

$$-\frac{\frac{1}{t_j}2^{-i-2}r_k}{s-\frac{1}{t_j}2^{-i-2}s_k}-\frac{\frac{1}{t_j}2^{-i-2}r_k}{s-\frac{1}{t_j}2^{-i-2}s_k^*}$$

$$\dots$$

$$=\frac{\frac{1}{t_j}r_k}{s-\frac{1}{t_j}s_k}+\frac{\frac{1}{t_j}r_k}{s-\frac{1}{t_j}s_k^*}. \tag{10}$$

There is left only one pole pair (10) in each C_j. Especially in C_1 the function $h_1(s)$ has left the poles

$$\frac{r_k}{s-s_k}+\frac{r_k}{s-s_k^*}. \tag{11}$$

Let t_j be a prime larger than 2. We can cancel the pole pair (10) of C_j by addding two pairs of poles to $h_1(s)$

$$-\frac{\frac{1}{t_j}r_k}{s-\frac{1}{t_j}s_k}-\frac{\frac{1}{t_j}r_k}{s-\frac{1}{t_j}s_k^*} \tag{11}$$

$$+\frac{2^{-1}\frac{1}{t_j}r_k}{s-2^{-1}\frac{1}{t_j}s_k}+\frac{2^{-1}\frac{1}{t_j}r_k}{s-2^{-1}\frac{1}{t_j}s_k^*}.$$

If $\frac{1}{t_j}r_k$ is fractional, the pole pair (10) of C_j must be cancelled, but we will show that it is cancelled even if $\frac{1}{t_j}r_k$ is an integer: Because of (5), if we add the poles (11) to $h_1(s)$, then there must be corresponding poles at $h_m(s)$

$$\frac{\frac{1}{mt_j}r_k}{s-\frac{1}{mt_j}s_k}+\frac{\frac{1}{mt_j}r_k}{s-\frac{1}{mt_j}s_k^*} \tag{12}$$

$$-\frac{\frac{1}{mt_j}2^{-1}r_k}{s-\frac{1}{mt_j}2^{-1}s_k}-\frac{\frac{1}{mt_j}2^{-1}r_k}{s-\frac{1}{mt_j}2^{-1}s_k^*}.$$

For a sufficiently large prime m the number $\frac{1}{mt_j}r_k$ is not an integer. Therefore this pole of $h_m(s)$ must be cancelled. It can only be cancelled by a pole of $h_1(s)$ and therefore the poles (11) are necessary. Therefore we must add the first pole pair in (11) even if $\frac{1}{t_j}r_k$ is an integer. The pole pair in C_j is cancelled by adding the pole pairs of (11).

We do not get new poles to each C_m. The new poles (12) are added to C_m only if t_j divides t_m. When we sum these new poles (12) to each such C_m, it is the same calculation as in (9). Most poles cancel and only one pole pair remains for each C_m, namely

$$-\frac{\frac{1}{t_m}r_k}{s-\frac{1}{t_m}s_k}-\frac{\frac{1}{t_m}r_k}{s-\frac{1}{t_m}s_k^*}. \tag{13}$$

Let $j = 1$, so $t_j = 3$. We add the poles (11) to $h_1(s)$ and there remains the poles (13). Adding (13) to (10) the new poles of $h_1(s)$

$$-\frac{\frac{1}{t_j}r_k}{s - \frac{1}{t_j}s_k} - \frac{\frac{1}{t_j}r_k}{s - \frac{1}{t_j}s_k^*}$$

cancels the remaining pole pair in C_j

$$\frac{\frac{1}{t_j}r_k}{s - \frac{1}{t_j}s_k} + \frac{\frac{1}{t_j}r_k}{s - \frac{1}{t_j}s_k^*}.$$

More generally, we notice that for each C_m such that 3 divides t_m the pole pair (10) has been cancelled. Especially, the remaining pole at C_2 has been cancelled.

But now comes a complication. We continue the process by adding to $h_1(s)$ a pole that cancels the remaining pole at C_3 where $t_3 = 5$, the smallest prime larger than t_2. Thus, we add the poles (11) for $j = 3$. The remaining pole pair in (10) in the set C_3 is cancelled, but in each C_m where $15 = 3 \cdot 5$ divides t_m we have a new pole pair. That is, the original pole pair (10) in C_m was cancelled when we added (11) with $t_j = t_2 = 3$ and in this process added the poles also to C_m where t_m is divisible by $15 = 3 \cdot 5$, but now we add $t_j = t_3 = 5$ and again have a pole pair at C_m since t_m is divisible by $5 \cdot 3$. The remaining pole at C_3 was cancelled, but we made new poles to every C_m where t_m has the factor 15.

Continuing this process by adding to $h_1(s)$ poles where $t_j = t_4 = 7$ as in (11) we cancel the remaining pole at C_4 in (10), but make new poles to C_m where t_m has the factors $4 \cdot 3$ or $4 \cdot 5$. Continuing the process by adding poles (11) to $h_1(s)$ for each prime number t_j in the increasing order we cancel the remaining pole in each C_j where t_j is a prime. At the same time we are creating new poles to each C_m where t_m has two prime factors larger than two. This is the first step of the pole cancellation process.

In the second step we add poles to $h_1(s)$ that have t_j a product of two primes larger than two and select r-values that cancel the remaining poles of C_j for every

t_j that is a product of two primes larger than two. Again we add new poles to C_m where t_m has more than two prime factors that are larger than two.

In the nth step we add to $h_1(s)$ poles which are products of n primes larger than two, and select r-values that cancel the poles of C_j for each t_j that is a product of exactly n primes larger than two.

This process continues and on each step we must add poles to $h_m(s)$ where m is a so large number that the resulting r value for $h_m(s)$ is fractional. Such a pole must be cancelled and it can only be cancelled by making a next step (11) by adding poles to $h_1(s)$. Thus, this process cannot stop. It does not stop even if the r value of a pole in C_j that we want to cancel is a positive integer. There is always a large $m = t_j m_1$ that will also be created because of adding the poles to $h_1(s)$ in (11). This C_m has a fractional r value and must be cancelled. Therefore the remaining pole of C_j is always cancelled by a new pole of $h_1(s)$.

The new poles that we are adding in each step to $h_1(s)$ are on each step closer and closer to $s = 0$. The r-values of the poles that we have to add to $h_1(s)$ become very large in absolute value when the number n of steps grows and the new poles added to $h_1(s)$ approach $s = 0$. The absolute value of r grows because the numbers of the remaining C_m on each step have many factors and we add as many poles as there are factors to those poles that do not get cancelled in each step. We cannot reach $s = 0$, but every C_j will have the poles completely cancelled at some step.

As a conclusion, $h_1(s)$ must have an infinite number of poles because this pole cancellation process cannot stop. Only some poles of $h_1(s)$ remain uncancelled in the sum $\sum_{j=1}^{\infty} h_j(s)$.

The cancellation of poles on the x-axis is the same, only there is one pole and not a pole pair. The procedure is obtained by setting $Im\{s_k\} = 0$ in the described process and removing the part with s_k^*. There is a pole of $h_1(s)$ on the x-axis at $s = \frac{1}{2}$ with the r-value $\frac{1}{2}$. This is so because $h_1(s)$ has a pole at $s = 1$ with the r-value -1. Consequently $h_2(s)$ has a pole with r-value $-\frac{1}{2}$ at $s = \frac{1}{2}$. This pole must be cancelled and can only be cancelled by $h_1(s)$. Therefore there must be the pole of $h_1(s)$ at $s = \frac{1}{2}$, but it is cancelled and the only pole remaining on the x-axis from this sequence is the pole of $h_1(s)$ at $s = 1$. However, $h_1(s)$ may have other uncancelled poles with a positive integer r-value on the x-axis. If there are such poles of $h_1(s)$, then they are not cancelled by this process.

We will not prove that the described pole cancellation process is the only possible process, though this claim may be true. However, every possible pole cancellation process has the same feature: every pole at $h_1(s)$ requires a corresponding pole at each $h_m(s)$. If m is sufficiently large, such a pole has a noninteger r-value and must be cancelled. Cancelling such a pole by adding (i.e., noticing that the pole exists) a pole to any $h_j(s)$ always implies adding a new pole to $h_1(s)$ and this again requires new poles to all $h_m(s)$. Again some of these new poles have noninteger r-values and must be cancelled. This process cannot stop, thus $h_1(s)$ must have an infinite number of poles. Only a subset of the poles of $\sum_{j=1}^{\infty} h_j(s)$ remain uncancelled. □

Lemma 3. *The uncancelled poles of $\sum_{j=1}^{\infty} h_j(s)$ in $Re\}s\} > 0$ are one pole at $s = 1$ and an infinite set of pole pairs at symmetric places in the positive and negative imaginary axis at $Re\{s\} = \frac{1}{2}$.*

Proof. Let us consider a function $f(s)$ that has a first-order pole at s_0 and write $z_1 = s - s_0$. The function $f(s)$ does not have a Taylor series at s_0, but the function $z_1 f(z1 + s_0)$ has a Taylor series at $z_1 = 0$ and $f(s)$ can be expressed as

$$f(s) = \frac{c_{-1}}{z_1} + \sum_{k=0}^{\infty} c_k z_1^k.$$

Let us evaluate $f(s)$ at another point at $s_0 + l$, $l > 0$, by first writing $z_1 = l - z_2$ where $|z_1| << 1$, inserting $z_1 = l - z_2$ to the series expression of $f(s)$, and then considering the result when $|z_2| << 1$. The function

$$f_1(z_1) = f(z_1 + s_0) - \frac{c_{-1}}{z_1}$$

has the Taylor series at $z_1 = l - z_2$ where $|z_1| << 1$ as

$$f_1(l - z_2) = \sum_{m=0}^{\infty} c_m (l - z_2)^m$$

$$= \sum_{m=0}^{\infty} \sum_{i=0}^{m} \frac{m!}{i!(m-i)!} l^i (-z_2)^{m-i} c_m$$

$$= \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{(k+i)!}{i!k!} l^i (-1)^k c_{k+i} z_2^k = \sum_{k=0}^{\infty} b_k z_2^k.$$

Thus

$$b_k = \sum_{i=0}^{\infty} \frac{(k+i)!}{i!k!} l^i (-1)^k c_{k+i}.$$

As

$$c_k = \frac{1}{k!} \frac{d^k}{dz_1^k} f_1(s)|_{z_1=0}$$

we can express

$$b_k = \left(\sum_{i=0}^{\infty} \frac{1}{i!} l^i \frac{d^i}{dz_1^i} \right) \frac{1}{k!} (-1)^k \frac{d^k}{dz_1^k} f_1(s)|_{z_1=0}.$$

If there is no pole of $f(s)$ at $s_0 + l$, the function

$$f_1(l - z_2) = \sum_{k=0}^{\infty} b_k z_2^k$$

is analytic and defined by its Taylor series as powers of z_2 where the series converges. The pole of $f(s)$ at c_{-1} can be evaluated as a Taylor series of z_2 at $s_0 + l$ as

$$\frac{c_{-1}}{l - z_2} = \frac{c_{-1}}{l} \frac{1}{1 - z_2 l^{-1}} = \frac{c_{-1}}{l} \sum_{k=0}^{\infty} \left(\frac{z_2}{l} \right)^k.$$

We can subtract a set of first-order poles of $f(s)$ in points $s_j \in A$ and define

$$f_1(z_1) = f(s) - \sum_{j \in A} \frac{r_j}{s - s_j}$$

where $r_j = c_{-1,j}$ and express

$$s - s_k = (s - s_0) - (s_j - s_0) = z_1 - s_j + s_0 = l - z_2 - (s_j - s_0).$$

At the point $s_0 + l$ the set of poles is

$$\sum_{j \in A} \frac{r_j}{s - s_j} = \sum_{j \in A} \frac{r_j}{l - z_2 - (s - s_0)} = x \sum_{j \in A} \frac{r_j}{1 - p_j x} \tag{14}$$

where $p_j = s_j - s_0$ and $x = (l - z_2)^{-1}$. Let us select $s_0 = 0$ for easier notations. Thus, $p_j = Re\{s_j\}$ is the x-coordinate of the pole s_j. Let us consider

$$f(s) = \sum_{k=1}^{k_1} \frac{r_k}{s - s_k} + f_1(s)$$

$$f_1(s) = -\sum_{j=1}^{j_{\max}} \ln(p_j) p_j^{-s}.$$

Let $l >> 1$. The Taylor series of the set of poles points s_k at s_0 in powers of z_1 is

$$-\sum_{i=0}^{\infty} \left(\sum_{k=1}^{k_1} r_k (s_k - s_0)^{-i-1} \right) z_1^i$$

and the Taylor series at $s_0 + l$ in powers of $z_2 = l - z_1$ is

$$\sum_{i=0}^{\infty} \left(\sum_{k=1}^{k_1} r_k (s_0 + l - s_k)^{-i-1} \right) z_2^i.$$

For each k the coefficient of the ith power of z_1 at s_0 is $c_i = r_k (s_k - s_0)^{-i-1}$ while the coefficient of z_2 at $s_0 + l$ is

$$p_i = r_k (s_0 + l - s_k)^{-i-1} = r_k l^{-i-1} + r_k (i+1)(s_k - s_0) l^{-i-2} + \cdots$$

The absolute value of the coefficient p_i of the Taylor series in powers of z_2 at s_0+l decreases as

$$|\sum_{k=1}^{k_1} r_k| l^{-i-1}$$

as a function of $l >> 1$. The part $f_1(s)$ of $f(s)$ satisfies

$$|f_1(s+l)| = \left| -\sum_{j=1}^{j_{\max}} \ln(p_j) p_j^{-s-l} \right| = \left| -\sum_{j=1}^{j_{\max}} \ln(p_j) p_j^{-s} e^{-l \ln p_j} \right|$$

$$\leq \left| e^{-l \ln 2} \right| \left| -\sum_{j=1}^{j_{\max}} \ln(p_j) p_j^{-s} \right| = e^{-l \ln 2} |f_1(s)|.$$

The absolute value of the coefficient p_i of the Taylor series in powers of z_2 at s_0+l decreases as

$$|p_i| \leq e^{-l \ln 2} |c_i|.$$

This is negative exponential decrease and much faster than the hyperbolic decrease for the set of poles.

Let us $k_1 \to \infty$. Then $f(s) = h_1(s)$. If $Re\{s\} = l >> 1$, we are far away of the pole at $s = 1$ and the sum in $h_1(s)$, where k_1 is replaced by infinity, converges absolutely. The absolute values of the Taylor series at $s_0 + l$ for the function $h_1(s)$ must decrease in negative exponential manner as a function of l. This implies that the poles partially cancel. The poles cannot completely cancel: a pole at s_k with $r = r_k$ can be completely cancelled only by a pole at s_k with $r = -r_k$. The sum of poles has all poles of its terms, but at $l >> 1$ there can be partial cancellation so that the Taylor series coefficients decrease faster as a function of l. Especially, when $l \to \infty$, the contribution from the poles must vanish.

The condition that the poles sum to zero when $l \to \infty$ is not a sufficient condition for the contribution of the poles to vanish and to leave the contribution of the negative exponential behavior of $f_1(s)$ to dominate at the limit $l \to \infty$. The sum of the poles clearly decreases as $O(x)$, $x = l^{-1}$, and goes to zero when

$x \to 0$ if the x-coordinate of every pole of $f(s)$ is smaller or equal to one, but this convergence to zero is not enough. The exponent of x grows faster than any power of x, thus the negative exponent of x decreases faster than any negative power of x. In order for the contribution of the poles to vanish and for the negative exponential terms of $f_1(s)$ to dominate, the coefficient of each power of x in the power series of the sum of the poles as a function of x must go to zero at least as $O(x)$. We require this type of stronger convergence from the sum of the poles of $f(s)$ as a function of x when $x \to 0$.

Lemma 2 shows that $h_1(s)$ has infinitely many poles, yet $h_1(s)$ has the behavior of the sum of negatively exponential terms when l is very large. This implies that the contribution of the poles of $h_1(s)$ must vanish when $l \to \infty$ in the sense that all coefficients of the power series of x go to zero at least as $O(x)$. Consequently the sum of the poles of every $h_j(s) = h_1(js)$ must vanish in the same sense in the limit $l \to \infty$. Therefore the sum of the poles of

$$f(s) = \frac{d}{ds} \ln \zeta(s) = \sum_{j=1}^{\infty} h_j(s) = -\sum_{j=1}^{\infty} (\ln p_j) p_j^{-1} \frac{1}{1 - p_j^{-ks}} = h_1(s) + \text{smaller terms}$$

must also vanish in the same sense when $l \to \infty$.

By Lemma 2 $f(s)$ can only have the following poles in $Re\{s\} > 0$:

(i) There is one pole with $r = -1$ at $s = 1$.

(ii) There is a set A of pole pairs of $h_1(s)$ at s_k and s_k^* where s_k has a nonzero imaginary part, and the r-value r_k is positive. All we know of s_k is that the real part of s_k is larger than zero and smaller than one, and that that there are poles s_k with the real part $\frac{1}{2}$.

(iii) There may be a set A_1 of poles $s_{k,1}$ of $h_1(s)$ with $r_{k,1}$ a positive integer and the pole s_k is real, $0 < s_k < 1$ and $s_k \neq \frac{1}{2}$. No such pole is known.

In Lemma 1 we did not continue $h_j(s)$ to the area $Re\{s\} \leq 0$. The function

$$f(s) = \sum_{j=1}^{\infty} h_j(s)$$

is analytically continued by

$$f(s) = \frac{d}{ds} \ln \zeta(s)$$

to $Re\{s\} \leq 0$ to all points where $\zeta(s) \neq 0$. The zeros of $\zeta(s)$ has zeros in the area $Re\{s\} \leq 0$ are the so called trivial zeros at even negative integers. They come from the formula

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$$

where $B_m = 0$ if $m > 1$ is odd. Zeta does not have a zero at $s = 0$. From the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin(2^{-1}\pi s) \Gamma(1-s) \zeta(1-s) \tag{15}$$

we can deduce that the trivial zeros are zeros of $\sin(2^{-1}\pi s)$ and therefore first-order zeros, thus, at a point $s_k = -2k$, $k > 0$, the function $f(s)$ has a first-order pole with the r-value 1.

Using the expression (14) instead of (4) for a pole or a pole pair (i.e., $s = s_0 + l$, $s_0 = 0$, $x = l^{-1}$) gives

$$\frac{r_k}{s - s_k} = \frac{x r_k}{1 - p_k x}$$

for a pole on the x-axis. A pole pair in the positive and negative y-axis can be written

$$\frac{r_k}{s - s_k} = \frac{x r_k}{1 - (1 + i\alpha_k) p_k x}$$

$$\frac{r}{s - s_k^*} = \frac{x r_k}{1 - (1 - i\alpha_k) p_k x}.$$

Here $x = (l - z_2)^{-1} > 0$ is a small real number if l is large, $p_k = Re\{s_k\}$ and α_k is chosen positive. The number l is the distance from $s_0 = 0$ to the observation point on the x-axis, $(l, 0)$, where the Taylor series with z_2 is evaluated and $|z_2| << 1$.

As z_2 is the variable of the Taylor series at $(l, 0)$, the expressions are valid for any small z_2 and we will select $z_2 = 0$ for easier notations. Thus, $x = l^{-1}$.

The pole (i) at $s = 1$ gives the power series of x where $p_k = 1$ and $r = -1$

$$\frac{xr}{1-(p_k x)} = \frac{-x}{1-x} = -x \sum_{m=0}^{\infty} x^m.$$

A pole at $s_k = -2k$, $k > 0$, is

$$\frac{r_k}{s - s_k} = \frac{1}{s+2k}.$$

We can evaluate the Taylor series of z_1 at s_0 and the Taylor series of z_2 at $s_0 + l$ for any such pole and for a finite sum of such poles:

$$\frac{1}{s_0 + z_1 + 2k} = \frac{1}{s_0 + 2k} \sum_{i=0}^{\infty} (-1)^i (s_0 + 2k))^{-1} z_1^i$$

$$\frac{1}{s_0 + l - z_2 + 2k} = \frac{1}{s_0 + l + 2k} \sum_{i=0}^{\infty} (s_0 + l + 2k))^{-1} z_2^i$$

but if we sum the index k to infinity, the series diverges at every point $s_0 + l$. Since $f(s)$ is finite at every point where $Re\{s_0\} + l > 1$ this means that we cannot evaluate the Taylor series at $s_0 + l$ by summing over k. We will cope with this difficulty by first evaluating the sum of these poles at $s_0 = 0$ and then moving the result to $s_0 + l$. Let $s_0 = 0$ and $z_1 = 0$. Notice that for a point $s_j = -k$ the pole at that point, with the r-value r, when evaluated to a Taylor series at $s = 0$ and z_1 is set to zero, is

$$\frac{r}{s - s_j} = \frac{r}{k}.$$

This is the inverse of a pole with the same r but with $s_j = k$ when evaluated to a Taylor series at $s = 0$ and z_1 is set to zero. As an example, $s_j = 1$ is the pole at $s = 1$ with $r = -1$. When evaluated at $s_0 = 0$ it is the inverse of a pole with $r = -1$ but $s_k = -1$. Thus, the pole at $s_k = -2k$ with $r = 1 > 0$ is the same at

$s_0 = 0$ as a a pole at $s_k = 2k$ with $r' = -1 < 0$. We see that the sum of all poles $s_k = -2k$ gives a negative contribution to the Taylor series of z_1 at $s_0 = 0$ when z_1 is set to zero.

The type of infinity of the sum of the poles $s = -2k$ at $s_0 = 0$ can be calculated. Using the facts that $\zeta(s)$ has a simple pole at $s = 1$

$$\zeta(s) = \frac{a}{s-1} + g(s)$$

where $g(s)$ is analytic at $s = 1$ and that $\lim_{s\to 1}(s-1)\zeta(s) = 1$, so $a = 1$, we can write

$$\zeta(1) = \lim_{s\to 1} \frac{1 + (s-1)f(1)}{s-1} = \lim_{s\to 1} \frac{1}{s-1} = \lim_{s\to 0} \frac{1}{s}$$

This result gives

$$\sum_{k=1}^{\infty} \frac{1}{2k} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^1} = \frac{1}{2}\zeta(1) = \lim_{s_0\to 0} \frac{1}{2} \frac{1}{s_0}.$$

Thus, the contribution from the sum of the poles at $s_k = -2k$ is a simple pole at $s_0 = 0$. The pole is a negative infinity $\lim_{s\to 0}(-1/(2s))$ at $s = 0$. The poles with $Re\{s_k\} \leq 0$ sum to a single pole $-1/2z_1$ at $s_0 = 0$ and yield the Taylor series of z_2

$$\frac{-0.5}{s_0 + l - z_2 + 2k} = \frac{-0.5}{l + 2k} \sum_{i=0}^{\infty} (l + 2k)^{-1} z_2^i$$

at $s_0 + l = l$. When we set $s = s_0 + l$, $s_0 = 0$ and $x = l^{-1}$ as in (14), the contribution from the poles at $s = -2k$, $k > 0$, at $s_0 + l$ is $-x/2$.

The poles (iii) of A_1 sum to a series $x \sum_{m=0}^{\infty} c_m x^m$ where every c_m is nonnegative and $c_{i+1} \neq c_i$ in the limit when $x \to 0$ because all of these poles are in the area $0 < s_k < 1$ and they are isolated and therefore do not have a concentration point at $s = 1$. It follows that they cannot be cancelled in $x \to 0$ by the the sum of poles in $Re\{s\} \leq 0$ giving the contribution $-x/2$ and the pole at $s = 1$ giving the contribution $-x/(1-x)$. Therefore the poles (iii) could only be cancelled by a sequence of poles of the type (ii), but the poles of (ii) also give a power series of x where the coefficient of every x^i is nonnegative. Thus, the poles of A_1 cannot be cancelled by any set of other poles and the set A_1 must be empty.

The coefficient of the power one of x can be cancelled by the corresponding coefficients of the pole at $s = 1$ and the poles in $Re\{s_k\} \leq 0$, but there are no other poles than the pole at $s = 1$ that can cancel the higher than power one coefficients of x coming from a pole pair (ii). Thus, the coefficient of every power $i > 1$ of x in the sum of pole pairs (ii) must be cancelled by the corresponding coefficient of x in the pole (i) at least to the degree of $O(x)$.

The two poles (ii) of a pole pair have a real sum:

$$\frac{xr_k}{1 - p_k(1 + i\alpha_k)x} + \frac{xr_k}{1 - p_k(1 - i\alpha_k)x} = xr_k \frac{2(1 - p_k x)}{1 - 2p_k x + (1 + \alpha_k^2)(p_k x)^2}.$$

We expand the sum S of the poles of a pole pair omitting the multiplier xr_k for simplicity in this calculation:

$$S = \frac{2(1 - p_k x)}{1 - 2p_k x + \alpha_k^2 (p_k x)^2} = \frac{2 - 2p_k x}{1 + \alpha_k^2 (p_k x)^2} \frac{1}{1 - 2p_k x \gamma_k^{-1}}$$

where $\gamma_k = 1 + \alpha_k^2 (p_k x)^2$.

$$= \frac{2 - 2p_k x}{\gamma_k} \sum_{i=0}^{\infty} (2p_k x \gamma_k^{-1})^i.$$

Writing $\beta_{k,i} = (2p_k)^i \gamma_k^{-i-1}$ we get

$$S = 2\sum_{i=0}^{\infty} \beta_{k,i} x^i - 2p_k \sum_{i=0}^{\infty} \beta_{k,i} x^{i+1} = \sum_{i=0}^{\infty} 2\beta_{k,i} x^i - 2p_k \sum_{i=1}^{\infty} \beta_{k,i-1} x^i$$

$$= 2\beta_{k,0} + \sum_{i=1}^{\infty} (2\beta_{k,i} - 2p_k \beta_{k,i-1}) x^i.$$

For $i > 0$

$$2\beta_i - 2p_k \beta_{k,i-1} = 2\frac{(2p_k)^{i-1}}{\gamma_k^i} (2p_k \gamma_k^{-1} - p_k)$$

$$= \frac{(2p_k)^i}{\gamma_k^{i+1}}(2-\gamma_k) = \beta_{k,i}(2-\gamma_k).$$

This gives an equation for every $i > 0$

$$2\beta_i - 2p_k\beta_{k,i-1} = 2\beta_{k,i} - \gamma_k\beta_{k,i}.$$

Inserting $\gamma_k = 1 + (\alpha_k p_k x)$

$$2p_k\beta_{k,i-1} = \gamma_k\beta_{k,i} = \beta_{k,i} + x^2(\alpha_k p_k)^2\beta_{k,i}.$$

For every k when $l >> 1$ and therefore $0 < x = l^{-1} << 1$ holds

$$2p_k\beta_{k,i-1} = \gamma_k\beta_{k,i} = \beta_{k,i} + O(x^2).$$

The coefficient of the the power x^i is

$$2\beta_{k,i} - 2p_k\beta_{k,i-1} = \beta_{k,i} + O(x^2). \tag{16}$$

The coefficient of the power of x^{i+1} in for the pole in $s = 1$ (i.e., in the power series of $x/(1-x)$) is -1 for every $i > 0$. The coefficient of x^{i+1} in the sum of poles (ii) is

$$\sum_{k\in A} r_k(2\beta_{k,i} - 2p_k\beta_{k,i-1})$$

where we have included the multiplier xr_k that was so far omitted. Summing the powers of i from $i = 2$ to $i = i_1 + 1$ and inserting (16) gives the equation where the coefficients of the pole pairs (ii) cancel the coefficients of the pole (i) to the degree of $O(x^2)$:

$$-i_1 = \sum_{i=2}^{i_1+1}(-1) = \sum_{i=2}^{i_1+1}\sum_{k\in A} r_k\beta_{k,i} + O(x^2). \tag{17}$$

For each k, when $x \to 0$, holds

$$\beta_{k,i} = 2p_k\beta_{k,i-1}. \tag{18}$$

If every $p_k = \frac{1}{2}$ the recursion equation (18) gives $\beta_{k,i+1} = \beta_{k,i}$ for every k. Then for every k the power series of x for $i > 1$ is of the form $x\beta_{k,1}(x + x^2 + x^3 + \cdots)$. This is the same form as the power series $x(x + x^2 + x^3 + \cdots)$ for the pole $s = 1$ for $i > 1$. The power series for the poles k for $i > 1$ add to one power series of the type $xb(x + x^2 + x^3 + \cdots)$. We see that if every $p_k = \frac{1}{2}$, the sum of poles (ii) do cancel the pole in $s = 1$ when $x \to 0$ and converge to the series of the pole in $s = 1$ with the same $O(x^2)$ speed in every power x^i for $i > 1$. The question is if every p_k must be $\frac{1}{2}$.

The form of (18) for a nonzero x is

$$\beta_{k,i} = 2p_k\beta_{k,i-1} + O(x^2). \tag{19}$$

Assume one p_k is not $\frac{1}{2}$. The functional equation (15) shows that if there exists a zero $s_0 = x_0 + iy_0$ of $\zeta(s)$ with $0 < x_0 < \frac{1}{2}$ then there exists a zero of $\zeta(s)$ at a symmetric point in $\frac{1}{2} < x < 1$. This implies that we can find $s_{k'}$ such that $2p_{k'} > 1$ in (19). The recursion (19) gives $\beta_{k',i} = \beta_{k',1}(2p_{k'})^i + O(x^2)$. In (17) the right side

$$-i_1 = \sum_{i=2}^{i_1+1}\sum_{k\in A} r_k\beta_{k,i} + O(x^2) \tag{20}$$

grows at least as fast as $\beta_{k',1}(2p_{k'})^{i_1}$ as a function of i_1 while the left side is linear in i_1. This is a contradiction. Thus, every p_k must be $\frac{1}{2}$.

Though the proof of Lemma 3 is already complete, let us look at the coefficients

$$\beta_i = \sum_{k\in A} r_k\beta_{k,i} \tag{21}$$

to check if we find a contradiction from the coefficients. Equation (20) implies that $\beta_i = 1$ for every $i > 0$. Is this a contradiction?

Inserting $p_k = 2^{-1}$ to (21) gives

$$\beta_i = \sum_{k \in A} r_k (1 + 2^{-2}(\alpha_k x)^2)^{-i-1}. \tag{22}$$

We notice that because $x \to 0$ the values of α_k must grow to infinity with k. We renumber the poles of (ii) so that (α_k) is a growing sequence and the sum $k \in A$ is the sum $k = 1$ to infinity. The set A is necessarily infinite.

As $p_k = \frac{1}{2}$ we can evaluate

$$2\beta_{k,i} - 2p_k \beta_{k,i-1} = \beta_{k,i}(2 - \gamma_k)$$

and get

$$\beta_{k,i} = \beta_{k,i-1}\left(1 - \frac{(0.5\alpha_k x)^2}{1 + (0.5\alpha_k x)^2}\right).$$

Let $l >> 1$ be fixed. If $\alpha_k >> l = x^{-1}$, then

$$\frac{(0.5\alpha_k x)^2}{1 + (0.5\alpha_k x)^2}$$

is close to one and $\beta_{k,i}$ is close to zero. This means that large values of α_k contribute very little to the Taylor series at $s_0 + l$. The sum in (22) can be finite. The recursion (18) for $p_k = \frac{1}{2}$ shows that for every $i > 0$ the value $\beta_{k,i}$ is the same when $x \to 0$. The sum can be one for every i when $l \to \infty$. There is no reason why every β_i, $i > 0$, could not be one, thus we do not find a contradiction from the coefficients. The value of every β_i, $i > 0$, must be one because the contribution from the poles of $f(s)$ must disappear when $l \to \infty$.

We have the remaining coefficient

$$2\beta_0 = 2\sum_{k=1}^{\infty} r_k (1 + 2^{-2}(\alpha_k x)^2)^{-1}. \tag{23}$$

All poles cancel in the limit $l \to \infty$ if $\beta_0 = 3/4$: the contribution at $s_0 + l$ of the poles in $Re\{s\} < 0$ is $-x/2$, the contribution from the pole at $s = 1$ is $-x/(1-x)$, the contribution from the poles of A_1 is 0, and the contribution of the pole pairs of A is approaching the series $x(3/2 + x + x^2 + x^3 + \cdots)$ when $x \to 0$ as $O(x^2)$ separately for the coefficient of each power i of x^i. These contributions sum to zero when $l \to \infty$:

$$-x/2 - \frac{x}{1-x} + 0 + x(3/2 + x + x^2 + x^3 + \cdots) = 0.$$

The convergence of the sum of these contributions to zero when x grows is $O(x^2)$ for the coefficient of each power i of x^i separately. This is enough for the negative exponential factors of $f_1(s)$ ($f_1(s)$ is the sum of $h_j(s)$ minus the poles as in the beginning of the proof of Lemma 3) to dominate when $l \to \infty$.

Consequently, β_0 must be 3/4 in the limit when $x \to 0$. This may seem a bit strange since $\beta_i = 1$ for every $i > 0$ in the limit $x \to 0$, but it must be so. It is not a contradiction: it is not possible to evaluate what the value (23) is by direct calculation, β_0 is not necessarily 1 even if the other β_i for $i > 0$ have the value 1 in the limit. □

Lemma 3 proves the Riemann Hypothesis.

Theorem 1. *If $\zeta(s_k) = 0$ and $0 < Re\{s_k\} < 1$ then $Re\{s_k\} = \frac{1}{2}$.*

Proof. The claim follows directly from Lemma 3. □

Notice that we get the value $Re\{s_k\} = p_k = \frac{1}{2}$ because the recursion (19) must yield the same form of the power series of x as the pole at $s = 1$ when $x \to 0$ in order for the the set of pole pairs to cancel the pole at $s = 1$ in the limit $l \to \infty$. We get (19) from the expansion of a pole pair. Thus, ultimately we get $Re\{s_k\} = \frac{1}{2}$ because pole pairs cancel a pole at $s = 1$ and therefore the real part of the poles in the pole pairs must be the half of one.

All known facts of the Riemann zeta function that are used in this proof can be found in [1]. The history and background of the Riemann Hypothesis are well described in the book [2]. As the problem is still open, recently published results do not add so much to the topic and as they are not needed in this proof, they are not referred to.

References

1. E. T. Whittaker and G. N. Watson, A Course in Modern Analysis, Cambridge, University Press, 1952.
2. K. Sabbagh, The Riemann Hypothesis, the greatest unsolved problem in mathematics, Farrar, Strauss and Giroux, New York, 2002.